COLLECTED PAPERS OF G. H. HARDY

INCLUDING JOINT PAPERS WITH J. E. LITTLEWOOD AND OTHERS

EDITED BY A COMMITTEE APPOINTED BY THE LONDON MATHEMATICAL SOCIETY

I

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VOLUME I

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The main object of this publication is to render more accessible the papers of the great mathematician, which in their original form appeared in many journals over a period of almost 60 years. The editors have kept in view a second object also; that of rendering the work useful to mathematicians generally by providing introductions to groups of papers, or comments where appropriate. These editorial additions, while not always systematic or exhaustive, will (it is hoped) assist the reader to view Hardy's papers in proper perspective.

The work will be completed in seven volumes.

G. H. Hardy

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VOLUME I

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PREFACE

The main object of this publication is to render more accessible the papers of a great mathematician, which in their original form appeared in many journals over a period of about 50 years. The editors have kept in view a second object also: that of rendering the work useful to mathematicians generally by providing introductions to groups of papers, or comments where appropriate. These editorial additions, while not always systematic or exhaustive, will (it is hoped) assist the reader to view Hardy's papers in proper perspective.

It is this second object which has led the editors to divide the papers into groups (or further into subgroups) in accordance with the nature of their subject-matter, instead of publishing them in chronological order. The editors have been very conscious of the difficulty of making such a classification, which is most acute in those instances in which a paper that is primarily on one topic has subsequently proved to be of great importance for another. There are cases in which our allocation of a paper to one section rather than to another has been in the nature of an arbitrary choice, but we hope that adequate cross-references are provided. It may be a matter for regret that our policy has sometimes resulted in distributing the papers of a series (such as 'Notes on the Integral Calculus') among several sections, but we believe that any arrangement which kept them together would have been less satisfactory.

We are grateful to Professor Littlewood for his permission to include all the Hardy–Littlewood papers, and for his approval of our policy of treating them on the same footing as Hardy's own papers.

THE EDITORS

ACKNOWLEDGEMENTS

THE editors are grateful to the following societies and publishers of journals who have kindly given permission for the reproduction of Hardy's papers. Details of the sources are given in the *List of papers by G. H. Hardy*, which appears at the end of each volume.

Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg: Messrs. A. Liebing (Neudruck Journalfranz), Würzburg.

Acta Mathematica: Institut Mittag-Leffler, Sweden.

American Mathematical Monthly: the Mathematical Association of America.

Annali della Scuola Normale Superiore de Pisa: The Director of the Scuola Normale Superiore.

Annals of Mathematics: the Editors.

British Association Reports: the British Association for the Advancement of Science.

Bulletin of the American Mathematical Society: the American Mathematical Society.

Bulletin of the Calcutta Mathematical Society: the Calcutta Mathematical Soceity.

Comptes Rendus de l'Académie des Sciences: Messrs. Gauthier-Villars, Paris.

Duke Mathematical Journal: the Editor.

Fundamenta Mathematicae: the Editor.

Journal für die reine und angewandte Mathematik: Messrs. Walter de Gruyter & Co., Berlin.

Journal of the Indian Mathematical Society: the Council of the Indian Mathematical Society.

Journal of the London Mathematical Society: the Council of the London Mathematical Society.

Matematisk Tidsskrift: Dansk Matematisk Forening, Copenhagen.

Matematiskfysiske Meddelelser: Det Kongelige Danske Videnskabernes Selskab, Copenhagen.

Mathematical Gazette: the Mathematical Association.

Mathematische Annalen: Springer-Verlag, Heidelberg.

Mathematische Zeitschrift: Springer-Verlag, Heidelberg.

Mind: the Editor.

Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen: die Akademie der Wissenschaften zu Göttingen.

Proceedings of the Cambridge Philosophical Society: the Cambridge Philosophical Society.

Proceedings of the London Mathematical Society: the Council of the London Mathematical Society.

Proceedings of the National Academy of Sciences: the National Academy of Sciences, Washington, D.C.

Proceedings of the Royal Society: the Council of the Royal Society.

Rendiconti del Circolo Matematico di Palermo: the Editor.

Tôhoku Mathematical Journal: the Editor.

Transactions of the American Mathematical Society: the American Mathematical Society.

Transactions of the Cambridge Philosophical Society: the Cambridge Philosophical Society.

EDITORIAL NOTE

The work will comprise seven volumes

FOR convenience of reference, papers are numbered according to years, e.g. 1912, 4. A complete list of Hardy's papers will be found at the end of this volume (pp. 683–99) and will be reproduced at the end of each volume. This list is based on that compiled by Titchmarsh (*Journal of the London Mathematical Society*, 25 (1950), 89–101).

The date of publication of a paper, where it differs from the year mentioned in the reference number, is given (for the sake of its historical interest) in the contents list of the volume containing the paper.

Where reference is made, in the corrections or comments, to the pages of a paper, the numbers used are those of the original pagination and not the consecutive page numbers of this volume. The joint papers with Ramanujan are reproduced from *The Collected Papers of S. Ramanujan* (Cambridge, 1927), and for these the 'original pagination' relates to that volume and not to the first publication in a journal.

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* Reprinted with slight changes from *Obituary Notices of Fellows of the Royal Society*, 6 (1949), 447–58.

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(*c*) *Goldbach's Problem*

(d) Inaugural Lecture

GODFREY HAROLD HARDY

1877–1947

GODFREY HAROLD HARDY was born on 7 February 1877, at Cranleigh, Surrey. He was the only son of Isaac Hardy, Art Master, Bursar and House Master of the preparatory branch of Cranleigh School. His mother, Sophia Hardy, had been Senior Mistress at the Lincoln Training College. Both parents were extremely able people and mathematically minded, but want of funds had prevented them from having a university training.

The future professor's interest in numbers showed itself early. By the time he was two years old he had persuaded his parents to show him how to write down numbers up to millions. When he was taken to church he occupied the time in factorizing the numbers of the hymns, and all through his life he amused himself by playing about with the numbers of railway carriages, taxi-cabs and the like.

He and his sister were brought up by enlightened parents in a typical Victorian nursery, and, as clever children do, he agonized his nurse with long arguments about the efficacy of prayer and the existence of Santa Claus: 'Why, if he gives me things, does he put the price on? My box of tools is marked 3s. 6d.' The Hardy parents had many theories about education. Their children had few books, but they had to be good literature. In the nursery G. H., who was slightly older than his sister, read to her such books as *Don Quixote*, *Gulliver's Travels* and *Robinson Crusoe*. They were never allowed to play with any toy that was broken and past repair. The nurse gave them some instruction in reading and writing, but they never had a governess, and on the whole were left to find things out for themselves.

A minute newspaper written by G. H. at the age of eight was unfortunately lost in the London blitz. It contained a leading article, a speech by Mr Gladstone, various tradesmen's advertisements, and a full report of a cricket match with complete scores and bowling analysis. He also embarked on writing a history of England for himself, but with so much detail that he never got beyond the Anglo-Saxons. Two exquisite little illustrations for this work have survived. He inherited artistic ability from his father, but it was crushed by bad teaching at Winchester. He had no interest in music.

As soon as he was old enough G. H. went to Cranleigh School, and by the time he was twelve he had passed his first public examination with distinctions in mathematics, Latin and drawing. By this time too he had reached the sixth form—the Cranleigh standard was at that time very low—so some of his work

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was sent to Winchester. He was offered a scholarship there on his mathematics alone, but was considered too young to go that year, and went the following year.

Apparently he was never taught mathematics in a class. Mr Clarke, Second Master at Cranleigh, and Dr Richardson, Head of 'College', always coached him privately. He was never enamoured of public school life. He was grateful to Winchester for the education it gave him, but the Spartan life in 'College' at that time was a great hardship, and he had one very bad illness.

There was some question of his going up to New College, but his mind was turned in the direction of Cambridge by a curious incident, which he has related in *A Mathematician's Apology*. He happened to read a highly coloured novel of Cambridge life called *A Fellow of Trinity*, by 'Alan St Aubin' (Mrs Frances Marshall), and was fired with the ambition to become, like its hero, a fellow of Trinity. He went up to Trinity College, Cambridge, as an entrance scholar in 1896, his tutor being Dr Verrall. He was first coached by Dr Webb, the stock producer of Senior Wranglers. He was so annoyed by Webb's methods that he even considered turning over to history, a love of which had been implanted in him by Dr Fearon, Headmaster of Winchester. However, his Director of Studies sent him to A. E. H. Love, and this, he considered, was one of the turning points of his life, and the beginning of his career as a 'real mathematician'. Love was, of course, primarily an applied mathematician; but he introduced Hardy to Jordan's *Cours d'Analyse*, the first volume of which had been published in 1882, and the third and last in 1887. This must have been Hardy's first contact with analysis in the modern sense, and he has described in *A Mathematician's Apology* how it opened his eyes to what mathematics really was.

Hardy was fourth wrangler in 1898, R. W. H. T. Hudson being Senior Wrangler, with J. H. Jeans and J. F. Cameron, later Master of Gonville and Caius, bracketed next. He took Part II of the Tripos in 1900, being placed in the first division of the first class, Jeans being then below him in the second division of the first class. In the same year he was elected to a Prize Fellowship at Trinity, and his early ambition was thus fulfilled. Hardy and Jeans, in that order, were awarded Smith's Prizes in 1901.

His life's work of research had now begun, his first paper apparently being that in the *Messenger of Mathematics*, **29**, 1900. It is about the evaluation of some definite integrals, a subject which turned out to be one of his permanent minor interests, and on which he was still writing in the last year of his life.

In 1906, when his Prize Fellowship was due to expire, he was put on the Trinity staff as lecturer in mathematics, a position he continued to hold until 1919. This meant that he had to give six lectures a week. He usually gave two courses, one on elementary analysis and the other on the theory of functions. The former included such topics as the implicit function theorem, the theory of unicursal curves and the integration of functions of one variable. This was doubtless the origin of his first Cambridge tract, *The Integration of Functions*

of a Single Variable. This work is so well known now that it is often forgotten that its systematization was due to Hardy. He also sometimes took small informal classes on elementary subjects, but he was never a 'tutor' in the Oxford sense.

In 1908 Hardy made a contribution to genetics which seems to be little known by mathematicians, but which has found its way into textbooks as 'Hardy's Law'. There had been some debate about the proportions in which dominant and recessive Mendelian characters would be transmitted in a large mixed population. The point was settled by Hardy in a letter to *Science*. It involves only some simple algebra, and no doubt he attached little weight to it. As it happens, the law is of central importance in the study of Rh-blood-groups and the treatment of haemolytic disease of the newborn. In the *Apology* Hardy wrote, 'I have never done anything "useful". No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.' It seems that there was at least one exception to this statement.

He was elected a Fellow of the Royal Society in 1910, and in 1914 the University of Cambridge recognized his reputation for research, already world-wide, by giving him the honorary title of Cayley Lecturer.

To this period belongs his well-known book *A Course of Pure Mathematics*, first published in 1908, which has since gone through numerous editions and been translated into several languages. The standard of mathematical rigour in England at that time was not high, and Hardy set himself to give the ordinary student a course in which elementary analysis was for the first time done properly. *A Course of Pure Mathematics* is hardly a *Cours d'Analyse* in the sense of the great French treatises, but so far as it goes it serves a similar purpose. It is to Hardy and his book that the outlook of present-day English analysts is very largely due.

He also played a large part in the reform of the old Cambridge Mathematical Tripos Part I, and in the abolition of the publication of the results in order of merit.

Another turning point in Hardy's career was reached about 1912, when he began his long collaboration with J. E. Littlewood. There have been other pairs of mathematicians, such as Phragmén and Lindelöf, or Whittaker and Watson, who have joined forces for a particular object, but there is no other case of such a long and fruitful partnership. They wrote nearly a hundred papers together, besides (with G. Pólya) the book *Inequalities*.

Soon afterwards came his equally successful collaboration with the Indian mathematician Ramanujan, though this was cut short six years later by Ramanujan's early death. An account of this association is given by Hardy in the introductions to Ramanujan's collected works and to the book *Ramanujan*. In a letter to Hardy in 1913, Ramanujan sent specimens of his work, which showed that he was a mathematician of the first rank. He came to England in 1914 and remained until 1919. He was largely self-taught, with no knowledge of modern rigour, but his 'profound and invincible originality' called out

Hardy's equal but quite different powers. Hardy said, 'I owe more to him than to any one else in the world with one exception, and my association with him is the one romantic incident in my life'.

Hardy was a disciple of Bertrand Russell, not only in his interest in mathematical philosophy, but in his political views. He sympathized with Russell's anti-war attitude, though he did not go to the lengths which brought Russell into collision with the authorities. In a little book *Bertrand Russell and Trinity*, which he had printed for private circulation in 1942, Hardy has described the Russell case and the storms that raged over it in Trinity. It was an unhappy time for those concerned, and one may think that it all would have been better forgotten. It must have been with some relief that, in 1919, he heard of his election to the Savilian Chair of Geometry at Oxford, and migrated to New College.

In the informality and friendliness of New College Hardy always felt completely at home. He was an entertaining talker on a great variety of subjects, and one sometimes noticed every one in common room waiting to see what he was going to talk about. Conversation was one of the games which he loved to play, and it was not always easy to make out what his real opinions were.

He played several games well, particularly real tennis, but his great passion was for cricket. He would read anything on this subject, and talk about it endlessly. His highest compliment was 'it is in the Hobbs class'. Even until 1939 he captained the New College Senior Common Room side against the Choir School and other opponents. He liked to recall the only occasion in the history of the Savilian chairs when one Savilian professor (himself) took the wicket of the other (H. H. Turner). The paper, 'A maximal theorem with function-theoretic applications', published in *Acta Math.* **54,** and presumably addressed to European mathematicians in general, contains the sentences, 'The problem is most easily grasped when stated in the language of cricket . . . Suppose that a batsman plays, in a given season, a given "stock" of innings . . .'.

A vivid account of Hardy's affection for cricket and of his life in his later Cambridge years is given by C. P. Snow, in an article entitled 'A mathematician and cricket', in *The Saturday Book, 8th Year*.

He liked lecturing, and was an admirable lecturer. His matter, delivery and hand-writing (a specimen of which appears on the dust-cover of *A Mathematician's Apology*) were alike fascinating. Though no original geometer, he fulfilled the conditions of his Oxford chair by lecturing on geometry as well as on his own subjects. He also lectured occasionally on mathematics for philosophers, and drew large audiences of Oxford philosophers to whom ordinary mathematics made no appeal. His Rouse Ball lecture on this subject, delivered at Cambridge in 1928, entitled *Mathematical Proof*, was published in *Mind*, **38**.

Hardy had singularly little appreciation of science, for one who was sufficiently nearly a scientist to be a Fellow of the Royal Society. In *A Mathematician's Apology* he is at some pains to show that real mathematics is useless,

or at any rate harmless. He says, 'It is true that there are branches of applied mathematics, such as ballistics and aerodynamics, which have been developed deliberately for war . . . but none of them has any claim to rank as "real". They are indeed repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable, and if he could not, who can?' His views on this subject were obviously coloured by his hatred of war, but in any case his whole instinct was for the purest of mathematics. I worked on the theory of Fourier integrals under his guidance for a good many years before I discovered for myself that this theory has applications in applied mathematics, if the solution of certain differential equations can be called 'applied'. I never heard him refer to these applications.

Nevertheless, he was a Fellow of the Royal Astronomical Society, which he joined in 1918 in order that he might attend the meetings at which the theory of relativity was debated by Eddington and Jeans. He even once, in 1930, took part in a debate on stellar structure, which involved R. H. Fowler's work on Emden's and allied differential equations. On this he made the characteristic remark that Fowler's work, being pure mathematics, would still be of interest long after all the physical theories which had been discussed had become obsolete. This prophecy has since been very largely fulfilled.

I first came into contact with him when I attended his advanced class at Oxford in 1920. The subjects which I remember specially as having been discussed at this class are Fourier series, continued fractions, and differential geometry, a commentary on R. H. Fowler's Cambridge tract. Whatever the subject was, he pursued it with an eager single-mindedness which the audience found irresistible. One felt, temporarily at any rate, that nothing else in the world but the proof of these theorems really mattered. There could have been no more inspiring director of the work of others.

He was always at the head of a team of researchers, both colleagues and students, whom he provided with an inexhaustible stock of ideas on which to work. He was an extremely kind-hearted man, who could not bear any of his pupils to fail in their researches. Many Oxford D.Phil. dissertations must have owed much to his supervision.

Hardy always referred to God as his personal enemy. This was, of course, a joke, but there was something real behind it. He took his disbelief in the doctrines of religion more seriously than most people seem to do. He would not enter a religious building, even for such a purpose as the election of a Warden of New College. The clause in the New College by-laws, enabling a fellow with a conscientious objection to being present in Chapel to send his vote to the scrutineers, was put in on his behalf.

He has been described as absent-minded, but I never saw any sign of this. If he dined at high table in tennis clothes it was because he liked to do so, not because he had forgotten what he was wearing. He had a way of passing in the street people whom he knew well without any sign of recognition, but this was due to a sort of shyness, or a feeling of the slight absurdity of a repeated conventional greeting.

His likes and dislikes, or rather enthusiasms and hates, have been listed as follows:

Enthusiasms

(i) Cricket and all forms of ball games.
(ii) America, though perhaps he only came into contact with the pleasanter side of it.
(iii) Scandinavia, its people and its food.
(iv) Detective stories.
(v) Good literature, English and French, especially history and biography.
(vi) Walking and mild climbing, especially in Scotland and Switzerland.
(vii) Conversation.
(viii) Odd little paper games, such as making teams of famous people whose names began with certain combinations of letters or who were connected with certain countries, towns or colleges. These were played for hours in hotels or on walks.
(ix) Female emancipation and the higher education of women (though he opposed the granting of full membership of the university to Oxford women).
(x) *The Times* cross-word puzzles.
(xi) The sun.
(xii) Meticulous orderliness, in everything but dress. He had a large library and there were piles of papers all about his rooms, but he knew where everything was and the exact position of each book in the shelves.
(xiii) Cats of all ages and types.

Hates

(i) Blood sports of all kinds, war, cruelty of all kinds, concentration camps and other emanations of totalitarian governments.
(ii) Mechanical gadgets; he would never use a watch or a fountain pen, and the telephone only under compulsion. He corresponded chiefly by prepaid telegrams and post cards.
(iii) Looking-glasses; he had none in his rooms, and in hotels the first thing he did in his room was to cover them over with bath-towels.
(iv) Orthodox religion, though he had several clerical friends.
(v) The English climate, except during a hot summer.
(vi) Dogs.
(vii) Mutton—a relic of his Winchester days, when they had by statute to eat it five days a week.
(viii) Politicians as a class.
(ix) Any kind of sham, especially mental sham.

He was an extraordinary mixture of out-of-the-way information and ignorance. 'What is a milliner? Would you call the Army and Navy Stores a

milliner's?' 'No hawking! (this on Brighton front); I shouldn't have thought they had to forbid that nowadays.' In doing a cross-word puzzle: 'The word comes to ladders, but the clue is about stockings'.

Returning to his mathematical career, I may refer here to the founding of the *Quarterly Journal of Mathematics* (Oxford series). Glaisher, the editor of the *Messenger of Mathematics* and the old *Quarterly Journal*, had died in 1928, and these two periodicals had come to an end. There was an obvious need for something to replace them, and it was largely due to Hardy that a new series of the *Quarterly Journal* was started in Oxford.

The London Mathematical Society occupied a leading place in his affections. He served on the Council from 1905–1908, joined it again in 1914, and from that time, except for two absences of a year each, in 1928–1929 (when he went to America) and 1934–1935, he was on it continuously until his final retirement in 1945. He was one of the secretaries from 1917 to 1926, President in 1926–1928 and again for a second term in 1939–1941, and secretary again from then until 1945. In his Presidential address (1928), *Prolegomena to a Chapter on Inequalities*, he boasted that he had been at every meeting both of the Council and of the Society, and sat through every word of every paper, since he became secretary in 1917. He was awarded the Society's De Morgan medal in 1929.

In 1928–1929 he was Visiting Professor at Princeton and at the California Institute of Technology, O. Veblen coming to Oxford in his place. In 1931 E. W. Hobson died, and Hardy returned to Cambridge as his successor in the Sadleirian chair of Pure Mathematics, becoming again a Fellow of Trinity.

Perhaps the most memorable feature of this period was the Littlewood-Hardy seminar or 'conversation class'. This was a model of what such a thing should be. Mathematicians of all nationalities and ages were encouraged to hold forth on their own work, and the whole thing was conducted with a delightful informality that gave ample scope for free discussion after each paper. The topics dealt with were very varied, and the audience was always amazed by the sure instinct with which Hardy put his finger on the central point and started the discussion with some illuminating comment, even when the subject seemed remote from his own interests.

He also lectured on the calculus of variations, a subject to which he had been drawn by his work on inequalities.

After his return to Cambridge he was elected to an honorary fellowship at New College. He held honorary degrees from Athens, Harvard, Manchester, Sofia, Birmingham, Edinburgh, Marburg, and Oslo. He was awarded a Royal Medal of the Royal Society in 1920, its Sylvester Medal in 1940, and the Copley Medal, its highest award, in 1947. He was President of Section A of the British Association at its Hull meeting in 1922, and of the National Union of Scientific Workers in 1924–6. He was an honorary member of many of the leading foreign scientific academies.

Some months before his death he was elected 'associé étranger' of the Paris Academy of Sciences, a particular honour, since there are only ten of these from all nations and scientific subjects. He retired from the Sadleirian chair in

1942, and died on 1 December 1947, the day on which the Copley Medal was due to be presented to him.

He was unmarried. He owed much to his sister, who provided him throughout his life with the unobtrusive support which such a man needs. Miss Hardy has supplied most of the personal information contained in this notice.

In addition to the books mentioned above, Hardy wrote three more Cambridge tracts, *Orders of Infinity* (1910), *The General Theory of Dirichlet's Series*, with M. Riesz (1915), and *Fourier Series*, with W. W. Rogosinski (1944). In 1934 he published *Inequalities* with J. E. Littlewood and G. Pólya, and in 1938 *The Theory of Numbers* with E. M. Wright. In 1940 followed *Ramanujan*, a collection of lectures or essays suggested by Ramanujan's work. His last book was on Divergent Series, and was completed but not published at the time of his death. His inaugural lecture at Oxford, *Some famous problems of the theory of numbers, and in particular Waring's problem*, was published in 1920. He was also one of the editors of the collected papers of Ramanujan, which were published in 1927.

The student of Hardy's style should also read his obituary notices* of Ramanujan, Mittag-Leffler, Bromwich, Paley, Hobson, Landau, W. H. Young, J. R. Wilton, and that of Glaisher at the end of the *Messenger of Mathematics*. These tributes to his late colleagues must have made every mathematician wish that he could have seen his own career described in the same generous terms.

Hardy was the author, or part author, of more than 300 original papers, covering almost every kind of analysis, which by their originality and quantity marked him as one of the leading mathematicians of his time. It is rarely possible to disentangle his own contributions from those of others. He liked collaboration, and much of his best work is to be found in joint papers, particularly those written with Littlewood and with Ramanujan. He used to say that each author of a joint paper gets much more than half the credit for it. No doubt the bulk of his work is greatly increased by these collaborations, but he was certainly the prime mover in much of it. He described himself as a problem-solver, and did not claim to have introduced any new system of ideas. Nevertheless, if we may judge by the references to his work in the writings of others, he had a profound influence on modern mathematics.

When he began research there was probably no one at hand who could give him the sort of supervision which he was to give to so many others, and it was some years before he found a problem of first-rate interest.

His early series of papers on Cauchy's principal values was overshadowed by the work of Lebesgue and others who were generalizing the integral in other directions. Nevertheless it contains some interesting formulae. Perhaps the most noteworthy are the inversion formulae

$$f(y)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{g(x)}{x-y}dx,\quad g(y)=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f(x)}{x-y}dx,$$

* For references, see p. 700. The notices will be reprinted in volume 7 of these *Collected Papers*.

which have come to be known as the formulae of 'Hilbert transforms'. In later years he wrote many papers on transforms and inversion formulae of various kinds. This work lies on the borders of the theory of integral equations; he never worked on the central Fredholm theory itself, though he lectured on it in his second period at Cambridge.

We next find him writing on the summation of divergent series, and this turned out to be one of the permanent interests of his life.

The theorem of Abel, that if

$$s_n = a_1 + a_2 + \dots + a_n \to s \tag{1}$$

then

$$\lim_{x \to 1-0} \sum_{n=1}^{\infty} a_n x^n = s \tag{2}$$

is classical; and in 1897 Tauber had proved a sort of converse, that (2) implies (1) if the coefficients a_n satisfy the condition $a_n = o(1/n)$, i.e. $na_n \to 0$. It was easy to prove the corresponding theorem for Cesàro summability, that if

$$\sigma_n = \frac{s_1 + \dots + s_n}{n} \to s \tag{3}$$

and $a_n = o(1/n)$, then again (1) follows. The simple and satisfactory appearance of the proofs of these theorems gives them an air of finality; but in 1909 Hardy proved that (3) implies (1) under the less restrictive condition $a_n = O(1/n)$, i.e. na_n is bounded. This result, often referred to as 'Hardy's theorem', was the first 'O-Tauberian theorem', the forerunner of a whole science of such theorems. It was followed in the next year by Littlewood's theorem that (2) implies (1) if $a_n = O(1/n)$. Later the two originators of the theory published a great deal of work on it together, and the whole matter has now been summed up in Hardy's last book.

His first important paper on Fourier series seems to be that in volume 12 of the *Proceedings of the London Mathematical Society* (1913, 4). The modern theory of Fourier series, depending on the Lebesgue integral, was then being constructed by Lebesgue, Fejér, W. H. Young and others, and it was Young's work that inspired Hardy particularly. The first theorem in the paper referred to is that any Fourier series is summable (C, δ) almost everywhere, for any positive δ, and there are many others in the same order of ideas. Later Hardy and Littlewood together developed a whole theory of 'Fourier constants' or coefficients, generalizing the Young-Hausdorff theorem that if $|f(x)|^p$ is integrable, where $1 < p \leqslant 2$, and a_n, b_n are its Fourier constants, then

$$\Sigma(|a_n|^{p'} + |b_n|^{p'}), \quad p' = p/(p-1),$$

is convergent. A typical Hardy-Littlewood theorem is that the integrability of $|f(x)|^q |x|^{q-2}$ over $(-\pi, \pi)$, where $q \geqslant 2$, implies the convergence of $\Sigma(|a_n|^q + |b_n|^q)$. A curious by-product of this analysis is that, if the two conjugate series

$$\Sigma(a_n \cos nx + b_n \sin nx), \qquad \Sigma(b_n \cos nx - a_n \sin nx)$$

are both Fourier series, then $\Sigma(|a_n| + |b_n|)/n$ is convergent.

The theory of the Riemann zeta-function had begun with the guesses of Riemann (1860), and the analysis of Hadamard and de la Vallée-Poussin, who proved the prime-number theorem (1896). The modern theory of the function had just been started by H. Bohr, Landau and Littlewood. The great puzzle of the theory was the 'Riemann hypothesis', that $\zeta(s)$ has all its complex zeros on the critical line $\Re(s)=\frac{1}{2}$. This presented all workers in the field, as it still does, with a perpetual challenge. It was Hardy who first gave any sort of answer to it, with the discovery, in 1914, that $\zeta(s)$ has at any rate an infinity of zeros on the critical line. The work was again carried on jointly with Littlewood, and it was proved that, if $N_0(T)$ denotes the number of complex zeros of $\zeta(s)$ with real part $\frac{1}{2}$ and imaginary part between 0 and T, then $N_0(T)>AT$ for some constant A (the total number of complex zeros in this region being asymptotic to $(T/2\pi)\log T$). It is only recently that this result has been surpassed by A. Selberg, with the proof that $N_0(T)>AT\log T$. Hardy used to say that any one who had a really new idea about the zeta-function must surely prove the Riemann hypothesis, but Selberg's work seems to have disproved this. Another of the main features of the Hardy-Littlewood analysis, the 'approximate functional equation', was discovered later to have been anticipated to a certain extent by Riemann himself, though the applications which they made of it go far beyond anything in Riemann.

Another subject to which Hardy made a fundamental contribution was that of the lattice-points in a circle. The number $R(x)$ of lattice-points in a circle of radius $\sqrt{x}$, i.e. of pairs of integers μ, ν, such that $\mu^2+\nu^2\leqslant x$, is roughly equal to the area πx of the circle, but closer approximations to $R(x)$ are difficult to make. It had been proved by Sierpinski that, if

$$R(x)=\pi x+\mathrm{P}(x),$$

then $\mathrm{P}(x)=O(x^{\frac{1}{3}})$, but the true order of $\mathrm{P}(x)$ was unknown.

Hardy obtained an exact formula for $R(x)$ as a series of Bessel functions. If x is not an integer this is

$$R(x)=\pi x-1+x^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{r(n)}{n^{\frac{1}{2}}}J_1\{2\pi\sqrt{(nx)}\},$$

where $r(n)$ is the number of solutions in integers of $\mu^2+\nu^2=n$. If x is an integer, $R(x)$ must be replaced by $R(x)-\frac{1}{2}r(x)$. This 'exact formula' is very striking, but it is not of much use in the problem of the order of $\mathrm{P}(x)$. If we could treat the series as a finite sum, the ordinary asymptotic formula for Bessel functions would give at once $\mathrm{P}(x)=O(x^{\frac{1}{4}})$. It is tempting to suppose that at any rate $\mathrm{P}(x)=O(x^{\frac{1}{4}+\epsilon})$, but nothing approaching this has ever been proved. What Hardy did prove was that each of the inequalities $\mathrm{P}(x)>Kx^{\frac{1}{4}}$, $\mathrm{P}(x)<-Kx^{\frac{1}{4}}$, is satisfied, with some K, for some arbitrarily large values of x. The true order of $\mathrm{P}(x)$ therefore lies somewhere between $x^{\frac{1}{4}}$ and $x^{\frac{1}{3}}$, and later research has done a little, but not much, to narrow this gap.

I must now describe Hardy's work on partitions, the 'circle method' in the analytic theory of numbers, and his association with Ramanujan. They wrote

five papers together, the most famous being that in volume 17 of the *Proceedings of the London Mathematical Society* (1918, 5), a section of which is reproduced on the dust-cover of *A Mathematician's Apology*. In this it was shown that $p(n)$, the number of unrestricted partitions of n, can not only be represented approximately by an asymptotic formula, but that it can be calculated exactly for any value of n. The 'circle method' on which this depends is, no doubt, Hardy's most original creation. It proceeds roughly as follows. The numbers $p(n)$ are the coefficients in the expansion

$$f(z)=1+\sum_{n=1}^{\infty}p(n)z^n=\frac{1}{(1-z)(1-z^2)(1-z^3)\ldots},$$

so that
$$p(n)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z^{n+1}}dz,$$

where Γ is a path enclosing the origin and lying entirely inside the unit circle, and is taken to be a concentric circle of radius just less than 1. The unit circle is a line of essential singularities of $f(z)$, but certain points are found to have a particularly strong influence on the integral, and it is from these that a dominant term is ultimately derived. In the Dirichlet series method for proving e.g. the prime-number theorem, the dominant term is easily identified, and almost the whole difficulty lies in showing that it *is* dominant. In the circle method this is not so, and a whole apparatus, involving the Farey dissection of the circle and the linear transformations of elliptic modular functions, is needed to produce the result. It is all the more astonishing that the analysis should have been carried to the point at which the exact value of $p(n)$ could be obtained.

Similar methods were applied later by Hardy and Littlewood to many other problems, particularly to the Waring problem of the expression of a number as a sum of given powers, and to problems involving primes. One such problem which had long defied analysis was Goldbach's theorem, or hypothesis, that any even number can be expressed as the sum of two primes. The success of the circle method in the study of $p(n)$ suggests a similar approach to Goldbach's theorem. Let $f(z)$ now denote Σz^p, where p runs through primes.

Then
$$\{f(z)\}^2=\Sigma a(n)z^n,$$

where $a(n)$ is the number of ways in which n can be expressed as the sum of two primes. If we can prove that $a(n)$ is always positive, Goldbach's theorem will follow, but the difficulties prove to be even more formidable than in the case of partitions. Actually it is a little easier to discuss

$$\{f(z)\}^3=\Sigma b(n)z^n,$$

where $b(n)$ is the number of ways of expressing n as the sum of three primes. Hardy and Littlewood showed that, if certain hypotheses of the type of the Riemann hypothesis are made, then $b(n)$ is ultimately positive, so that any sufficiently large odd number n is the sum of three primes. Later Vinogradoff,

by combining the essential ideas of the circle method with some entirely new ideas of his own, showed that all unproved hypotheses could be dispensed with. The whole method is perhaps the most remarkable example that at present exists of analysis carried through apparently insoluble difficulties to ultimate success. But the original Goldbach hypothesis still remains unproved.

Hardy had many other interests of which there is no space to speak at length here: orders of infinity, Diophantine approximation, Bessel functions, inequalities.

Hardy's work has had a profound influence throughout the whole of analysis. It has resulted in the complete remodelling of some parts of the subject, and has enriched other parts with new methods and theories of fundamental importance.

E. C. TITCHMARSH

1. DIOPHANTINE APPROXIMATION

INTRODUCTION TO PAPERS ON DIOPHANTINE APPROXIMATION

Practically all Hardy's researches on this subject were carried out in collaboration with Littlewood, the only exception being represented by 1919, 4. Apart from this paper and 1946, 1, all the papers appeared under the general title 'Some problems of Diophantine approximation', with various subtitles.

The series began with the famous communication to the 1912 Congress. This is largely a statement of results, with occasional indications of the method of proof. Re-reading it now, one is surprised to find how many important and fundamental discoveries had been made by the authors, and how many delicate distinctions between apparently similar questions they had already perceived. The results announced in 1912 were published for the most part in 1914, 2 and 1914, 3, but some of them not until 1922, 6.

Hardy and Littlewood were primarily interested in problems of distribution modulo 1. The simplest such problem is: given a function $f(n)$, can we say that the values of $f(n)$ for $n = 1, 2, \ldots$ are everywhere dense (mod 1), i.e. that their fractional parts are everywhere dense in $(0, 1)$? In 1914, 2 it was proved that this is the case if $f(n)$ is a polynomial with at least one irrational coefficient (other than the constant term). Results were also proved for the simultaneous distribution of the values of several polynomials; these results are generalizations of Kronecker's theorem, which is itself the particular case when the several polynomials are all of degree 1. More precisely, Kronecker's theorem† states that if $1, \theta_1, \ldots, \theta_m$ are linearly independent over the rationals, there exist integers n for which the numbers

$$n\theta_1, \ldots, n\theta_m$$

are arbitrarily near (mod 1) to any m prescribed numbers. Hardy and Littlewood proved that the same holds for the mp numbers

$$n^q\theta_j \quad (q = 1, \ldots, p;\ j = 1, \ldots, m),$$

and their method applied in principle to more general polynomials.

Once it is known that the values of a function $f(n)$ are everywhere dense (mod 1), the further question arises of their *uniformity* of distribution. For this we require that the frequency with which $f(n)$ falls (mod 1) into any given sub-interval of $(0, 1)$ shall be proportional to the length of that sub-interval. Two other problems which prove

† For some remarks about various results related to Kronecker's theorem, see the comments on 1914, 2.

to be closely related to the uniformity of distribution are those of estimating the sum

$$S(N) = \sum_{n=1}^{N} \{f(n)\}, \quad \text{where } \{t\} = t-[t]-\tfrac{1}{2}, \tag{1}$$

and the sum

$$s(N) = \sum_{n=1}^{N} e^{2\pi i f(n)}. \tag{2}$$

These various questions form the basic themes of most of the papers, and they are treated by a variety of methods. This is not surprising, for they are questions which lie very much on the borderline between the theory of numbers and analysis, and can often be approached from either side.

As early as 1912 Hardy and Littlewood had proved the uniformity of distribution (mod 1) of the values of a polynomial with an irrational coefficient, and had announced as a consequence the estimate $\zeta(1+it) = o(\log t)$ as $t \to \infty$. The proofs of these results never appeared, since Weyl's memoir of 1916 rendered their publication unnecessary (see 1916, 9). Weyl reduced the question of uniformity of distribution to that of estimating sums of the type (2), and gave a simple and powerful method for finding such estimates when $f(n)$ is a polynomial.

In 1914, 2 Hardy and Littlewood also laid the foundations of the 'metrical' theory of Diophantine approximation, in which results are proved to hold for almost all values of a real parameter, in the sense of Lebesgue measure.

The second big memoir (1914, 3) was entirely devoted to the study of the exponential sum (2) in the particular case

$$f(n) = \theta n^2 + \phi n.$$

It was proved that if θ is an irrational with bounded partial quotients in its continued fraction, then $s(N) = O(N^{\frac{1}{2}})$, and that this is best possible. Other results were deduced on other hypotheses concerning θ. The basic principle of the proofs was the so-called 'approximate functional equation of the ϑ-function', which enabled them to relate $s(N)$ to the continued fraction expansion of θ.

One possible application of Diophantine approximation which Hardy and Littlewood kept in mind was the provision of explicit examples to illustrate general theorems in the theory of functions or the theory of series, and to show to what extent they are best possible. Some such applications were given in 1914, 3; but in 1916, 3 other examples which are less intimately related to Diophantine approximation were shown to be equally effective.

Two other large memoirs (1922, 6 and 9) were devoted to the triangle problem. This is the problem of approximating to the number $N(\eta)$ of points with integral coordinates in the triangle

$$x > 0, \qquad y > 0, \qquad \omega x + \omega' y < \eta$$

as $\eta \to \infty$, where ω, ω' are fixed positive numbers whose ratio $\theta = \omega/\omega'$ is irrational.

It is easily seen that

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + S_1(\eta) + O(1),$$

where
$$S_1(\eta) = \sum_{n \leqslant \eta/\omega} \{n\theta - f'\}, \tag{3}$$

f' denoting the fractional part of η/ω'. Thus $S_1(\eta)$ is a sum similar to the sum $S(N)$ in (1).

Two methods were used in the study of $S_1(\eta)$, one elementary and the other analytical. The elementary method is based on a transformation formula and is on similar general lines to the method of 1914, 3. It was proved that

$$S_1(\eta) = o(\eta) \qquad \text{for any irrational } \theta,$$
$$S_1(\eta) = O(\log \eta) \quad \text{if } \theta \text{ has bounded partial quotients},$$

and that both of these are best possible. The analytical method uses contour integration and the double zeta-function

$$\zeta_2(s, a, \omega, \omega') = \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} (a + m\omega + m'\omega')^{-s}.$$

For the estimation of $S_1(\eta)$, the analytical method is no more effective than the elementary method, and is if anything slightly less powerful. But it led the authors to a remarkable explicit formula for $S_1(\eta)$ in the form of an infinite series; a formula which can be compared with that of Voronoi for the divisor problem or with that of Sierpiński for the circle problem.

The sum $\sum \{n\theta\}$, which is a particular case of (3), was deeply studied in the years 1922–5 by Hecke, Behnke, and Ostrowski, as well as by Hardy and Littlewood. The analytical character of the function

$$\phi(s) = \sum_{n=1}^{\infty} \{n\theta\} n^{-s} \tag{4}$$

depends very much on the arithmetical character of θ. When θ is a quadratic irrational, Hecke proved that $\phi(s)$ is meromorphic and specified its poles; and the same results were found by Hardy and Littlewood by a different method (1923, 3 and 4). When θ is any irrational, and λ is defined by

$$\lambda + 1 = \overline{\lim} \frac{\log q_{\nu+1}}{\log q_\nu}$$

(where the q_ν are the denominators of the convergents to θ), Hardy and Littlewood proved that the series (4) is convergent for

$$\Re s > \lambda/(\lambda+1)$$

and that $\phi(s)$ has the line
$$\Re s = \lambda/(\lambda+1)$$

as a line of singularities if $\lambda > 0$.

The paper 1919, 4, by Hardy alone, is not related to the rest of the work, but is of considerable historical interest. Here Hardy proved the basic property of the so-called Pisot–Vijayaraghavan numbers. Suppose θ (> 1) is any algebraic number and

λ is any real number except 0. Suppose that

$$\lambda\theta^n \to 0 \pmod 1$$

as $n \to \infty$. Then the conclusion is that θ is an algebraic integer with the property that all the algebraic conjugates of θ (whether real or complex) have absolute values less than 1, and λ is an algebraic number in the field generated by θ.

Further comments are given immediately after the individual papers.

I conclude by listing a few problems in the subject, connected (directly or indirectly) with the Hardy–Littlewood body of work, which are still unsolved.

(1) Little is known about the order of magnitude of

$$\sum_{n=1}^{N} e^{2\pi i n^3\theta}$$

as $N \to \infty$, where θ is a fixed irrational number of some specific type; say, with bounded partial quotients. One easily deduces from Weyl's inequality that for such θ the sum is $O(N^{3/4+\epsilon})$, but it is doubtful whether this is the full truth. There is the same problem for almost all θ. Any new results may well prove to be significant for Waring's problem.

(2) The tetrahedron problem, that is, the analogue in three dimensions of the triangle problem. The bounding plane is now

$$\omega x+\omega' y+\omega'' z = \eta,$$

where $\eta \to \infty$. It is appropriate to assume that ω, ω', ω'' are linearly independent over the rationals. It is easily proved that the error term is $o(\eta^2)$, but it is not known whether this is best possible.

(3) The nature of $\phi(s)$ in (4) when $\lambda = 0$ (or more particularly when θ has bounded partial quotients), but θ is not a quadratic irrational. It was conjectured in 1923, 4 that $\phi(s)$ has $\Re s = 0$ as a line of singularities, but this has never been proved.

(4) In 1930, 3 it is proved that if $\theta = \sqrt{(a^2+1)}$, where a is an odd integer, then

$$\sum_{n=1}^{N} \frac{1}{n \sin n\pi\theta} = O(1)$$

as $N \to \infty$. The proof of this remarkable result is curiously indirect; it involves contour integration and the use of Cesàro means of arbitrarily high order. In the same paper it is stated that for any quadratic irrational θ, the above sum is

$$A(\theta)\log N + O(1)$$

as $N \to \infty$, where $A(\theta) = 0$ for the special values of θ just mentioned, but is not always 0. The problem is to give a simpler and more direct proof of these results.

(5) Littlewood's problem on simultaneous Diophantine approximation: to prove (if it is true) that for any real θ, ϕ and any $\epsilon > 0$ there is a positive integer n satisfying

$$n|\sin n\pi\theta \sin n\pi\phi| < \epsilon.$$

For references, see Davenport, *Mathematika*, 3 (1956), 131–5.

(6) Khintchine's problem on uniform distribution. If S is a subset of $(0, 1)$ with measure $|S|$ in the sense of Riemann (or Jordan), then the frequency of those n for which $n\theta$ lies in $S \pmod 1$ is $|S|$, and this holds for *every* irrational θ. The problem is to prove (if it is true) that the same holds for *almost all* θ if S has measure $|S|$ in the sense of Lebesgue. See Khintchine, *Math. Z.* 18 (1923), 289–306.

H. D.

Abbreviated titles

In the comments which follow the individual papers, references to Cassels's Tract, to Hardy and Wright, and to Koksma are meant to refer to:

J. W. S. Cassels, *An introduction to Diophantine approximation* (Cambridge Mathematical Tract No. 45), Cambridge, 1957.

G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 4th ed., 1960.

J. F. Koksma, *Diophantische Approximationen* (Ergebnisse der Math. IV, 4), Springer, Berlin, 1936.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION

BY G. H. HARDY AND J. E. LITTLEWOOD.

1. Let us denote by $[x]$ and (x) the integral and fractional parts of the real number x, so that

$$(x) = x - [x], \quad 0 \leqq (x) < 1.$$

Let θ be an irrational number, and α any number between 0 and 1 (0 included). Then it is well known that it is possible to find a sequence of positive integers $n_1, n_2, n_3, \ldots$ such that

$$(n_r\theta) \to \alpha$$

as $r \to \infty$. Now let $f(n)$ denote a positive increasing function of n, integral when n is integral, such as

$$n, n^2, n^3, \ldots, 2^n, 3^n, \ldots, n!, 2^{n^2}, \ldots, 2^{2^n}, \ldots,$$

and let f_r denote the value of $f(n)$ for $n = n_r$. The result just stated suggests the following question, which seems to be of considerable interest:—*For what forms of $f(n)$ is it true that, for any irrational value of θ, and any value of α such that $0 \leqq \alpha < 1$, a sequence n_r can be found such that*

$$(f_r\theta) \to \alpha\,?$$

It is easy to see that, when the increase of $f(n)$ is sufficiently rapid, the result suggested will not generally be true. Thus, if $f(n) = 2^n$, and θ is a number which, when expressed in the binary scale, shows at least k 0's following upon every 1, it is plain that

$$(2^n\theta) < \tfrac{1}{2} + \lambda_k,$$

where λ_k is a number which can be made as small as we please by increasing k sufficiently. There is thus an "excluded interval" of values of α, the length of which can be made as near to $\frac{1}{2}$ as we please. If $f(n) = 3^n$ we can obtain an excluded interval whose length is as near $\frac{2}{3}$ as we please, and so on, while if $f(n) = n!$ it is (as is well known) possible to choose θ so that $(n!\,\theta)$ has a unique limit. Thus

$$(n!\,e) \to 0.$$

2. The first object of this investigation has been to prove the following theorem:—

Theorem 1. *If $f(n)$ is a polynomial in n, with integral coefficients, then a sequence can be found for which $(f_r\theta) \to \alpha$.*

1912, 4 (with J. E. Littlewood) *Proceedings of the 5th International Congress of Mathematicians*, Cambridge, 1912, i. 223–9.

We shall give the proof in the simple case in which

$$f(n) = n^2,$$

a case which is sufficient to exhibit clearly the fundamental ideas of our analysis. Our argument is based on the following general principle, which results from the work of Pringsheim and London on double sequences and series*:

If $$f_{r,s}, \quad \phi_{r,s}, \quad \dots$$

are a finite number of functions of the positive integral variables r, s; and if

$$\lim_{s\to\infty}\lim_{r\to\infty} f_{r,s} = a, \quad \lim_{s\to\infty}\lim_{r\to\infty} \phi_{r,s} = b, \ \dots;$$

then we can find a sequence of pairs of numbers

$$(r_1, s_1), \quad (r_2, s_2), \quad (r_3, s_3) \ \dots$$

such that $r_i \to \infty$, $s_i \to \infty$ *and* $f_{r_i, s_i} \to a$, $\phi_{r_i, s_i} \to b$, $\dots$, *as* $i \to \infty$.

We shall first apply this principle to prove that a sequence n_r can be found so that

$$(n_r\theta) \to 0, \quad (n^2_r\theta) \to 0$$

simultaneously. We shall, in the argument which follows, omit the brackets in $(n_r\theta)$, etc., it being understood always that integers are to be ignored.

We can choose a sequence n_r so that $n_r\theta \to 0$. The corresponding values $n^2_r\theta$ are infinite in number, and so have at least one limiting point ξ; ξ may be positive or zero, rational or irrational. We can (by restricting ourselves to a subsequence of the n_r's) suppose that

$$n_r\theta \to 0, \quad n^2_r\theta \to \xi.$$

If $\xi = 0$, we have what we want. If not we write

$$f_{r,s} = (n_r + n_s)\,\theta, \quad \phi_{r,s} = (n_r + n_s)^2\,\theta.$$

Then
$$\lim_{s\to\infty}\lim_{r\to\infty} f_{r,s} = \lim_{s\to\infty} n_s\theta = 0,$$
$$\lim_{s\to\infty}\lim_{r\to\infty} \phi_{r,s} = \lim_{s\to\infty} (\xi + n^2_s\theta) = 2\xi.$$

Hence, by the general principle, we can pick out a new sequence p_r such that

$$p_r\theta \to 0, \quad p^2_r\theta \to 2\xi.$$

Repeating the argument, with $n_r + p_s$ in the place of $n_r + n_s$, we are led to a sequence q_r such that

$$q_r\theta \to 0, \quad q^2_r\theta \to 3\xi;$$

and it is plain that by proceeding in this way sufficiently often we can arrive at a sequence $n_{r,k}$ such that

$$n_{r,k}\,\theta \to 0, \quad n^2_{r,k}\,\theta \to k\xi,$$

for any integral value of k.

Now whatever number ξ is, rational or irrational, we can find a sequence k_s such that

$$k_s\xi \to 0$$

as $s \to \infty$. Then

$$\lim_{s\to\infty}\lim_{r\to\infty} n_{r,k_s}\,\theta = \lim_{s\to\infty} 0 = 0,$$
$$\lim_{s\to\infty}\lim_{r\to\infty} n^2_{r,k_s}\,\theta = \lim_{s\to\infty} k_s\xi = 0.$$

* Pringsheim, *Sitzungsberichte der k. b. Akademie der Wiss. zu München*, vol. 27, p. 101, and *Math. Annalen*, vol. 53, p. 289; London, *Math. Annalen, ibid.*, p. 322.

Applying the general principle once more we deduce a sequence of values of n for which $(n\theta) \to 0$, $(n^2\theta) \to 0$ simultaneously.

When we have proved that there is a sequence n_r for which $n_r^2\theta \to 0$, it is very easy to define a sequence $\nu_r n_r$, where ν_r is an integer depending on r, which gives any arbitrary α as a limit. We thus complete the proof of Theorem 1 in the case $f(n) = n^2$. An analogous method may be applied in the case of the general power n^k. As in the course of this proof we obtain a sequence for which

$$n\theta \to 0,\quad n^2\theta \to 0,\quad \ldots,\quad n^k\theta \to 0$$

simultaneously, we thus prove the theorem when $\alpha = 0$ for the general polynomial $f(n)$. The extension to the case $\alpha > 0$ may be effected on the same lines as in the case $f(n) = n^k$, but it is more elegant to complete the proof by means of the theorems of the next section.

It may be observed that the relation

$$n\theta \to 0$$

may be satisfied *uniformly* for all values of θ, rational or irrational; that is to say, given any positive ϵ, a number $N(\epsilon)$ can be found such that

$$n\theta < \epsilon$$

for every θ and some n, which depends on ϵ and θ but is less that $N(\epsilon)$. Similar results may be established for $n^2\theta$, $n^3\theta$, ... The chief interest of this result lies in the fact that it shows that there must be *some* function $\phi(n)$, independent of θ, which tends to zero as $n \to \infty$ and is such that for every θ there is an infinity of values of n for which

$$n^2\theta < \phi(n)^*.$$

3. The following generalisation of the theorem quoted at the beginning of § 1 was first proved by Kronecker†:—

If θ, ϕ, ψ, ... are any number of linearly independent irrationals (i.e. if no relation of the type

$$a\theta + b\phi + c\psi + \ldots = 0,$$

where a, b, c, ... are integers, not all zero, holds between θ, ϕ, ψ, ...), and if α, β, γ, ... are any numbers between 0 and 1 (0 included), then a sequence n_r can be found such that

$$n_r\theta \to \alpha,\quad n_r\phi \to \beta,\quad n_r\psi \to \gamma,\quad \ldots.$$

This theorem, together with the results of § 2, at once suggest the truth of the following theorem:—

Theorem 2. *If θ, ϕ, ψ, ... are linearly independent irrationals, and*

$$\alpha_l,\ \beta_l,\ \gamma_l,\ \ldots\ (l = 1, 2, \ldots, k)$$

* It is well known that, in the case of $n\theta$, $\phi(n)$ may be taken to be $1/n$. No such simple result holds when $\alpha > 0$: exception has to be made of certain aggregates of values of θ. On the other hand, if θ is a fixed *irrational*, the relation $n\theta \to \alpha$ holds uniformly *with respect to* α. All these results suggest numerous generalisations.

† *Werke*, vol. 3, p. 31. The theorem has been rediscovered independently by various authors, e.g. by Borel, F. Riesz, and Bohr (see for example Borel, *Leçons sur les séries divergentes*, p. 135, and F. Riesz, *Comptes Rendus*, vol. 139, p. 459).

k sets of numbers all lying between 0 and 1 (0 included), then it is possible to find a sequence of values of n for which

$$n\theta \to \alpha_1,\ n\phi \to \beta_1,\ n\psi \to \gamma_1,\ \ldots,$$
$$n^2\theta \to \alpha_2,\ n^2\phi \to \beta_2,\ n^2\psi \to \gamma_2,\ \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n^k\theta \to \alpha_k,\ n^k\phi \to \beta_k,\ n^k\psi \to \gamma_k,\ \ldots.$$

This theorem we prove by means of two inductions, the first from the case of k sets $\alpha_l, \beta_l, \gamma_l, \ldots$ to the case of $k+1$ sets *in which the numbers of the last set are all zero*, the second from this last case to the general case of $k+1$ sets. The principles which we employ do not differ from those used in the proof of the simpler propositions discussed in § 2.

4. The investigations whose results are summarised in the preceding sections were originally begun with the idea of obtaining further light as to the behaviour of the series

$$\Sigma e^{n^2\theta\pi i},\ \Sigma e^{n^3\theta\pi i},\ \ldots$$

from the point of view of convergence, summability, and so forth. If we write*

$$s_n^{(2)} = \sum_{\nu \leqq n} e^{(\nu-\frac{1}{2})^2\theta\pi i},\quad s_n^{(3)} = \sum_{\nu \leqq n} e^{\nu^2\theta\pi i},\quad s_n^{(4)} = \sum_{\nu \leqq n} (-1)^{\nu-1} e^{\nu^2\theta\pi i}$$

it is obvious that, if s_n is any one of $s_n^{(2)}, \ldots$, then $s_n = O(n)$. If θ is *rational*, either $s_n = O(1)$ or $s_n = An + O(1)$, where A is a constant: the cases may be differentiated by means of the well known formulae for "Gauss's sums." Similar remarks apply to the higher series in which (e.g.) ν^2 is replaced by $\nu^3, \nu^4, \ldots$. The results of the preceding sections have led us to a proof of

Theorem 3. *If θ is irrational, then $s_n = o(n)$: the same result is true for the corresponding higher sums.*

The argument by which we prove this theorem has a curious and unexpected application to the theory of the Riemann ζ-function; it enables us to replace Mellin's result $\zeta(1+ti) = O(\log|t|)$† by

$$\zeta(1+ti) = o(\log|t|).$$

Theorem 4. *Theorem 3 is the best possible theorem of its kind, that is to say the $o(n)$ which occurs in it cannot be replaced by $O(n\phi)$, where ϕ is any definite function of n, the same for all θ's, which tends to zero as $n \to \infty$.*

But although Theorem 3 contains the most that is true for *all* irrational θ's, it is possible to prove much more precise results for special classes of θ's. Here we use methods of a less elementary (though in reality much easier) type than are required for Theorem 3, the proof of which is intricate.

In Chap. 3 of his *Calcul des Résidus*‡ M. Lindelöf gives a very elegant proof of the formula

$$\sum_0^{q-1} e^{\nu^2 p\pi i/q} = \sqrt{\left(\frac{iq}{p}\right)} \sum_0^{p-1} e^{-\nu^2 q\pi i/p}$$

* The notation is chosen so as to run parallel with Tannery and Molk's notation for the ϑ-functions: n is not necessarily an integer.

† Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 167.

‡ pp. 73 *et seq.*

of Genocchi and Schaar. Here p and q are integers of which one is even and the other odd. By a suitable modification of Lindelöf's argument, we establish the formula

$$s_n^{(\lambda)}(\theta) = \sqrt{\left(\frac{i}{\theta}\right)} s_{n\theta}^{(\lambda_1)}\left(-\frac{1}{\theta}\right) + \frac{O(1)}{\sqrt{\theta}},$$

where θ is an irrational number, which we may suppose to lie between -1 and 1, λ is one of 2, 3, 4, λ_1 a corresponding one of the same numbers, and $O(1)$ stands for a function of n and θ less in numerical value than an absolute constant.

We observe also that the substitution of $\theta + 1$ for θ merely permutes the indices 2, 3, 4, and that the substitution of $-\theta$ for θ changes s_n into its conjugate. If now we write θ in the form of a simple continued fraction

$$\frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 + \dots},$$

and put

$$\theta = \frac{1}{a_1 + \theta_1}, \quad \theta_1 = \frac{1}{a_2 + \theta_2}, \dots$$

we obtain

$$\begin{aligned} s_n^{(\lambda)}(\theta) &= \sqrt{\left(\frac{i}{\theta}\right)} s_{n\theta}^{(\lambda_1)}(-\theta_1) + \frac{O(1)}{\sqrt{\theta}} \\ &= \sqrt{\left(\frac{1}{\theta\theta_1}\right)} s_{n\theta\theta_1}^{(\lambda_2)}(\theta_2) + O(1)\left\{\frac{1}{\sqrt{\theta}} + \frac{1}{\sqrt{(\theta\theta_1)}}\right\} \\ &= \sqrt{\left(\frac{i}{\theta\theta_1\theta_2}\right)} s_{n\theta\theta_1\theta_2}^{(\lambda_3)}(-\theta_3) + O(1)\left\{\frac{1}{\sqrt{\theta}} + \frac{1}{\sqrt{(\theta\theta_1)}} + \frac{1}{\sqrt{(\theta\theta_1\theta_2)}}\right\}, \end{aligned}$$

and so on. We can continue this process until $n\theta\theta_1\theta_2 \dots < 1$, when the first term vanishes, and we are left with an upper limit for $|s_n|$ the further study of which depends merely on an analysis of the continued fraction.

We thus arrive at easy proofs of Theorems 3 and 4 for $k = 2$. We can also prove

Theorem 5. *If the partial quotients a_n of the continued fraction for θ are limited* then $s_n(\theta) = O(\sqrt{n})$. In particular this is true if θ is a quadratic surd, pure or mixed.*

5. The question naturally arises whether Theorem 5 is the best possible of its kind. The answer to this question is given by

Theorem 6. *If θ is any irrational number, it is possible to find a constant H and an infinity of values of n such that*

$$|s_n(\theta)| > H\sqrt{n}.$$

The same is true of all Cesàro's means formed from the series.

The attempt to prove this theorem leads us to a problem which is very interesting in itself, namely that of the behaviour of the modular functions

$$\Sigma q^{(n-\frac{1}{2})^2}, \quad \Sigma q^{n^2}, \quad \Sigma(-1)^{n-1} q^{n^2}$$

as q tends along a radius vector to an "irrational place" $e^{\theta\pi i}$ on the unit circle. If $f(q)$ denotes any one of these functions, it is trivial that

$$f(q) = O\{(1 - |q|)^{-\frac{1}{2}}\}.$$

* This hypothesis may be generalised widely.

If q tends to a *rational* place, it is known that $f(q)$ tends to a limit or becomes definitely infinite of order $\frac{1}{2}$. By arguments depending upon the formulae of transformation of the ϑ-functions, and similar in principle to, though simpler than, those of § 4, we prove

Theorem 7. *When q tends to any irrational place on the circle of convergence,*

$$f(q) = o\{(1-|q|)^{-\frac{1}{2}}\}.$$

No better result than this is true in general. If $q \to e^{\theta\pi i}$, where θ is one of the irrationals defined in Theorem 5, then

$$f(q) = O\{(1-|q|)^{-\frac{1}{4}}\}.$$

Further, whatever be the value of θ, we can find a constant H and an infinity of values of $|q|$, tending to unity, such that

$$|f(q)| > H\{(1-|q|)^{-\frac{1}{4}}\}.$$

In so far as these results assign *upper* limits for $|f(q)|$, they could be deduced from our previous theorems. But the remaining results are new, and Theorem 6 is a corollary of the last of them. Another interesting corollary is

Theorem 8. *The series*

$$\Sigma n^{-a} e^{(n-\frac{1}{2})^2\theta\pi i}, \quad \Sigma n^{-a} e^{n^2\theta\pi i}, \quad \Sigma(-1)^n n^{-a} e^{n^2\theta\pi i},$$

where θ is irrational, and $a \leqq \frac{1}{2}$, can never be convergent, or summable by any of Cesàro's means.

On the other hand, if $a > \frac{1}{2}$, these series are each certainly convergent for an everywhere dense set of values of θ. They are connected with definite integrals of an interesting type: for example

$$\sum_1^\infty \frac{(-1)^{n-1}}{n} e^{n^2\theta\pi i} = \sqrt{\left(\frac{i}{\pi}\right)} \int_0^\infty e^{-ix^2} \log(4\cos^2 \varpi x)\, dx,$$

where $\varpi = \sqrt{(\theta\pi)}$, whenever the series is convergent.

6. We have also considered series of the types $\Sigma(n\theta)$, $\Sigma(n^2\theta)$, ... It is convenient to write

$$\{n\theta\} = (n\theta) - \tfrac{1}{2}, \quad s_n = \sum_{\nu \leqq n} \{\nu\theta\}.$$

Arithmetic arguments analogous to those used in proving Theorems 3 and 4 lead to

Theorem 9. *If θ is any irrational number, then $s_n = o(n)$. The same result holds for the series in which ν is replaced by $\nu^2, \nu^3, \ldots, \nu^k, \ldots$*. Further, this result is the best possible of its kind.*

* This result, in the case $k=1$, has (as was kindly pointed out to us by Prof. Landau) been given by Sierpinski (see the *Jahrbuch über die Fortschritte der Math.*, 1909, p. 221). Similar results hold for the function

$$x + a - [x+a] - \tfrac{1}{2}$$

which reduces to $\{x\}$ for $a=0$.

When $k=1$, we can obtain more precise results analogous to those of §§ 4, 5. The series $\Sigma\{n\theta\}$ behaves, in many ways, like the series $\Sigma e^{n^2\theta\pi i}$. The *rôle* of the formula of Genocchi and Schaar is now assumed by Gauss's formula

$$\sum_1^{\frac{1}{2}(q-1)}\left[\frac{\nu p}{q}\right]+\sum_1^{\frac{1}{2}(p-1)}\left[\frac{\nu q}{p}\right]=\tfrac{1}{4}(p-1)(q-1),$$

where p, q are odd integers. Taking this formula as our starting point we easily prove Theorem 9 in the case $k=1$. Further, we obtain

Theorem 10. *If θ is an irrational number of the type defined in Theorem 5, then $s_n=O(\log n)$.*

This corresponds to Theorem 5. When we come to Theorem 6 the analogy begins to fail. We are not able to show that, for every irrational θ (or even for every θ of the special class of Theorem 5), s_n is sometimes effectively of the order of $\log n$. The class in question includes values of θ for which this is so, but, for anything we have proved to the contrary, there may be values of θ for which $s_n=O(1)$. And when we consider, instead of s_n, the corresponding Cesàro mean of order 1, this phenomenon does actually occur. While engaged on the attempt to elucidate these questions we have found a curious result which seems of sufficient interest to be mentioned separately. It is that

$$\sum_{\nu\leqq n}\{\nu\theta\}^2=\tfrac{1}{12}n+O(1)$$

for all irrational values of θ. When we consider the great irregularity and obscurity of the behaviour of $\Sigma\{\nu\theta\}$, it is not a little surprising that $\Sigma\{\nu\theta\}^2$ (and presumably the corresponding sums with higher *even* powers) should behave with such marked regularity.

7. The exceedingly curious results given by the transformation formulae for the series $\Sigma e^{n^2\theta\pi i}$, $\Sigma\{n\theta\}$ suggest naturally the attempt to find similar formulae for the higher series. It is possible, by a further modification of Lindelöf's argument, to obtain a relation between the two sums

$$\sum^{n} e^{\nu^3\theta\pi i},\quad \sum^{m}\mu^{-\frac{1}{4}}e^{-K\mu^{3/2}\pi i},$$

where $K=\sqrt{(32/27\theta)}$. The relation thus obtained gives no information about the first series that is not trivial. We can however deduce the non-trivial result

$$\sum_1^{n} e^{\nu^{3/2}\theta\pi i}=O(n^{\frac{3}{4}}).$$

Similar remarks apply to the higher series $\Sigma e^{n^k\theta\pi i}$ and to the series $\Sigma\{n^k\theta\}$, where $k>1$. But it does not seem probable that we can make much progress on these lines with any of our main problems.

In conclusion we may say that (with the kind assistance of Dr W. W. Greg, Librarian of Trinity College, and Mr J. T. Dufton, of Trinity College) we have tabulated the values of $(n^2\theta)$ for the first 500 values of n, in the cases

$$\theta=\frac{1}{\sqrt{10}}=\cdot 31622776\ldots,\ \theta=e.$$

The distribution of these values shows striking irregularities which encourage a closer scrutiny.

COMMENTS

This communication to the 1912 Congress† is mainly a summary of the principal results of 1914, 2 and 1914, 3, though the proofs of Theorems 9 and 10 were not published until 1922, 6 and 1922, 9.

§ 4. The footnote to Theorem 5 does not mean that the same result can be proved under a more general hypothesis, but that other hypotheses could be made about θ which would imply similar (but weaker) results. Such results were given in 1914, 3 and 1922, 5.

§ 6. The statement concerning $\Sigma\{\nu\theta\}^2$ is erroneous and was corrected in the last sentence of 1922, 6. The correct form appears as Theorem 11 of 1922, 9.

§ 7. For some remarks on more general transformation formulae, such as that for $\Sigma e^{\nu^3\theta\pi i}$ mentioned in the text, see the comments on 1914, 3.

The final sentence does not seem to have given rise to any further investigation, and it would be of interest to know in what senses the fractional parts of $n^2\theta$ are less well distributed than those of $n\theta$.

† Some of the results had been briefly communicated to the London Mathematical Society at its meeting on 8 February 1912 (see *Proc.* 11 (1912), xxi–xxii).

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION.

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I.

The fractional part of $n^k \theta$.

1.0 — Introduction.

1.00. Let us denote by $[x]$ and (x) the integral and fractional parts of x, so that

$$(x) = x - [x], \qquad 0 \leq (x) < 1.$$

Let θ be an irrational number, and α any number such that $0 \leq \alpha < 1$. Then it is well known that it is possible to find a sequence of positive integers $n_1, n_2, n_3, \cdots$ such that

$$(n_r \theta) \rightarrow \alpha \tag{1.001}$$

as $r \rightarrow \infty$.

It is necessary to insert a few words of explanation as to the meaning to be attributed to relations such as (1.001), here and elsewhere in the paper, in the particular case in which $\alpha = 0$. The formula (1.001), when $\alpha > 0$, asserts that, given any positive number ε, we can find r_0 so that

$$-\varepsilon < (n_r \theta) - \alpha < \varepsilon \qquad (r > r_0).$$

The points $(n_r \theta)$ may lie *on either side of* α. But $(n_r \theta)$ is never negative, and so, in the particular case in which $\alpha = 0$, the formula, if interpreted in the obvious manner, asserts *more* than this, viz. that

$$0 \leq (n_r \theta) < \varepsilon \qquad (r > r_0).$$

 1914, 2 (with J. E. Littlewood) *Acta Mathematica*, 37, 155–91.

The obvious interpretation therefore gives rise to a distinction between the value $\alpha = 0$ and other values of α which would be exceedingly inconvenient in our subsequent analysis.

These difficulties may be avoided by agreeing that, when $\alpha = 0$, the formula (1.001) is to be interpreted as meaning '*the set of points* $(n_r \theta)$ *has, as its sole limiting point or points, one or both of the points 1 and 0*', that is to say as impying that, for any r greater than r_0, *one or other* of the inequalities

$$0 \leq (n_r \theta) < \varepsilon, \quad 1 - \varepsilon < (n_r \theta) < 1$$

is satisfied. In the particular case alluded to above, this question of interpretation happens to be of no importance: our assertion is true on either interpretation. But in some of our later theorems the distinction is of vital importance.

Now let $f(n)$ denote a positive increasing function of n, integral when n is integral, such as

$$n, n^2, n^3, \cdots 2^n, 3^n, \cdots, n!, 2^{n^2}, \cdots 2^{2^n}, \cdots.$$

The result stated at the beginning suggests the following question, which seems to be of considerable interest: — *For what forms of* $f(n)$ *is it true that, for any irrational* θ, *and any value of* α *such that* $0 \leq \alpha < 1$, *a sequence* (n_r) *can be found such that*

$$(f(n_r)\theta) \to \alpha? \tag{1.002}$$

It is easy to see that when the increase of $f(n)$ is sufficiently rapid the result suggested will not always be true. Thus if $f(n) = 2^n$ and θ is a number which, expressed in the binary scale, shows at least k 0's following upon every 1, it is plain that

$$(2^n \theta) < \frac{1}{2} + \lambda_k,$$

when λ_k is a number which can be made as small as we please by increasing k sufficiently. There is thus an »excluded interval» of values of α, the length of which can be made as near to $\frac{1}{2}$ as we please. If $f(n) = 3^n$ we can obtain an excluded interval whose length is as near to $\frac{2}{3}$ as we please, and so on; while if $f(n) = n!$ it is (as is well known) possible to choose θ so that $(n!\theta)$ tends to a unique limit. Thus $(n!e) \to 0$.

At the end of the paper we shall return to the general problem. The immediate object with which this paper was begun, however, was to determine whe-

ther the relation (1.002) always holds (if θ is irrational) when $f(n)$ is a power of n, and we shall be for the most part concerned with this special form of $f(n)$.

1.01. The following generalisation of the theorem expressed by (1.001) was first proved by KRONECKER.[1]

Theorem 1.01. *If $\theta_1, \theta_2, \cdots \theta_m$ are linearly independent irrationals (i. e. if no relation of the type*

$$a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m + a_{m+1} = 0,$$

where $a_1, a_2, \cdots a_{m+1}$ are integers, not all zero, holds between $\theta_1, \theta_2, \cdots \theta_m$), and α_1, $\alpha_2, \cdots \alpha_m$ are numbers such that $0 \leq \alpha_p < 1$, then a sequence (n_r) can be found such that

$$(n_r\theta_1) \to \alpha_1, (n_r\theta_2) \to \alpha_2, \cdots, (n_r\theta_m) \to \alpha_m,$$

as $r \to \infty$. Further, in the special case when all the α's are zero, it is unnecessary to make any restrictive hypothesis concerning the θ's, or even to suppose them irrational.

This theorem at once suggests that the solution of the problem stated at the end of 1.00 may be generalised as follows.

Theorem 1.011. *If $\theta_1, \theta_2, \cdots \theta_m$ are linearly independent irrationals, and the α's are any numbers such that $0 \leq \alpha < 1$, then a sequence (n_r) can be found such that*

$$(1.011) \quad \begin{cases} (n_r\theta_1) \to \alpha_{11}, (n_r\theta_2) \to \alpha_{12}, \cdots, (n_r\theta_m) \to \alpha_{1m}; \\ (n_r^2\theta_1) \to \alpha_{21}, (n_r^2\theta_2) \to \alpha_{22}, \cdots, (n_r^2\theta_m) \to \alpha_{2m}; \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (n_r^k\theta_1) \to \alpha_{k1}, (n_r^k\theta_2) \to \alpha_{k2}, \cdots, (n_r^k\theta_m) \to \alpha_{km}. \end{cases}$$

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[1] KRONECKER, *Berliner Sitzungsberichte,* 11 Dec. 1884; *Werke,* vol. 3, p. 49.

A number of special cases of the theorem were known before. That in which all the α's are zero was given by DIRICHLET (*Berliner Sitzungsberichte,* 14 April 1842, *Werke,* vol. 1, p. 635). Who first stated explicitly the special theorems in which $m = 1$ we have been unable to discover. DIRICHLET (l. c.) refers to the simplest as »längst bekannt»: it is of course an immediate consequence of the elementary theory of simple continued fractions. See also MINKOWSKI, *»Diophantische Approximation»,* pp. 2, 7. KRONECKER's general theorem has been rediscovered independently by several writers. See *e. g.* BOREL, *Leçons sur les séries divergentes,* p. 135; F. RIESZ, *Comptes Rendus,* 29 Aug. 1904. Some of the ideas of which we make most use are very similar to those of the latter paper. It should be added that DIRICHLET's and KRONECKER's theorems are presented by them merely as particular cases of more general theorems, which however represent extensions of the theory in a direction different from that with which we are concerned.

A number of very beautiful applications of KRONECKER's theorem to the theory of the RIEMANN ζ-function have been made by H. BOHR.

Further, if the α's are all zero, it is unnecessary to suppose the θ's restricted in any way.

1.02. This theorem is the principal result of the paper: it is proved in section 1.2. The remainder of the paper falls into three parts. The first of these (section 1.1) consists of a discussion and proof of KRONECKER's theorem. We have thought it worth while to devote some space to this for two reasons. In the first place our proof of theorem **1.011** proceeds by induction from k to $k+1$, and it seems desirable for the sake of completeness to give some account of the methods by which the theorem is established in the case $k=1$. In the second place the theorem for this case possesses an interest and importance sufficient to justify any attempt to throw new light upon it; and the ideas involved in the various proofs which we shall discuss are such as are important in the further developments of the theory. We believe, moreover, that the proof we give is considerably simpler than any hitherto published.

The second of the remaining parts of the paper (section 1.3) is devoted to the question of the *rapidity* with which the numbers $(n^x \theta_p)$ in the scheme (1.011) tend to their respective limits. Our discussion of the problems of this section is very tentative, and the results very incomplete;[1] and something of the same kind may be felt about the paper as a whole. We have not solved the problems which we attack in this paper with anything like the definiteness with which we solve those to which our second paper is devoted. The fact is, however, that the first paper deals with questions which, in spite of their more elementary appearance, are in reality far more difficult than those of the second. Finally, the last section (1.4) contains some results the investigation of which was suggested to us by an interesting theorem proved by F. BERNSTEIN.[2] The distinguishing features of these results are that they are concerned with a single irrational θ and with sequences which are not of the form $(n^k\theta)$, and that they hold for *almost all* values of θ, *i. e.* for all values except those which belong to an exceptional and unspecified set of measure zero.

1.1 — Kronecker's Theorem.

1.10. KRONECKER's theorem falls naturally into two cases, according as to whether or not all the α's are zero. We begin by considering the simpler case,

[1] Some of the results that we do obtain, however, are important from the point of view of applications to the theory of the series $\Sigma\, e^{n^k\theta i}$ and that of the RIEMANN ζ-function. It was in part the possibility of these applications that led us to the researches whose results are given in the present paper. The applications themselves will, we hope, be given in a later paper.

[2] *Math. Annalen*, vol. 71, p. 421.

when all the α's are zero. Unlike most of the theorems with which we are concerned, this is not proved by induction, and there is practically no difference between the cases of one and of several variables. The proof given is DIRICHLET'S.

Let $\bar{x}$ denote the number which differs from x by an integer and which is such that $-\frac{1}{2} < \bar{x} \leq \frac{1}{2}$. Then the theorem to be proved is equivalent to the theorem that, given any integers q and N, we can find an n not less than N and such that

$$|\overline{n\theta_1}| \leq 1/q, \; |\overline{n\theta_2}| \leq 1/q, \; \cdots, \; |\overline{n\theta_m}| \leq 1/q.$$

Let us first suppose that $N = 1$. Let R be the region in m-dimensional space for which each coordinate ranges from 0 to 1. Let the range of each coordinate be divided into q equal parts: R is then divided into q^m parts. Consider now the $q^m + 1$ points

$$(\nu\theta_1), \; (\nu\theta_2), \; \cdots, \; (\nu\theta_m); \; (\nu = 0, 1, 2, \cdots q^m).$$

There must be one part of R which contains two points; let the corresponding values of ν be ν_1 and ν_2. Then clearly

$$|\overline{(\nu_1 - \nu_2)\theta_1}| \leq 1/q, \; |\overline{(\nu_1 - \nu_2)\theta_2}| \leq 1/q, \; \cdots, |\overline{(\nu_1 - \nu_2)\theta_m}| \leq 1/q,$$

and
$$|\nu_1 - \nu_2| \geq 1.$$

We have therefor only to take $n = |\nu_1 - \nu_2|$. We observe that we have also

$$n \leq q^m,$$

a result to which we shall have occasion to return in section 1.3.

If $N > 1$ we have only to consider the points $(\nu N\theta_1), (\nu N\theta_2), \cdots$ instead of the points $(\nu\theta_1), (\nu\theta_2), \cdots$.

1.11. We turn now to the case when the α's are not all necessarily zero. In this case the necessity of the hypothesis that the θ's are linearly independent is obvious, for the existence of a linear relation between the θ's would plainly involve that of a corresponding relation between the α's; naturally, also, the added restriction makes the theorem much more difficult than the one just proved.

Our proof proceeds by induction from m to $m + 1$; it is therefore important to discuss the case $m = 1$. The result for this case may be proved in a variety of ways, of which we select four which seem to us to be worthy of separate dis-

cussion. These proofs are all simple, and each has special advantages of its own. It is important for us to consider very carefully the ideas involved in them with a view to selecting those which lend themselves most readily to generalisation. For example, it is essential that our proof should make no appeal to the theory of continued fractions.

(*a*). The first proof is due to KRONECKER. It follows from the result of 1.10, with $m = 1$, or from the theory of continued fractions, that we can find an arbitrarily large q such that

$$\theta = \frac{p}{q} + \frac{\delta}{q^2},$$

and so

$$q\theta - p = \delta/q. \tag{1.111}$$

where

$$|\delta| < 1.$$

It is possible to express any integer, and in particular the integer $\{q\alpha\}$ nearest to $q\alpha$, in the form

$$q n_1 + p n$$

where n and n_1 are integers, and $|n| \leq q/2$. From the two equations

$$q\theta - p = \delta/q, \; q n_1 + p n = \{q\alpha\}$$

we obtain

$$q(n\theta + n_1) = \frac{n\delta}{q} + q\alpha + \frac{1}{2}\delta_1, \qquad |\delta_1| < 1,$$

and so

$$-1 < q(n\theta + n_1 - \alpha) < 1,$$

or

$$|(n\theta) - \alpha| < 1/q.$$

If we write $\nu = n + q$ and use (1.111), we see that

$$|(\nu\theta) - \alpha| < 2/q, \qquad q/2 < \nu < 3q/2;$$

so that

$$|(\nu\theta) - \alpha| < 3/\nu$$

for some value of ν between $q/2$ and $3q/2$. This evidently establishes the truth of the theorem.

If we attempt to extend this proof to the case of several variables we find nothing to correspond to the equation

$$\{q\alpha\} = qn_1 + pn.$$

But KRONECKER's proof has, as against the proofs we shall now discuss, the very important advantage of furnishing a definite result as to the order of the approximation, a point to which we shall return in 1.3.

(b). Let ε be an arbitrary positive constant. By the result of 1.10, we can find an n such that $0 < \theta_1 < \varepsilon$ or $1 - \varepsilon < \theta_1 < 1$, where $\theta_1 = (n\theta)$. Since θ is irrational, θ_1 is not zero. Let us suppose that $0 < \theta_1 < \varepsilon$; the argument is substantially the same in the other case. We can find an m such that

$$m\theta_1 \leq \alpha < (m+1)\theta_1,$$

$$|m\theta_1 - \alpha| < \theta_1;$$

and so

$$|(nm\theta) - \alpha| < \varepsilon,$$

which proves the theorem.

(c).[1] Let S denote the set of points $(n\theta)$. S', its first derived set, is closed. It is moreover plain that, if α is not a point of S', then neither is $(\alpha + n\theta)$ nor $(\alpha - n\theta)$.

The theorem to be proved is clearly equivalent to the theorem that S' consists of the continuum $(0,1)$. Suppose that this last theorem is false. Then there is a point α which is not a point of S', and therefore an interval containing α and containing[2] no point of S'. Consider I, the greatest possible such interval containing α.[3] The interval obtained by translating I through a distance θ, any number of times in either direction,[4] must, by what was said above, also contain no point of S'. But the interval thus obtained cannot overlap with I, for then I would not be the »greatest possible» interval of its kind.

[1] This proof was discovered independently by F. RIESZ, but, so far as we know, has not been published.

[2] In its interior, in the strict sense.

[3] The existence of such a »greatest possible» interval is easily established by the classical argument of DEDEKIND.

[4] Taking the congruent interval in $(0, 1)$. This interval may possibly consist of two separate portions $(0, \xi_1)$, and $(\xi_2, 1)$.

Hence, if we consider a series of $[1/\delta]$ translations, where δ is the length of I, it is clear that two of the corresponding $[1/\delta]+1$ intervals must coincide. Clearly this can only happen if θ is rational, which is contrary to our hypothesis.

(*d*). We argue as before that, if the theorem is false, there is an interval I, of length 2ε and middle point α, containing no point of S'. By the result of 1.10 we can find n so that, if $\theta_1=(n\theta)$, then $0<\theta_1<\varepsilon$ or $1-\varepsilon<\theta_1<1$.

By the reasoning used in (*c*) it appears that the interval obtained by translating I through a distance θ_1, any number of times in either direction, must contain no point of S'. But since each new interval overlaps with the preceding one it is clear that after a certain number of translations we shall have covered the whole interval 0 to 1 by intervals containing no point of S', and shall thus have arrived at a contradiction.

1.12. Let us compare the three last proofs. It is clear that (*b*) is considerably the simplest, and that (*d*) appears to contain the essential idea of (*b*) together with added difficulties of its own. It appears also that, in point of simplicity, there is not very much to choose between (*c*) and (*d*), and that (*c*) has a theoretical advantage over (*d*) in that it dispenses the assumption of the theorem for the case $\alpha=0$, an assumption which is made not only in (*b*) and (*d*), but also in (*a*). When, however, we consider the theorem for several variables, it seems that (*b*) does not lend itself to direct extension at all, that the complexity of the region corresponding to I in (*c*) leads to serious difficulties, and that (*d*) provides the simplest line of argument. It is accordingly this line of argument which we shall follow in our discussion of the general case of KRONECKER's theorem.

1.13. We pass now to the general case of KRONECKER's theorem. We shall give a proof by induction. For the sake of simplicity of exposition we shall deduce the theorems for three independent irrationals θ, φ, ψ, from that for two. It will be obvious that the same proof gives the general induction from n to $n+1$ irrationals.

We wish to show that if we form the set S of points within the cube $0\leq x\leq 1, 0\leq y\leq 1, 0\leq z\leq 1$, which are congruent with

$$(\theta, \varphi, \psi),\ (2\theta, 2\varphi, 2\psi), \cdots\cdots (n\theta, n\varphi, n\psi), \cdots$$

then every point of the cube is a point of the first derived set S'. It is plain that, if (α, β, γ) is not a point of S, then neither is $((\alpha+n\theta), (\beta+n\varphi), (\gamma+n\psi))$ nor $((\alpha-n\theta), (\beta-n\varphi), (\gamma-n\psi))$. If now our theorem is not true, there must exist a sphere, of centre (α, β, γ) and radius ϱ, which contains[1] no point of S'. By

[1] Within or upon the boundary.

the result of 1.10, there is an n such that the distance δ of $((n\theta), (n\varphi), (n\psi))$ or $(\theta_1, \varphi_1, \psi_1)$ from one of the vertices of the cube is less than $\varrho/\sqrt{2}$. Let us suppose, for example, that the vertex in question is the point (0, 0, 0). Consider the straight line

$$(1.131) \qquad \frac{x-\alpha}{\theta_1} = \frac{y-\beta}{\varphi_1} = \frac{z-\gamma}{\psi_1},$$

and the infinite cylinder of radius δ with this line as axis. It is clear that the finite cylinder C obtained by taking a length δ on either side of (α, β, γ) is entirely contained in the sphere and therefore contains no point of S'. Hence the cylinder obtained by translating C through $(\theta_1, \varphi_1, \psi_1)$, any number of times in either direction, also contains no point of S', so that, since each new position of C overlaps with the preceding, the whole of the infinite cylinder, or rather of the congruent portions of the cube, is free from points of S'.

Let us now consider the intersections of the totality of straight lines in the cube, which are congruent with portions of the axis of the cylinder, with an arbitrary plane $x = x_0$. We shall show that they are everywhere dense in the square in which the plane cuts the cube, whence clearly follows that no point of the cube is a point of S', and so a contradiction which establishes the theorem.

The intersections (y, z) are congruent with the intersections of the axis (1.131) with

$$x = x_0 + \nu, \qquad (\nu = \cdots, -2, -1, 0, 1, 2, \cdots),$$

and so they are the points congruent with

$$\beta + \frac{(x_0-\alpha)\varphi_1}{\theta_1} + \frac{\nu\varphi_1}{\theta_1}, \qquad \gamma + \frac{(x_0-\alpha)\psi_1}{\theta_1} + \frac{\nu\psi_1}{\theta_1}.$$

But, under our hypothesis, φ_1/θ_1 and ψ_1/θ_1 are linearly independent irrationals, and so, by the theorem for two irrationals, this set of points is everywhere dense in the square. The proof is thus completed.

1.14. We add two further remarks on the subject of KRONECKER's theorem, in which, for the sake of simplicity of statement, we confine ourselves to the case of two linearly independent irrationals θ, φ.

(*a*) Suppose that $0 < \alpha < 1$, $0 < \beta < 1$. KRONECKER's theorem asserts the existence of a sequence (n_s) such that $(n_s\theta) \to \alpha$, $(n_s\varphi) \to \beta$. Let us choose a sequence of points

$$(\alpha_\mu, \beta_\mu), \qquad (\mu = 1, 2, 3, \cdots),$$

such that

$$\alpha_\mu > \alpha, \ \beta_\mu > \beta, \ \ \alpha_\mu \to \alpha, \ \beta_\mu \to \beta.$$

There is, for any value of μ, a sequence $(n_{s\mu})$ such that

$$(n_{s\mu}\theta) \to \alpha_\mu, \ (n_{s\mu}\varphi) \to \beta_\mu,$$

as $s \to \infty$. From this it is easy to deduce the existence of a sequence (n_r) for which $(n_r\theta)$ and $(n_r\varphi)$ tend to the limits α and β and are always greater than those limits, so that the direction of approach to the limit is in each case from the right hand side.[1] Similarly, of course, we can establish the existence of a sequence giving, for either θ or φ, either a right-handed or a left-handed approach to the limit.

If we apply similar reasoning to the case in which α or β or both are zero we see that, when θ and φ are linearly independent irrationals, we may abandon the convention with respect to the particular value 0 which was adopted in 1.00, and assert that there is a sequence for which $(n\theta) \to \alpha$ and $(n\varphi) \to \beta$, α and β having any values between 0 and 1, both values included, and the formulae having the ordinary interpretation. This result is to be carefully distinguished from that of 1.10. The latter is, the former is *not*, true without restriction on the θ's, as may be seen at once by considering the case in which $\varphi = -\theta$.

(b). It is easy to deduce from KRONECKER's theorem a further theorem, which may be stated as follows:[2] *if we take any portion γ of the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, bounded by a finite number of regular curves, and of area δ; and if we denote by $N_\gamma(n)$ the number of the points*

$$((\nu\theta), (\nu\varphi)), \qquad (\nu = 1, 2, \cdots n),$$

which fall inside γ; then

$$N_\gamma(n) \backsim \delta n$$

as $n \to \infty$.

This result, when compared with the various theorems of this paper, suggests a whole series of further theorems. The proofs of these appear likely to be very difficult, and we have, up to the present, considered only the case of a single irrational θ. We have proved that, if $N_\gamma(n)$ denotes the number of the points

$$(\nu^\varkappa\theta), \qquad (\nu = 1, 2, \cdots n),$$

[1] The reasoning by which this is established is essentially the same as that of 1.20.

[2] This is a known theorem. For a proof and references see the tract 'The Riemann Zeta-function and the Theory of Prime Numbers', by H. BOHR and J. E. LITTLEWOOD, shortly to be published in the *Cambridge Tracts in Mathematics and Mathematical Physics*.

which fall inside a segment γ of $(0, 1)$, of length δ, then $N_\gamma(n) \sim \delta n$. This result may be compared with that of Theorem **1.483** at the end of the paper. But results of this character will find a more natural place among our later investigations than among those of which we are now giving an account.

1.2. — The generalisation of Kronecker's theorem.

1.20. We proceed now to the proof of theorem **1.011**. Our argument is based on the following general principle, which results from the work of PRINGSHEIM and LONDON on double sequences and series.[1]

1.20. *If*

$$\lim_{r_1 \to \infty} \lim_{r_2 \to \infty} \cdots \lim_{r_k \to \infty} f_p(r_1, r_2, \cdots r_k) = A_p, \; (p = 1, 2, \cdots m),$$

then we can find a sequence of sets $(r_{1n}, r_{2n}, \cdots r_{kn})$ *such that, as* $n \to \infty$,

$$r_{qn} \to \infty, \; (q = 1, 2, \cdots k),$$

and

$$f_p(r_{1n}, r_{2n}, \cdots r_{kn}) \to A_p, \; (p = 1, 2, \cdots m).$$

We shall show that, if this principle is true for all values of m and a particular k, then it is true for $k+1$. As it is plainly true for $k = 1$, we shall thus have proved it generally.

We shall abbreviate '$\lim_{r_1 \to \infty} \lim_{r_2 \to \infty} \cdots \lim_{r_k \to \infty}$' into '$\lim_{r_1, r_2, \cdots r_k}$', or, when there is no danger of confusion, into 'lim'.

Let

$$\lim_{r_1, r_2, \cdots r_k} f_p(r_1, r_2, \cdots r_{k+1}) = f_p(r_{k+1}).$$

Then by hypothesis

$$f_p(r_{k+1}) \to A_p$$

as $r_{k+1} \to \infty$. Let us choose an integer $r_{k+1, n}$, greater than 2^n, for which

$$|f_p(r_{k+1, n}) - A_p| < 2^{-n-1}, \quad (p = 1, 2, \cdots m).$$

By the principle for k variables, we can find $r_{1n}, r_{2n}, \cdots r_{kn}$, all greater than 2^n, and such that

[1] PRINGSHEIM, *Münchener Sitzungsberichte*, vol. 27, p. 101, and *Math. Annalen*, vol. 53, p. 289; LONDON, *Math. Annalen*, vol. 53, p. 322.

$$|f_p(r_{1n}, r_{2n}, \cdots r_{kn}, r_{k+1,n}) - f_p(r_{k+1,n})| < 2^{-n-1}, \ (p = 1, 2, \cdots m).$$

We thus obtain a sequence of sets $(r_{1n}, r_{2n}, \cdots r_{k+1,n})$, such that every member of the n^{th} set is greater than 2^n and

$$|f_p(r_{1n}, r_{2n}, \cdots r_{k+1,n}) - A_p| < 2^{-n}, \ (p = 1, 2, \cdots m).$$

This sequence evidently gives us what we want.

An important special case of the principle is the following:

1.201. *If for all values of t we can find a sequence $n_{1t}, n_{2t}, \cdots, n_{rt}, \cdots$ such that*

$$f_p(n_{rt}) \to A_{pt}, \quad (p = 1, 2, \cdots m),$$

as $r \to \infty$, and if

$$A_{pt} \to A_p, \quad (p = 1, 2, \cdots m),$$

as $t \to \infty$, then there is a sequence (n_s) such that

$$f_p(n_s) \to A_p, \quad (p = 1, 2, \cdots m),$$

as $s \to \infty$.

This is in reality merely a case of the principle that a limiting-point of limiting-points is a limiting-point.

1.21. We consider first the case in which all the α's are zero, and the θ's are unrestricted. In this case the proof is comparatively simple.

Theorem 1.21. *There is a sequence (n_r) such that, as $r \to \infty$*

$$(n_r^{\varkappa} \theta_p) \to 0, \quad (\varkappa = 1, 2, \cdots k; \ p = 1, 2, \cdots m).$$

We prove this theorem by induction from k to $k + 1$: we have seen that it is true when $k = 1$. We suppose then that there is a sequence (μ_s) such that

$$(\mu_s^{\varkappa} \theta_p) \to 0, \quad (\varkappa = 1, 2, \cdots k; \ p = 1, 2, \cdots m). \tag{1.211}$$

The sequence

$$(\mu_s^{k+1} \theta_1), \ (\mu_s^{k+1} \theta_2), \cdots (\mu_s^{k+1} \theta_m), \quad (s = 1, 2, \cdots),$$

has at least one limiting point $\varphi_1, \varphi_2, \cdots \varphi_m$; hence, by restricting ourselves to a subsequence selected from the sequence (μ_s), we can obtain a sequence (ν_s) such that, as $s \to \infty$,

$$(\nu_s^{\varkappa} \theta_p) \to 0, \ (\varkappa \leq k); \ (\nu_s^{k+1} \theta_p) \to \varphi_p; \ (p = 1, 2, \cdots m).$$

We then have, for $\varkappa \leq k+1$,

$$\lim_{s_1, s_2, \cdots s_\lambda} ((\nu_{s_1} + \nu_{s_2} + \cdots + \nu_{s_\lambda})^\varkappa \theta_p) = \sum_{q=1}^{\lambda} \lim (\nu_{s_q}^\varkappa \theta_p) + \sum C \lim (\nu_{s_1}^{\varkappa_1} \nu_{s_2}^{\varkappa_2} \cdots \nu_{s_\lambda}^{\varkappa_\lambda} \theta_p),$$

where the C's are constants, $\varkappa_1 + \varkappa_2 + \cdots + \varkappa_\lambda = \varkappa$, and $\varkappa_q \leq k$. In virtue of (1.211) we can evaluate at once every repeated limit on the right hand side, and it is clear that we obtain $\lambda \varphi_p$ or 0 according as $\varkappa = k+1$ or $\varkappa \leq k$. It follows from the general principle **1.20** that we can find a sequence $(n_{r\lambda})$, $(r = 1, 2, \cdots)$, such that, as $r \to \infty$,

$$(n_{r\lambda}^\varkappa \theta_p) \to 0, \ (\varkappa \leq k); \ (n_{r\lambda}^{k+1} \theta_p) \to \lambda \varphi_p; \ (p = 1, 2, \cdots m).$$

But, by theorem **1.01**, we can find a sequence (λ_s) such that

$$(\lambda_s \varphi_p) \to 0, \quad (p = 1, 2, \cdots m);$$

and we have only to apply the principle **1.201** to obtain the theorem for $k+1$.

1.22. We pass now to the general case when the α's are not all zero. We have to prove that *if $\theta_1, \theta_2, \cdots \theta_m$ are linearly independent irrationals, there is a sequence (n_r) such that, as $r \to \infty$,*

$$(n_r^\varkappa \theta_p) \to \alpha_{\varkappa p}, \ (\varkappa = 1, 2, \cdots k; \ p = 1, 2, \cdots m).$$

We shall prove this by an induction from k to $k+1$ which proceeds by two steps.

(i). We assume the existence, for a particular k, any number m of θ's, and any corresponding system of α's, of a sequence giving the scheme of limits

$$\begin{array}{c|cccc} & \theta_1 & \theta_2 & \cdots & \theta_m \\ \hline n & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ n^2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ n^k & \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{km} \end{array}$$

and we prove the existence, for any number m of θ's, and any corresponding system of α's, of a sequence giving the scheme

$$\begin{array}{c|cccc} & \theta_1 & \theta_2 & \dots & \theta_m \\ \hline n & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ n^2 & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ n^k & \alpha_{k1} & \alpha_{k2} & \dots & \alpha_{km} \\ n^{k+1} & 0 & 0 & \dots & 0\,. \end{array}$$

It will be understood that neither m, nor the θ's, nor the α's are necessarily the same in these two schemes, all of them being arbitrary.

(ii). We then show that we can pass from the last written scheme of limits to the general scheme in which the elements of the last row also are arbitrary.

1.23. *Proof of the first step.* To fix our ideas we shall show that we can pass from a sequence (n_r) giving[1]

$$n_r\theta \to \alpha_1,\ n_r\varphi \to \beta_1,\ n_r\psi \to \gamma_1,\ n_r\chi \to \delta_1,\ n_r\omega \to \eta_1,\ n_r\tau \to \zeta_1,$$

$$n_r^2\theta \to \alpha_2,\ n_r^2\varphi \to \beta_2,\ n_r^2\psi \to \gamma_2,\ n_r^2\chi \to \delta_2,\ n_r^2\omega \to \eta_2,\ n_r^2\tau \to \zeta_2,$$

to a sequence (m_r) giving

$$m_r\theta \to \alpha_1,\ m_r\varphi \to \beta_1,$$

$$m_r^2\theta \to \alpha_2,\ m_r^2\varphi \to \beta_2,$$

$$m_r^3\theta \to 0,\ m_r^3\varphi \to 0.$$

It will be clear that the argument is in reality of a perfectly general type.

Suppose we are given $\alpha'_1, \alpha'_2, \beta'_1, \beta'_2$, and that $\theta, \varphi, \alpha'_1, \alpha'_2, \beta'_1, \beta'_2$, are linearly independent irrationals. Then by hypothesis we can find a sequence giving the scheme

$$n_r\theta \to \alpha'_1,\ n_r\varphi \to \beta'_1,\ n_r\alpha'_1 \to 0,\ n_r\beta'_1 \to 0,\ n_r\alpha'_2 \to 0,\ n_r\beta'_2 \to 0,$$

$$n_r^2\theta \to \alpha'_2,\ n_r^2\varphi \to \beta'_2,\ n_r^2\alpha'_1 \to 0,\ n_r^2\beta'_1 \to 0.$$

Further, the set of points $(n_r^3\theta, n_r^3\varphi)$ has at least one limiting-point (λ, μ), and, by restricting ourselves to a subsequence of (n_r), we may suppose that we have also

$$n_r^3\theta \to \lambda,\ n_r^3\varphi \to \mu.$$

[1] In what follows we shall omit the brackets in $(n\theta, \dots$; it is of course to be understood that integers are to be ignored.

We express all this by saying that we can find a sequence (n_r) giving the scheme

$$(1.221)\qquad \begin{cases} \alpha'_1,\ \beta'_1,\ 0,\ 0,\ 0,\ 0, \\ \alpha'_2,\ \beta'_2,\ 0,\ 0, \\ \lambda,\ \ \mu. \end{cases}$$

The sequence (k_r), where $k_r = 2n_r$, gives us the scheme

$$(1.222)\qquad \begin{cases} 2\alpha'_1,\ 2\beta'_1,\ 0,\ 0,\ 0,\ 0, \\ 4\alpha'_2,\ 4\beta'_2,\ 0,\ 0, \\ 8\lambda,\ \ 8\mu. \end{cases}$$

By the general principle **1.20**, we can find a sequence (l_r) giving the scheme

$$\begin{cases} \lim (n_{r_1} + n_{r_2} + \cdots + n_{r_8})\,\theta,\ \lim (n_{r_1} + n_{r_2} + \cdots + n_{r_8})\,\varphi,\ \dots\dots \\ \lim (n_{r_1} + n_{r_2} + \cdots + n_{r_8})^2\,\theta,\qquad \dots\dots\qquad ,\ \dots\dots \\ \lim (n_{r_1} + n_{r_2} + \cdots + n_{r_8})^3\,\theta,\qquad \dots\dots\qquad , \end{cases}$$

where 'lim' stands for $\lim\limits_{r_1, r_2, \cdots r_8}$.

Consider the repeated limit

$$\lim_{r_1, r_2, \cdots r_8} (n_{r_1} + n_{r_2} + \cdots + n_{r_8})^3\,\theta,$$

which is easily evaluated with the aid of the table (1.221). The limit of a term $n^3_{r_i}\theta$ is λ: that of a 'cross-term'

$$n^a_{r_i} n^b_{r_j} n^c_{r_k}\theta \quad (a + b + c = 3;\ a, b, c < 3;\ i < j < k)$$

is zero, since $n^c_{r_k}\theta$ tends to an α' or a β', and $n^b_{r_j}\alpha'$ and $n^b_{r_j}\beta'$ tend to zero. Thus we obtain the repeated limit 8λ. In all the other repeated limits the cross-terms give zero in the same way, and we see that the sequence (l_r) gives the scheme

$$\begin{matrix} 8\alpha'_1,\ 8\beta'_1,\ 0,\ 0,\ 0,\ 0, \\ 8\alpha'_2,\ 8\beta'_2,\ 0,\ 0, \\ 8\lambda\ ,\ 8\mu. \end{matrix}$$

Consider now the repeated limits

$$\lim_{r, r_1, \cdots r_m} (l_r + k_{r_1} + k_{r_2} + \cdots + k_{r_m})^{\varkappa} \chi$$

where

$$\chi = \theta, \varphi, \alpha'_1, \beta'_1, \alpha'_2, \beta'_2; \ \varkappa = 1, 2, 3.$$

All the cross-terms contribute zero as before, and we obtain the scheme

$$\begin{array}{l} (8 + 2m)\alpha'_1, \ (8 + 2m)\beta'_1, \ 0, 0, 0, 0, \\ (8 + 4m)\alpha'_2, \ (8 + 4m)\beta'_2, \ 0, 0, \\ (8 + 8m)\lambda \ , \ (8 + 8m)\mu, \end{array}$$

or

$$\begin{array}{l} 6\alpha'_1 + (m+1)2\alpha'_1, \ 6\beta'_1 + (m+1)2\beta'_1, \ 0, 0, 0, 0, \\ 4\alpha'_2 + (m+1)4\alpha'_2, \ 4\beta'_2 + (m+1)4\beta'_2, \ 0, 0, \\ \quad (m+1)8\lambda \ , \qquad (m+1)8\mu. \end{array}$$

It is possible, then, to find a sequence giving this scheme. But now, since it is possible to find a sequence of m's such that

$$(m+1)\psi \to 0, \ (\psi = 2\alpha'_1, 2\beta'_1, 4\alpha'_2, 4\beta'_2, 8\lambda, 8\mu),$$

it follows (in virtue of the principle **1.20**) that we can find a sequence giving the scheme

$$\begin{array}{l} 6\alpha'_1, 6\beta'_1, \ 0, 0, 0, 0, \\ 4\alpha'_2, 4\beta'_2, \ 0, 0, \\ \ 0 \ , \ 0. \end{array}$$

This gives us what we want (and something more) provided it is possible to choose

$$\alpha'_1 = \frac{1}{6}\alpha_1, \ \beta'_1 = \frac{1}{6}\beta_1, \ \alpha'_2 = \frac{1}{4}\alpha_2, \ \beta'_2 = \frac{1}{4}\beta_2.$$

This is the case provided $\theta, \varphi, \alpha_1, \beta_1, \alpha_2, \beta_2$ are linearly independent irrationals: it remains only to show that this restriction on $\alpha_1, \beta_1, \alpha_2, \beta_2$ may be removed. It is obvious, in virtue of the principle **1.20**, that this may be done provided we can find a sequence $(\alpha_{1n}, \beta_{1n}, \alpha_{2n}, \beta_{2n})$ such that, for each n, $\theta, \varphi, \alpha_{1n}, \beta_{1n}, \alpha_{2n}, \beta_{2n}$ are linearly independent irrationals, and such that

$$\alpha_{1n} \to \alpha_1, \ \beta_{1n} \to \beta_1, \ \alpha_{2n} \to \alpha_2, \ \beta_{2n} \to \beta_2.$$

Now it is easy to see that there must be points $(\alpha_{1n}, \beta_{1n}, \alpha_{2n}, \beta_{2n})$ interior to the 'cube' with $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ as centre and of side 2^{-n}, and exterior to that

with the same centre and of side 2^{-n-1}, and such that $\theta, \varphi, \alpha_{1n}, \beta_{1n}, \alpha_{2n}, \beta_{2n}$ are linearly independent irrationals. By selecting one such point corresponding to each value of n we obtain a sequence of the kind desired.[1]

1.24. *Proof of the second step.* Here also we shall consider a special case for simplicity: the argument is really general. We shall show that we can pass from a sequence giving the scheme

$$\begin{array}{c|cccc} & \theta & \varphi & \psi & \chi \\ \hline n & \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ n^2 & \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ n^3 & 0 & 0 & & \end{array}$$

to one giving

$$\begin{array}{c|cc} & \theta & \varphi \\ \hline n & \alpha_1 & \beta_1 \\ n^2 & \alpha_2 & \beta_2 \\ n^3 & \alpha_3 & \beta_3. \end{array}$$

As in 1.23, we may suppose, without real loss of generality, that $\theta, \varphi, \alpha_1, \beta_1$ are linearly independent irrationals. Let (n_r) be a sequence giving

$$\begin{array}{c|cccc} & \theta & \varphi & \alpha_1 & \beta_1 \\ \hline n & \frac{1}{2}\alpha_1 & \frac{1}{2}\beta_1 & \alpha_2 & \beta_2 \\ n^2 & 0 & 0 & \frac{2}{3}\alpha_3 & \frac{2}{3}\beta_3 \\ n^3 & 0 & 0 & & \end{array}$$

---

[1] This argument depends ostensibly on ZERMELO's 'Auswahlsprinzip' (or WHITEHEAD and RUSSELL's 'Multiplicative Axiom'). This difficulty can however be surmounted with a little trouble. It should perhaps be observed that we have ignored several similar points early in the paper: in all of these the difficulty is comparatively trivial, and we have only called attention to it in the present instance because it occurs in a more serious form than is usual in constructive mathematics.

An alternative line of argument from that in the text proceeds as follows. It is easy to show that if at most a finite number of primes are omitted, any four of the sequence $\log 2, \log 3, \log 5, \log 7, \log 11, \dots$, together with θ and φ, form a set of six linearly independent irrationals. Moreover it can be deduced from known results concerning the distribution of the primes that we can find a sequence $(\log p_n, \log q_n, \log r_n, \log s_n)$, where p_n, q_n, r_n, and s_n are primes, such that

$$(\log p_n) \to \alpha_1,\ (\log q_n) \to \beta_1,\ (\log r_n) \to \alpha_2,\ (\log s_n) \to \beta_2.$$

Then

$$\lim_{r,s}(n_r + n_s)\theta = \lim_{r}\left(\frac{1}{2}\alpha_1 + n_r\theta\right) = \alpha_1$$

$$\lim_{r,s}(n_r + n_s)^2\theta = \lim_{r}(n_r\alpha_1 + n_r^2\theta) = \alpha_2,$$

$$\lim_{r,s}(n_r + n_s)^3\theta = \lim_{r}\left(\frac{3}{2}n_r^2\alpha_1 + n_r^3\theta\right) = \alpha_3;$$

with similar results for φ. It follows by the principle **1.20** that there is a sequence giving the desired scheme, and the proof of the induction, and therefore that of the theorem, is completed.

1.3. — The order of the approximation.

1.30. We have proved that under certain conditions we can find a sequence (n_r) such that

$$(1.301) \qquad (n_r^\varkappa\theta_p) \to \alpha_{\varkappa p} \quad (\varkappa = 1, 2, \cdots k;\ p = 1, 2, \cdots m).$$

There are a number of interesting questions which may be asked with regard to the *rapidity* with which the scheme of limits is approached.

The relations (1.301) assert that, if we are given λ, there is a function $\Phi(k, m;\ \theta_1, \theta_2, \cdots\theta_m;\ \alpha_{11}, \alpha_{12}, \cdots\alpha_{km};\ \lambda)$[1] such that

$$|(n^\varkappa\theta_p) - \alpha_{\varkappa p}| < 1/\lambda$$

for some $n < \Phi$. It is hardly necessary to observe, after the explanations of 1.00, that this inequality requires a modification when $\alpha_{\varkappa p} = 0$, which may be expressed roughly by saying that $\alpha_{\varkappa p}$ is then to be regarded as a two-valued symbol capable of assuming indifferently the values 0 and 1.

(i) Does Φ necessarily depend on the θ's and α's: can we for example, find a Φ independent of the α's? It will be seen that this last question is answered in the affirmative.

(ii) Can we assert anything concerning the order of Φ *qua* function of λ, the variables θ and α being supposed fixed? The same question may be asked concerning any Φ which is independent of the α's; it should be observed, moreover, that the best answer to the latter question does not necessarily give the best answer to the former.

[1] For shortness we shall write this $\Phi(k, m, \theta, \alpha, \lambda)$.

Our attempts to answer these questions have not been successful, and such results as we have been able to obtain are of a negative character. The question then arises as to whether we can obtain more definite results by imposing restrictions on the θ's or the α's, by supposing for example that all the α's are zero, or that the θ's belong to some special class of irrationals.

(iii) The relations (1.301) imply the truth of the following assertion: there is a function $\varphi(k, m, \theta, \alpha, n)$ which tends to infinity with n, and is such that

$$|(n^{\varkappa}\theta_p) - \alpha_{\varkappa p}| < 1/\varphi$$

for an infinity of values of n. A series of questions may then be asked concerning φ similar to those which we have stated with reference to Φ.

1.31. We shall begin by proving two theorems which are connected with the questions (i). The first of them deals with the case in which all the α's are zero, and it will be convenient to use in its statement, as in 1.10, not the function (x), but the allied function $\bar{x}$.

Theorem 1.31. *There is a function $\Phi(k, m, \lambda)$, depending only on k, m, and λ, such that*

$$|\overline{n^{\varkappa}\theta_p}| < 1/\lambda, \quad (\varkappa = 1, 2, \cdots k;\ p = 1, 2, \cdots m),$$

for some $n < \Phi$.

For suppose that this theorem is false. Then to every r corresponds a set of θ's, say ${}_r\theta_1, {}_r\theta_2, \cdots {}_r\theta_m$, such that the inequalities

$$(1.311) \qquad |\overline{n^{\varkappa}\,{}_r\theta_p}| < 1/\lambda$$

are not all true unless $n > r$. The set of points $({}_r\theta_1, {}_r\theta_2, \cdots {}_r\theta_m)$ has at least one limiting point $(\Theta_1, \Theta_2, \cdots \Theta_m)$, and by restricting ourselves to a subsequence of r's we can make

$${}_r\theta_p \to \Theta_p, \quad (p = 1, 2, \cdots m).$$

From this it follows that we can choose a number n_r which tends to infinity with r but so slowly that

$$(1.312) \qquad n_r^k |{}_r\theta_p - \Theta_p| < 1/2\lambda, \quad (p = 1, 2, \cdots m).$$

Clearly we may suppose that $n_r \leq r$, and so we have, for an infinity of values of r, $n_r \leq r$ and

$$(1.313) \qquad n^{\varkappa} |{}_r\theta_p - \Theta_p| < 1/2\lambda, \quad (n \leq n_r;\ \varkappa = 1, 2, \cdots k;\ p = 1, 2, \cdots m).$$

From (1.311) and (1.313) it follows that the inequalities

$$|\overline{n^\varkappa \theta_p}| < 1/2\lambda$$

cannot all be true unless $n > n_r$, and so, since $n_r \to \infty$, cannot be true for any value of n. This contradicts Theorem **1.011**.

In the case $k = 1$ it is possible to assert much more than this. It is known, and is proved in 1.10, that in this case we may take

$$\Phi = ([\lambda] + 1)^m \tag{1.314}$$

This problem, in fact, may be regarded as completely solved. When $k > 1$, however, the case is very different. We have not even succeeded in finding a definite function $\Phi(\lambda)$, the same for all θ's, such that

$$|\overline{n^2 \theta}| \leq 1/\lambda$$

for $n \leq \Phi$. It would be not unnatural to suppose that the »best possible» function[1] Φ is less than $K\lambda$, where K is an absolute constant. But we have been unable to prove this or indeed any definite result as to its order in λ.

1.32. **Theorem 1.32.** *If the θ's are linearly independent irrationals, it is possible to find a function $\Phi(k, m, \theta, \lambda)$, independent of the α's, such that*

$$|(n^\varkappa \theta_p) - \alpha_{\varkappa p}| < 1/\lambda, \quad (\varkappa = 1, 2, \cdots k; \; p = 1, 2, \cdots m)$$

for some $n < \Phi$.

That this theorem is true for the special case $k = 1$, $m = 1$, follows from the argument (*a*) in 1.11. It is easily proved in the most general case by an argument resembling, but simpler than, that of 1.31.

If the theorem is untrue, it is possible to find a sequence of sets $({}_r\alpha_{\varkappa p})$ $(r = 1, 2, \cdots)$ for which the inequalities of the theorem do not all hold unless $n > r$. The sequence of sets has at least one limiting set $(\bar{\alpha}_{\varkappa p})$: let us choose r so that

$$|{}_r\alpha_{\varkappa p} - \bar{\alpha}_{\varkappa p}| < 1/2\lambda, \quad (\varkappa = 1, 2, \cdots k; \; p = 1, 2, \cdots m).$$

Then clearly the inequalities

$$|(n^\varkappa \theta_p) - \bar{\alpha}_{\varkappa p}| < 1/2\lambda$$

cannot all be true unless $n > r$, and so, since r is arbitrarily large, cannot all be true for any n. This contradicts Theorem **1.011**.

[1] That is, the function which has, for each value of λ, the least possible value. For the existence of this function it is necessary that the sign $\leq$ above should not be replaced by $<$.

1.33. Let us consider more particularly the case in which $k = 1$.

The equation (1.314) suggests that it may in this case be possible to choose for Φ a function of the form

$$\Omega(m, \theta, \alpha) \lambda^m.$$

This we believe to be improbable, but we have not succeeded, even when $m = 1$, in obtaining a definite proof. What is certain is that no corresponding result is true of the Φ of Theorem **1.32**. It is impossible to choose a function $\Omega(m, \theta)$ independent of λ, and a function $\psi(m, \lambda)$ independent of the θ's, in such a way that the Φ of this theorem may be taken to be of the form

$$\Phi = \Omega(m, \theta) \psi(m, \lambda).$$

This is shown by the following theorem.[1]

Theorem 1.33. *Let $\psi(\lambda)$ be an arbitrary function of λ which tends steadily to infinity with λ. Then it is possible to find irrational numbers θ for which the assertion 'there is a function*

$$\Phi(\theta, \lambda) = \Omega(\theta) \psi(\lambda)$$

such that, when λ is chosen, the inequality

$$|(n\theta) - \alpha| < 1/\lambda$$

is satisfied, for every α, by some n less than Φ' is false.

Suppose that the assertion in question is true. Taking $\alpha = 1/\lambda$, we see that

(1.331) $$0 < (n\theta) < 2/\lambda$$

for some n less then Φ.

Let p_ν/q_ν be the ν-th convergent to the simple continued fraction

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

which represents θ, so that $p_1 = 1$, $q_1 = a_1$; and let us consider the system of 'intermediate convergents'

$$\frac{p_{2n,r}}{q_{2n,r}} = \frac{p_{2n} + r\, p_{2n+1}}{q_{2n} + r\, q_{2n+1}}, \quad (0 \leq r \leq a_{2n+2}),$$

[1] In proving a result of this negative character we may evidently confine ourselves to the special case in which $m = 1$.

intercalated between p_{2n}/q_{2n} and p_{2n+2}/q_{2n+2}. These fractions are all less than θ and increase with r. Also

$$(1.332) \qquad \theta - \frac{p_{2n,r}}{q_{2n,r}} = \frac{a'_{2n+2} - r}{q_{2n,r}\, q'_{2n+2}},$$

where a'_{2n+2} is the complete quotient corresponding to a_{2n+2}, and

$$q'_{2n+2} = a'_{2n+2}\, q_{2n+1} + q_{2n}.$$

Let

$$(1.333) \qquad \lambda_n = \frac{2\, q'_{2n+2}}{a'_{2n+2} - s},$$

where s is a particular value of r which we shall fix in a moment. We shall suppose a_{2n+2} large, and s also large, but small in comparison with a_{2n+2}. In these circumstances λ_n will be approximately equal to $2\, q_{2n+1}$.

We shall now prove that if

$$(1.334) \qquad 0 < (Q\theta) < 2/\lambda_n$$

then

$$(1.335) \qquad Q > q_{2n,s}.$$

From (1.334) it follows that there is a fraction P/Q such that

$$(1.336) \qquad 0 < \theta - \frac{P}{Q} < \frac{2}{\lambda_n Q}.$$

On the other hand

$$(1.337) \qquad \theta - \frac{p_{2n,s}}{q_{2n,s}} = \frac{2}{\lambda_n\, q_{2n,s}}$$

If P/Q actually gave a better approximation by defect to θ than $p_{2n,s}/q_{2n,s}$, it would follow at once that $Q > q_{2n,s}$. We may therefore suppose the contrary; and then it follows from (1.336) and (1.337) that

$$0 < \frac{p_{2n,s}}{q_{2n,s}} - \frac{P}{Q} < \frac{2}{\lambda_n Q},$$

Hence

$$0 < p_{2n,s}\, Q - q_{2n,s}\, P < 2\, q_{2n,s}/\lambda_n.$$

But

$$\frac{1}{2}\lambda_n = \frac{a'_{2n+2}\, q_{2n+1} + q_{2n}}{a'_{2n+2} - s} > q_{2n+1},$$

and

$$q_{2n,s} = q_{2n} + s q_{2n+1} < (s+1) q_{2n+1}.$$

Hence $p_{2n,s} Q - q_{2n,s} P$ is less than $s + 1$, and so

(1.338) $$p_{2n,s} Q - q_{2n,s} P = \varrho \quad (0 \leq \varrho \leq s).$$

On the other hand

$$p_{2n,s}\, q_{2n,s-\varrho} - q_{2n,s}\, p_{2n,s-\varrho} = \varrho;$$

and so

$$p_{2n,s}(Q - q_{2n,s-\varrho}) = q_{2n,s}(P - p_{2n,s-\varrho}).$$

Hence either $Q = q_{2n,s-\varrho}$, or $Q - q_{2n,s-\varrho}$ is divisible by $q_{2n,s}$; and the latter hypothesis plainly involves that $Q > q_{2n,s}$.

On the other hand, if $Q = q_{2n,s-\varrho}$, then $P = p_{2n,s-\varrho}$, and

$$\theta - \frac{P}{Q} = \frac{a'_{2n+2} - s + \varrho}{q_{2n,s-\varrho}\, q'_{2n+2}} \geq \frac{2}{\lambda_n\, q_{2n,s-\varrho}},$$

$$(Q\theta) \geq 2/\lambda_n,$$

which contradicts (1.337). Hence in any case $Q > q_{2n,s}$.

It is now easy to complete the proof of the theorem. We have *a fortiori* $Q > s$. Also, if a_{2n+2} is large, and s large, but small in comparison with a_{2n+2}, λ_n will clearly be less than $4 q_{2n+1}$. We may suppose for definiteness that $s = [\sqrt{a_{2n+2}}]$.

We choose a value of θ such that the inequality

$$a_{2n+2} > \{\psi(4\, q_{2n+1})\}^4$$

is satisfied for an infinity of values of n. Then

$$s > \frac{1}{2}\{\psi(4\, q_{2n+1})\}^2.$$

But if Q, and *a fortiori* s, is less than Φ, we must have

$$\frac{1}{2}\{\psi(4\, q_{2n+1})\}^2 < \Omega(\theta)\, \psi(\lambda_n) < \Omega(\theta)\, \psi(4\, q_{2n+1});$$

and this is obviously impossible when n is sufficiently large. This completes the proof of the theorem.

It should be observed that the success of our argument depends entirely on our initial choice of α in such a way that $(n\theta)$ is small. It would not be enough that $\overline{n\theta}$ should be small, that is to say that $(n\theta)$ should be nearly equal to either 0 or 1: this can of course be secured by choice of an n less than Φ, Φ being indeed independent of θ.

1.34. We turn now for a moment to the questions concerning φ. If we have found a function $\Phi(\lambda)$ which is continuous and monotonic, the inverse function is plainly a φ. The converse, however, is not true, and we cannot, from the existence of a φ of given form, draw any conclusion as to the order of Φ for *all* values of λ. This is clear from the fact that, to put it roughly, the existence of φ asserts an inequality which need only hold very occasionally, and which therefore gives us information as to the behaviour of Φ only for occasional values of λ. Thus the existence of a Φ asserts much more than that of the corresponding φ. Since moreover it will appear (in the third paper of the series) that in applications of the present theory it is always the properties of Φ, and not those of φ, which are relevant, we are justified in regarding theorems concerning φ as of rather minor importance. There are, however, one or two results which are worth noticing, and which are not deductions from the corresponding results concerning Φ. It should be observed that whereas we wish Φ to increase as slowly as possible, we wish φ to increase as rapidly as possible.

Theorem 1.340. *It is possible to choose the α's so that $\varphi(m, \theta, \alpha, n)$ increases with arbitrary rapidity. Moreover the α's may be chosen in an arbitrarily small neighbourhood of any set $(\alpha_1, \alpha_2, \cdots \alpha_m)$.*

We omit the proof of this theorem, which is easy.

Theorem 1.341. *If $k = 1$ and $m = 1$, then, provided only that θ is irrational, we may take*

$$\varphi(n) = \frac{1}{3} n$$

(a function independent of both θ and α).

This follows at once from the argument (a) of 1.11. It is natural to suppose that, when $m > 1$, we may take

$$\varphi(n) = \omega(m) \sqrt[m]{n},$$

where $\omega(m)$ depends only on m. But this we have not been able to prove.

A comparison of Theorems **1.33** and **1.341** shows very clearly the difference between theorems involving Φ and those involving φ, and the greater depth and difficulty of the former.

1.35. Theorem **1.33** shows that it is hopeless to expect any such simple result concerning Φ as is asserted concerning φ in Theorem **1.341**. It is however possible to obtain theorems which involve Φ and correspond to Theorem **1.341**, if we suppose that certain classes of irrationals (as well as the rationals) are excluded from the range of variation of θ. In the two theorems which follow it is supposed that $m = 1$ and $k = 1$.

Theorem 1.350. *Let θ be confined to the class of irrationals whose partial quotients are limited, a set which is everywhere dense. Then we may take*

$$\Phi = \lambda \Omega(\theta).$$

Theorem 1.351. *Let θ be confined to the class of irrationals whose partial quotients a_n satisfy, from a certain value of n onwards, the inequality*

$$a_n < n^{1+\delta} \ (\delta > 0).$$

Then we may take,

$$\Phi = \lambda (\log \lambda)^{1+\delta'} \Omega(\theta)$$

where δ' is any number greater than δ.

The interest of the last theorem lies in the fact that the set in question is of measure 1,[1] so that we may take Φ to be of the form $\lambda (\log \lambda)^{1+\epsilon} \Omega(\theta)$,[2] where ϵ is an arbitrarily small positive number, for *almost all* values of θ.

The proofs of these theorems are simple and depend merely on an adaptation of KRONECKER's argument reproduced in 1.11. Suppose first that the partial quotients of θ are limited. We can choose H so that, when λ is assigned, there is always a denominator q_m of a convergent to θ such that

$$2\lambda \leq q_m < H\lambda \tag{1.350}$$

We take $q = q_m$. It follows from KRONECKER's argument that there is for any α a number ν such that

$$|(\nu\theta) - \alpha| < 2/q, \quad q/2 < \nu < 3q/2,$$

and so

$$|(\nu\theta) - \alpha| < 1/\lambda$$

for some ν less than a constant multiple of λ.

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[1] By a theorem of BOREL and BERNSTEIN. See BOREL, *Rendiconti di Palermo*, vol. 27, p. 247, and *Math. Ann.*, vol. 72, p. 578; BERNSTEIN, *Math. Ann.*, vol. 71, p. 417.

[2] It is not difficult to replace $\lambda(\log\lambda)^{1+\epsilon}$ by $\lambda \log \lambda (\log\log\lambda)^{1+\epsilon}$, or by the corresponding but more complicated functions of the logarithmic scale.

The proof of Theorem **1.351** is very similar. We suppose that

$$q_{m-1} \leq 2\lambda < q_m,$$

and so

$$q_m/m^{1+\delta} \leq 2\lambda < q_m.$$

There is a constant ϱ such that $q_m > e^{\varrho m}$; and from these facts it follows easily that

$$q_m < \lambda (\log \lambda)^{1+\delta'}$$

for sufficiently large values of λ. The proof may now be completed in the same manner as that of Theorem **1.350**.

It is natural to suppose that these theorems have analogues when $m > 1$. But our arguments, depending as they do on the theory of continued fractions, do not appear to be capable of extension.

1.4. — The general sequence $(f(n)\theta)$ and the particular sequence $(a^n\theta)$

1.40. We return now to the general sequence $(f(n)\theta)$: it will be convenient to write λ_n for $f(n)$. We suppose then that (λ_n) is an arbitrary increasing sequence of numbers whose limit is infinity.[1]

It would be natural to attempt to prove that, if θ is irrational and α is any number such that $0 \leq \alpha < 1$, a sequence (n_r) can be found such that

$$(\lambda_{n_r}\theta) \rightarrow \alpha;$$

but we saw in 1.00 that this statement is certainly false, for example when $\lambda_n = 2^n$ or $\lambda_n = n!$

The result which is in fact true was suggested to us by a theorem of Bernstein,[2] which runs as follows:

If λ_n is always an integer, then the set of values of θ for which

$$(\lambda_n\theta) \rightarrow 0$$

is of measure zero.

This result, when considered in conjunction with what we have already proved, at once suggests the following theorem.

[1] In the introductory remarks of 1.00 we stated our main problem subject to the restriction that λ_n is an integer. No such restriction, however, is required in what follows.

[2] F. Bernstein, *loc. cit.*

Theorem 1.40. *The set of values of θ, for which the set of points $(\lambda_n \theta)$ is not everywhere dense in the interval* (0, 1), *is of measure zero.*

In other words, the main question asked in 1.00 may be answered affirmatively if we make exception of a set of measure zero.

1.41. The proof will be based upon the following lemma.

Lemma 1.41. *Suppose that a finite number of intervals are excluded from the continuum* (0, 1), *and that the length of the remainder S is l. Let a be any number betwen* 0 *and* 1, *and consider the set T of $[\lambda]$ intervals of length δ/λ $(\delta < 1)$ whose centres are at the points*

$$\frac{a}{\lambda}, \frac{a+1}{\lambda}, \ldots\ldots, \frac{a+[\lambda]-1}{\lambda}.^{1}$$

Then the length of the common part of S and T is

$$\delta l + \varepsilon_\lambda,$$

where $\varepsilon_\lambda \to 0$ as $\lambda \to \infty$.

The truth of the lemma is almost obvious. A formal proof may be given as follows. Let the lengths of the intervals excluded from S be $l_1, l_2, \ldots, l_p$. If now we extend each of these intervals a distance $1/2\lambda$ at each end,[2] we obtain a system of p intervals of length

$$l'_s = l_s + \frac{1}{\lambda}, \quad (s = 1, 2, \cdots p).$$

We denote what is left of (0, 1) by S'.

If $(a+r)/\lambda$ falls in S', the whole of the corresponding interval of T falls in S. Hence the part of S inside T has a length not less than $\mu\delta/\lambda$, where μ is the number of points $(a+r)/\lambda$ in S'. If $\nu_1, \nu_2, \ldots, \nu_p$ are the numbers of these points which fall in the intervals excluded from S', we have

$$\mu + \Sigma \nu_s = [\lambda], \quad (\nu_s - 1)/\lambda \leq l'_s;$$

and so

$$\begin{aligned}\mu = [\lambda] - \Sigma \nu_s &> \lambda - 1 - p - \lambda \Sigma l'_s \\ &= \lambda - p - 1 - \lambda \Sigma \left(l_s + \frac{1}{\lambda}\right) \\ &= l\lambda - 2p - 1,\end{aligned}$$

[1] It is of course to be understood that an interval, or a part of an interval, which falls outside (0, 1), is to be replaced by the congruent interval inside.

[2] We suppose λ large enough to ensure that this extension does not cause any overlapping. If any part of an extended interval should fall outside (0, 1), as will happen if an interval contains 0 or 1, we of course replace this part by the congruent part of (0, 1).

since $\Sigma l_s = 1 - l$. Hence the length in question is greater than

$$\delta l - \frac{(2p+1)\delta}{\lambda}.$$

A similar argument, which we may leave to the reader, furnishes a corresponding upper limit for the length; and the lemma follows. It is plain that $\varepsilon_\lambda = O(1/\lambda)$.

1.42. We can now prove the following theorem, which is a generalisation of Bernstein's, but is itself contained in Theorem 1.40.

Theorem 1.42. *If* I *is any interval contained in* (0, 1), *the set* Θ *of points* θ *such that no one of the points* $(\lambda_n \theta)$ *falls inside* I, *is of measure zero.*

Let a be the centre of I and δ its length; and let T_m be the set T of the lemma, with $\lambda = \lambda_m$. If, for any value of m, θ falls in T_m, then $(\lambda_m \theta)$ falls in I, and so θ belongs to the set complementary to Θ.

Let

$$S_n = T_1 + T_2 + \cdots + T_n,$$

and let l_n be the length of S_n. Finally let $l_n \to l$ as $n \to \infty$. We have to show that $l = 1$.

We now apply the lemma, taking S to be the set $\overline{S}_n$ complementary to S_n, and T to be T_m. If m is large enough, the length of the common part $(\overline{S}_n, T_m)$ of $\overline{S}_n$ and T_m is greater than

$$\delta(1 - l_n) - \varepsilon.$$

Any point which belongs either to this set or to S_n itself belongs to *some* S_ν. Hence

$$l > l_n + \delta(1 - l_n) - \varepsilon;$$

and so

$$l \geq l + \delta(1 - l) - \varepsilon:$$

which is impossible unless $l = 1$.

1.43. We can now complete the proof of Theorem 1.40. Let E_n be the set of values of θ such that some one of the intervals

$$\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right), \ldots\ldots, \left(\frac{n-1}{n}, 1\right)$$

contains no point $(\lambda_n \theta)$. Then E_n is of measure zero, and so

$$E = E_1 + E_2 + E_3 + \dots\dots$$

is of measure zero.

If now the set $(\lambda_m \theta)$ is not everywhere dense in $(0, 1)$, there is an interval i which contains no $(\lambda_m \theta)$. We can choose n so that some interval $\left(\frac{r}{n}, \frac{r+1}{n}\right)$ falls inside i. Then θ belongs to E_n and so to E. Thus the theorem is established.

1.44. Perhaps the most interesting special sequence falling under the general type $(f(n)\theta)$ is that in which $f(n) = a^n$, where a is a positive integer. When θ is expressed as a decimal in the scale of a, the effect of multiplication by a is merely to displace the digits. To study the properties of the sequence $(a^n \theta)$ is therefore equivalent to studying the distribution of the digits in the expression of θ in the scale of a: it is to this fact that this form of $f(n)$ owes its peculiar interest.

Let b be one of the possible digits $0, 1, 2, \dots, a-1$, and let $p(n, m)$ denote the number of decimals of n figures whose digits include exactly m b's. Then

$$p(n, m) = \frac{n!}{m!\,(n-m)!}(a-1)^{n-m}. \tag{1.441}$$

We write

$$\mu = m - \frac{n}{a}; \tag{1.442}$$

so that μ is the excess of the number of b's above the average.

We shall base our investigation on a series of lemmas.

Lemma 1.441. *Given any positive number δ, we can find a positive number ε such that*

$$p(n, m) < \frac{a+\delta}{\sqrt{2\pi(a-1)n}} e^{-(\alpha-\delta)\mu^2/n} a^n \tag{1.443}$$

where

$$\alpha = \frac{a^2}{2(a-1)},$$

for $|\mu| < \varepsilon n$ and all sufficiently large values of n.

We omit the proof of this lemma, which depends merely on a straightforward application of STIRLING's Theorem.

Lemma 1.442. *Given any positive number ε, we can find a positive number ζ such that*

$$p(n, m) < a^n e^{-\zeta n}$$

for $|\mu| \geq \varepsilon n$ *and all sufficiently large values of* n.

Suppose, *e. g.*, $\mu > \frac{1}{2}\varepsilon n$. Then

$$\frac{p(n, m+1)}{p(n, m)} = \frac{n-m}{(a-1)(m+1)} < \frac{a - 1 - \frac{1}{2}a\varepsilon}{(a-1)(1 + \frac{1}{2}a\varepsilon)} < 1;$$

and from this it is easy to deduce the truth of the lemma when $\mu > \varepsilon n$. A similar proof applies when $\mu < -\varepsilon n$.

Lemma 1.443. *Let c be a positive constant. Then*

(1.4431)
$$\overline{\lim_{n \to \infty}}\, a^{-n} \sum_{|\mu| < c\sqrt{n}} p(n, m) < 1,$$

(1.4432)
$$\overline{\lim_{n \to \infty}}\, a^{-n} \sum_{\mu > -c\sqrt{n}} p(n, m) < 1,$$

(1.4433)
$$\overline{\lim_{n \to \infty}}\, a^{-n} \sum_{\mu < c\sqrt{n}} p(n, m) < 1.$$

Of these three inequalities the first is plainly a consequence of either the second or third. It will be enough to prove the second.

We have

$$a^{-n} \sum_{\mu > -c\sqrt{n}} p(n, m) = a^{-n} \sum_{-c\sqrt{n}}^{\varepsilon n} + a^{-n} \sum_{\varepsilon n}^{(a-1)n/a} = S_1 + S_2,$$

say. By Lemma 1.442,

$$S_2 < \frac{(a-1)n}{a} e^{-\zeta n} \to 0.$$

And by Lemma 1.441,

$$S_1 < \frac{a + \delta}{\sqrt{2\pi(a-1)n}} \sum_{-c\sqrt{n}}^{\varepsilon n} e^{-(a-\delta)\mu^2/n}$$

$$< \frac{a+\delta}{\sqrt{2\pi(a-1)n}}\left\{2+\int_{-c\sqrt{n}}^{\infty} e^{-(a-\delta)\mu^2/n}\,d\mu\right\}.$$

The term of order $1/\sqrt{n}$ may be ignored. The remainder is less than

$$\frac{a+\delta}{\sqrt{2\pi(a-1)}}\int_{-c}^{\infty} e^{-(a-\delta)\xi^2}\,d\xi,$$

which is less than 1. Thus the lemma is proved. In a similar manner we can prove

Lemma 1.444. *If ν is a function of n such that $\nu/\sqrt{n}\to\infty$, then*

$$a^{-n}\sum_{|\mu|<\nu} p(n,m) < K\left\{\frac{\sqrt{n}}{\nu}e^{-(a-\delta)\nu^2/n}+a^{-n}\right\},$$

where K depends only on a.

1.45. We are now in a position to prove our main theorems. We observe first that all irrational[1] numbers θ between 0 and 1, whose decimals have just m b's in their first n figures, may be included in a set of intervals whose total length is

$$a^{-n}p(n,m).$$

For let $\theta_1, \theta_2, \ldots, \theta_q$, where $q=a^n$, denote the terminating decimals of n figures. The set of intervals $(\theta_r, \theta_r+a^{-n})$ just fills up the whole interval (0, 1). Among the numbers θ_r there are $p(n,m)$ which have just m b's, which we may call $\xi_1, \xi_2, \ldots, \xi_p$; and the set of intervals (ξ_s, ξ_s+a^{-n}) fulfils our requirements.

Theorem 1.45. *Let δ be any positive number. Then the set of numbers θ for which*

$$\varlimsup_{n\to\infty}\frac{|\mu|}{\sqrt{n\log n}}<\sqrt{\frac{1}{\alpha}}+\delta$$

is of measure 1.

Let $\bar{S}$ denote the complementary set. Any number belonging to $\bar{S}$ satisfies

$$\mu>\left(\sqrt{\frac{1}{\alpha}}+\delta'\right)\sqrt{n\log n}=\nu_n$$

for an infinity of values of n, δ' being any positive number less than δ.

[1] The end points of the intervals will be *rational* numbers satisfying the condition. In what follows we may confine ourselves to irrational values of θ, since the rational values form in any case a set of measure zero.

All θ's for which this inequality is true for a particular n may be enclosed in a set of intervals whose total length is

$$a^{-n} \sum_{|\mu| < \nu_n} p(n, m). \tag{1.451}$$

We can choose a positive number δ'' such that

$$2\delta' \sqrt{a} - \frac{\delta''}{a} - \frac{2\delta'\delta''}{\sqrt{a}} > 0,$$

and then choose n_1 so that the expression (1.451) is less than

$$K\left\{\frac{\sqrt{n}}{\nu_n} e^{-(a-\delta'')\nu_n^2/n} + a^{-n}\right\}$$

for $n \geq n_1$. To prove the theorem it is enough to show that the result of summing this expression for $n = n_1, n_1 + 1, \ldots\ldots$ can be made as small as we please by choice of n_1; and it is obvious that this conclusion cannot be affected by the presence of the term a^{-n}. But

$$\frac{\sqrt{n}}{\nu_n} e^{-(a-\delta'')\nu_n^2/n} < e^{-(a-\delta'')\left(\frac{1}{\sqrt{a}}+\delta'\right)^2 \log n}$$

$$= n^{-1-\delta'''},$$

where

$$\delta''' > (a - \delta'')\left(\frac{1}{a} + \frac{2\delta'}{\sqrt{a}}\right) - 1$$

$$= 2\delta' \sqrt{a} - \frac{\delta''}{a} - \frac{2\delta'\delta''}{\sqrt{a}} > 0;$$

and plainly

$$\sum_{n_1}^{\infty} n^{-1-\delta'''}$$

can be made as small as we please by choice of n_1. Thus the theorem is proved.

Theorem 1.45 includes as a particular case

Theorem 1.451. *If n_b is the number of b's in the first n figures of the expression of θ as a decimal in the scale of a, then*

$$n_b \sim n/a$$

for almost all values of θ.

1.46. Theorem **1.45** shows that the deviation, from the average n/a, of the number of occurrences of a particular figure b in the first n places, is not in general of an order materially greater than $\sqrt{n}$.[1] If we were to suppose that there was a *steady* deviation from the average (instead of a merely occasional deviation), we would naturally obtain a more precise result. Thus reasoning analogous to, but simpler than, that which led to theorem 1.45, leads also to

Theorem 1.46. *If $\varphi(n) \to \infty$ with n, then the set of θ's for which*

$$|\mu(n)|/\sqrt{n}\,\varphi(n) \to \infty$$

is of measure zero.

This theorem, however, is included in a much more interesting and general theorem which we shall now proceed to prove, which, to put it roughly, assigns a *lower* limit for the deviation in either direction.

1.47. **Theorem 1.47.** *If c is any positive constant, the set of θ's for which*

$$\mu(n) > -c\sqrt{n},$$

and the set for which $\mu(n) < c\sqrt{n}$, are of measure zero.

Let

$$c_0 = c\prod_{m=1}^{\infty}\left(1 + \frac{1}{2^m}\right).$$

By Lemma **1.443**, there is a positive number δ_{c_0} such that

$$\overline{\lim_{n\to\infty}}\, a^{-n}\sum_{\mu > -c_0\sqrt{n}} p(n, m) = 1 - \delta_{c_0}.$$

And if $c \leq c_1 < c_0$, it is clear that

$$\overline{\lim_{n\to\infty}}\, a^{-n}\sum_{\mu > -c_1\sqrt{n}} p(n, m) = 1 - \delta_{c_1},$$

where

$$\delta_c \geq \delta_{c_1} \geq \delta_{c_0}.$$

Let E_c be the set of the theorem. We can enclose E_c in a set of intervals of total length

[1] It follows from the elements of the theory of errors that the 'most probable error' is of order $\sqrt{n}$.

$$a^{-n_1} \sum_{\mu > -c\sqrt{n_1}} p(n_1, m) < 1 - \frac{1}{2}\delta_c.$$

Consider now any one of the

$$N = \Sigma\, p(n, m)$$

intervals of this set, each of which is of length a^{-n_1}; and let $\xi = (a^{n_1}\theta)$. As θ ranges in the interval in question, ξ ranges in the whole interval $(0, 1)$.

If θ belongs to E_c, the corresponding ξ has the property that

$$\mu(n') > -c\sqrt{n_1 + n'}$$

for all values of n'; and so, if n' is large enough compared with n_1,

$$\mu(n') > -c'\sqrt{n'},$$

where

$$c' = c(1 + 2^{-n_1-1}).$$

We may now enclose the ξ's in a set of intervals whose total length is less than

$$1 - \frac{1}{2}\delta_{c'};$$

and therefore we may enclose the θ's which lie in the particular interval under consideration in a set of intervals whose total length is less than $a^{-n_1}(1 - \frac{1}{2}\delta_{c'})$. If we do this for each of the N intervals, we have enclosed the θ's in a set of intervals of length less than

$$\left(1 - \frac{1}{2}\delta_c\right)\left(1 - \frac{1}{2}\delta_{c'}\right).$$

Repeating this argument, it is clear that we can enclose the θ's in a set of intervals of total length less than

$$\left(1 - \frac{1}{2}\delta_c\right)\left(1 - \frac{1}{2}\delta_{c'}\right)\left(1 - \frac{1}{2}\delta_{c''}\right)\cdots\left(1 - \frac{1}{2}\delta_{c^{(\nu)}}\right),$$

where

$$c^{(\nu)} = c(1 + 2^{-n_1-1})(1 + 2^{-n_2-1})\ldots(1 + 2^{-n_\nu-1}),$$

the indices n_ν being integers which tend to infinity with ν, as rapidly as we please. Plainly $c^{(\nu)} < c_0$ and so

$$\delta_{c^{(\nu)}} \geq \delta_{c_0},\ 1 - \frac{1}{2}\,\delta_{c^{(\nu)}} \leq 1 - \frac{1}{2}\,\delta_{c_0},$$

$$\left(1 - \frac{1}{2}\,\delta_c\right)\left(1 - \frac{1}{2}\,\delta_{c'}\right)\cdots\left(1 - \frac{1}{2}\,\delta_{c^{(\nu)}}\right) \leq \left(1 - \frac{1}{2}\,\delta_{c_0}\right)^{\nu+1}.$$

As this tends to zero as $\nu \to \infty$, our theorem is proved.

From Theorem **1.47** we can at once deduce

Theorem 1.471. *The set of θ's such that to each θ corresponds a c for which $\mu(n) > -c\sqrt{n}$ is of measure zero.*

Let E_c denote the set of Theorem **1.47**. The set of this theorem is plainly the sum of the sets $E_1, E_2, E_3, \ldots$; and so is of measure zero.

1.48. So far we have considered merely the occurrence of a particular digit b in the decimal which represents θ. But our results are easily extended so as to give analogous information concerning the occurrence of any combination of digits. The method by which this extension is effected is quite simple in principle, and it will be sufficient to show its working in a special case.

Consider the succession 317 of digits, in the scale of 10. In the scale of 1000, the number 317 corresponds to a single digit τ; and, if θ is expressed in the scale of 1000, it will, by theorem **1.451**, be almost always true that the number n_τ of occurrences of τ, among the first n figures, satisfies the relation

$$n_\tau \backsim \frac{n}{1000}.$$

Now the combination 317, in the expression of θ in the scale of 10, will occur when, and only when, the digit τ occurs in the expression of one or other of the three numbers

$$\theta,\ 10\,\theta,\ 100\,\theta$$

in the scale of 1000. Hence it is almost always true that the number of occurrences of the combination 317, in the first n digits of the expression of θ in the scale of 10, is asymptotically equivalent to

$$\frac{n}{1000}.$$

We may now, without further preface, enunciate the following theorems.

Theorem 1.48. *It is almost always true that, when a number θ is expressed in any scale of notation, the number of occurrences of any digit, or any combination of digits, is asymptotically equivalent to the average number which might be expected.*

Theorem 1.481. *It is almost always true that the deviation from the average, in the first n places, is not of order exceeding $\sqrt{n \log n}$.*

Theorem 1.482. *It is almost always true that the deviation, in both directions, is sometimes of order exceeding $\sqrt{n}$.*

Theorem 1.483. *The number of the first n numbers $(a^{\nu}\theta)$ which fall inside an interval of length δ included in the interval* (0, 1) *is almost always asymptotically equivalent to δn.*

The last theorem is merely a translation of theorem **1.48** into different language, and a corresponding form may of course be given to theorems **1.481** and **1.482**.

1.49. Throughout this section (1.4) we have confined ourselves to results concerning a single irrational θ. Some of our theorems, however, have obvious many-dimensional analogues. It will be sufficient, for the present, to mention the following, which are generalisations of Theorems **1.40** and **1.483** respectively.

The interval (0, 1) is now replaced by an m-dimensional 'square'.

Theorem 1.49. *The set of values $(\theta_1, \theta_2 \cdots \theta_m)$, for which the points $(\lambda_n \theta_1, \lambda_n \theta_2, \cdots \lambda_n \theta_m)$ are not everywhere dense in the square, is of measure zero.*

Theorem 1.491. *The number of the first n points $(a^{\nu}\theta_1, a^{\nu}\theta_2, \cdots a^{\nu}\theta_m)$, which fall inside a portion of the square, of area δ, is almost always asymptotically equivalent to δn.*

We leave the proofs to the reader. The first theorem may be proved by an obvious adaptation of the proof of Theorem **1.40**, and the second deduced from Theorem **1.483** by a process of correlation very similar to that employed n 1.48.

CORRECTIONS

p. 175. The last displayed formula but one should read:

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots.$$

p. 177. The parenthesis in (1.338) should read: $(0 \leqslant \rho \leqslant s)$.

p. 185, Lemma 1.444. Under the sign of summation, read: $|\mu| > \nu$.

p. 185, last formula. Read: $|\mu|$.

p. 186, (1.451). Read: $|\mu| > \nu_n$.

p. 188. In the second displayed formula, read n_1 for n.

COMMENTS

§ 1.00. A simpler notation, which has come into use since the time of this paper, renders unnecessary the use of fractional parts and the special convention when $\alpha = 0$. Instead of writing

$$-\epsilon < (n_r\theta)-\alpha < \epsilon$$

one now writes

$$-\epsilon < n_r\theta-\alpha < \epsilon \pmod 1, \quad \text{or} \quad |n_r\theta-\alpha| < \epsilon \pmod 1.$$

§ 1.01. As mentioned in the footnote on p. 157, the form of Kronecker's theorem which is stated by Hardy and Littlewood as Theorem 1.01, and generalized by them in Theorem 1.011, is only a particular case of the original theorem of Kronecker (though the general case is deducible from it). The original theorem gave necessary and sufficient conditions for a system of linear equations (not generally homogeneous) to have an arbitrarily good approximate solution in integers. The condition in Theorem 1.01 that $\theta_1,\ldots,\theta_m$ shall be linearly independent irrationals is a particular case of Kronecker's condition on the *arithmetical rank* of a system of linear equations. It is customary now to express the condition in the form: $1, \theta_1,\ldots,\theta_m$ are linearly independent over the rationals.

A good account of Kronecker's theorem, in the form considered by Hardy and Littlewood, is given in ch. 23 of Hardy and Wright. Here several later proofs are given and their merits compared. This account supersedes most of § 1.1 of the present paper.

Theorem 1.011 implies the existence for any $\epsilon > 0$ (under the conditions stated) of infinitely many integers n such that

$$|n^K\theta_p-\alpha_{K,p}| < \epsilon \pmod 1 \quad \text{for } 1 \leqslant p \leqslant m,\ 1 \leqslant K \leqslant k.$$

P. Szüsz (*Acta Math. Acad. Sci. Hungaricae*, 4 (1953), 115–18) has proved that there exists a number $M(\theta_1,\ldots,\theta_m; k)$ such that there is an integer n with the property in every interval $(t, t+M)$.

§ 1.14 (*a*). We have here an early result on one-sided, or asymmetrical, Diophantine approximation. It will be seen that Hardy and Littlewood realized the important distinction (in the case of homogeneous approximation, when the α_μ and β_μ are zero) between approximation from one side and from both sides. For an investigation into this question, see C. A. Rogers, *Proc. London Math. Soc.* (2), 52 (1951), 186–90.

§ 1.14 (*b*). The two theorems stated here are early results on uniform distribution (mod 1). In the first, it is to be understood that $1, \theta, \phi$ are linearly independent, as in Kronecker's theorem. The second result includes as a particular case the uniform distribution of $\nu\theta \pmod 1$ for any irrational θ. This seems to have been first proved by P. Bohl in 1909 in work on perturbation problems, and independently by Sierpiński and Weyl at about the same time (see Koksma, p. 92).

In his important memoir of 1916 (*Math. Annalen*, 77 (1916), 313–52) Weyl gave his criterion for uniform distribution in terms of exponential sums, and this provided a powerful technique for proving the uniformity of distribution of many sequences. On the relationship between this memoir and the work of Hardy and Littlewood, see 1916, 9. The method of exponential sums also enabled Weyl to prove a stronger form of Theorem 1.011 asserting simultaneous uniform distribution; see Koksma, p. 93, Satz 10.

The Bohr–Littlewood Cambridge tract, announced in footnote 2 on p. 164, never appeared. But the Bohr–Littlewood manuscript formed the basis for Titchmarsh's tract on the Riemann zeta-function and (to some extent) for Ingham's tract on the distribution of primes. Titchmarsh's tract developed later into *The theory of the Riemann zeta-function* (Oxford, 1951), and this contains the applications of Kronecker's theorem to the zeta-function mentioned in footnote 1 on p. 158 of the present paper.

More recently, Kronecker's theorem was the starting-point for an important body of work by Turán (see his book: *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest, 1953). If $1, \theta_1, \ldots, \theta_m$ are linearly independent, Kronecker's theorem asserts that for any real $\alpha_1, \ldots, \alpha_m$ and any $\epsilon > 0$ there exist integers ν for which

$$\left|1+\sum_{j=1}^{m} e^{i(\nu\theta_j-\alpha_j)}\right| > m+1-\epsilon.$$

But it is impossible to give any bound for ν, or to name any interval in which some ν must fall, unless the hypothesis is strengthened by assuming some quantitative measure for the degree of linear independence of $\theta_1, \ldots, \theta_m$. The general idea of Turán's work is that if one is content with a much weaker inequality, it is possible (under reasonable conditions) to give good limits for ν and (what is equally important) to omit the requirement of linear independence. The work has applications to the zeta-function and to a variety of questions in analysis and in analytic number theory.

§ 1.31. The specific problem (p. 174) of finding a function $\Phi(\lambda)$ such that for any θ there is a solution of

$$|n^2\theta| \leqslant \lambda^{-1} \pmod 1$$

with $n \leqslant \Phi(\lambda)$ has been answered most effectively to date by Heilbronn (*Quart. J. of Math.* 19 (1948), 249–56). He proved that if λ is large (as one may suppose) the result holds with $\Phi(\lambda) = \lambda^{2+\delta}$, where δ is any fixed positive number. A similar and more general result, but one that is less precise for this particular problem, had been found by Vinogradov in 1927 (see Koksma, p. 119). The conjecture, described by Hardy and Littlewood as 'not unnatural', that perhaps one could take $\Phi(\lambda)$ to be proportional to λ, is, however, false. For take $\theta = a/p$, where p is a large prime of the form $4x+1$ and a is a quadratic non-residue $(\bmod\, p)$. On choosing λ so that $\Phi(\lambda) = p-1$, we should deduce from the conjecture that there is a quadratic non-residue $(\bmod\, p)$ which is bounded independently of p, and this is obviously false.

§ 1.33. The conjecture at the beginning of this section, which Hardy and Littlewood say they believe to be improbable, is in fact false. This is a particular consequence of results found later by Blichfeldt (see Koksma, p. 86, Satz 6).

§ 1.34. There were earlier results, not mentioned by Hardy and Littlewood, of the same character as Theorem 1.341. Thus Chebyshev proved in 1866 that there are infinitely many $n > 0$ for which

$$|n\theta-\alpha| < 2/n \pmod 1 \tag{1}$$

(Koksma, footnote on p. 76). This is equivalent to $\phi(n) = \frac{1}{2}n$ in the present notation. Minkowski proved that provided α is not of the form $n_1\theta - m_1$, there are infinitely many integers n for which

$$|n\theta-\alpha| < \frac{1}{4|n|} \pmod 1;$$

here n is not restricted to positive values. This result is best possible of its kind. The best possible result when n is restricted to positive values was found by Cassels (*Math. Annalen*, 127 (1954), 288–304); the constant which replaces 2 in (1) above is $\frac{27}{28\sqrt{7}}$, and the proof is difficult.

The conjecture concerning $\phi(n)$ when $m > 1$ is false, as follows from the results of Blichfeldt mentioned above. There is, in fact, a vital difference in problems of non-homogeneous approximation between the one-dimensional case and the many-dimensional case.

§ 1.4. This section is an early contribution to the subject now called 'metrical' Diophantine approximation. Almost the only previous papers were those of Borel and Bernstein quoted on p. 179. For a recent account of the subject, see Cassels's *Tract*, ch. 7. The later part of the section, beginning with § 1.44, is concerned with 'normal decimals', and on this subject the reader should consult ch. 9 of Hardy and Wright.

The notion of an exceptional set of Lebesgue measure zero can be refined by considering the Hausdorff measure of the exceptional set. See, for example, Eggleston, *Proc. London Math. Soc.* (2), 54 (1952), 42–93.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION.

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II.

The trigonometrical series associated with the elliptic ϑ-functions.

2. 0. — Introduction.

2. 00. The series

$$2\sum_{1}^{\infty} q^{\left(n-\frac{1}{2}\right)^2}, \quad 1+2\sum_{1}^{\infty} q^{n^2}, \quad 1+2\sum_{1}^{\infty}(-1)^n q^{n^2},$$

where $q=e^{\pi i\tau}$, are convergent when the imaginary part of τ is positive, and represent the elliptic ϑ-functions

$$\vartheta_2(0, \tau), \quad \vartheta_3(0, \tau), \quad \vartheta_4(0, \tau).^1$$

When τ is a real number x, the series become oscillating trigonometrical series which, if we neglect the factor 2 and the first terms of the second and third series, may be written in the forms

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[1] The notation is that of TANNERY and MOLK's *Théorie des fonctions elliptiques*. We shall refer to this book as *T. and M.*

$$\sum e^{\left(n-\frac{1}{2}\right)^2 \pi i x}, \quad \sum e^{n^2 \pi i x}, \quad \sum (-1)^n e^{n^2 \pi i x}.$$

These series, the real trigonometrical series formed by taking their real or imaginary parts, and the series derived from them by the introduction of convergence factors, possess many remarkable and interesting properties. It was the desire to elucidate these properties which originally suggested the researches whose results are contained in this series of papers, and it is to their study that the present paper is devoted.[1]

2. 01. We shall write

$$(2.011) \qquad s_n^2 = \sum_{\nu \leq n} e^{\left(\nu-\frac{1}{2}\right)^2 \pi i x}, \quad s_n^3 = \sum_{\nu \leq n} e^{\nu^2 \pi i x}, \quad s_n^4 = \sum_{\nu \leq n} (-1)^\nu e^{\nu^2 \pi i x}.$$

It is obvious that, if s_n is any one of s_n^2, s_n^3, s_n^4, then

$$(2.012) \qquad s_n = O(n).$$

Our object is to obtain more precise information about s_n; and we shall begin by a few remarks about the case in which x is rational. In this case s_n is always of one or other of the forms

$$O(1), \quad An + O(1),$$

where A is a constant. It is not difficult to discriminate between the different cases; it will be sufficient to consider the simplest of the three sums, viz. s_n^3.

We suppose, as plainly we may do without loss of generality, that x is positive. Then x is of one or other of the forms

$$\frac{2\lambda+1}{2\mu}, \quad \frac{2\lambda}{4\mu+1}, \quad \frac{2\lambda+1}{2\mu+1}, \quad \frac{2\lambda}{4\mu+3},$$

according as the denominator of $\xi = \frac{1}{2}x$ is congruent to 0, 1, 2, or 3 to modulus 4.

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[1] Some of the properties in question are stated shortly in our paper 'Some problems of Diophantine Approximation' published in the *Proceedings of the fifth International Congress of Mathematicians*, Cambridge, 1912.

Now it is easy to verify that

$$\sum_{0}^{s-1} e^{2\nu^2 \pi i r/s}$$

is of the forms

$$(\pm 1 \pm i)\sqrt{s}, \quad \pm \sqrt{s}, \quad 0, \quad \pm i\sqrt{s}$$

according as $s \equiv 0, 1, 2, 3 \pmod{4}$; and from this it follows immediately that s_n^2 is of the forms

$$(\pm 1 \pm i) An + O(1),$$

$$\pm An + O(1),$$

$$O(1),$$

$$\pm iAn + O(1),$$

in these four cases. Thus, for example, the series

$$\sum \cos(\nu^2 \pi x)$$

oscillates finitely if x is of the form $(2\lambda + 1)/(2\mu + 1)$ or $2\lambda/(4\mu + 3)$, and diverges if x is of the form $(2\lambda + 1)/2\mu$ or $2\lambda/(4\mu + 1)$.[1]

2. 1. — O and o Theorems.

2. 10. We pass to the far more difficult and interesting problems which arise when x is irrational. The most important and general result which we have proved in this connexion is that

(2. 101) $$s_n = o(n)$$

for any irrational x. This result may be established by purely elementary reasoning which can be extended so as to show that such series as

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[1] This result (or rather the analogous result for the sine series) is stated by Bromwich, *Infinite Series*, p. 485, Ex. 10. We have been unable to find any complete discussion of the question, but the necessary materials well be found in Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie*, pp. 285 *et seq.* See also Riemann, *Werke*, p. 249; Genocchi, *Atti di Torino*, vol. 10, p. 985.

(2. 102)
$$\sum e^{n^3\pi i x}, \quad \sum e^{n^4\pi i x}, \ldots$$

also possess the same property. We do not propose to include this proof in the present paper. Although elementary, it is by no means particularly easy; and it will find a more natural place in a paper dealing with the higher series (2. 102). In the present paper we shall establish the equation (2. 101) by arguments of a more transcendental, though really simpler, character, which depend ultimately on the formulae for the linear transformation of the ϑ-functions, and will be found to give much more precise results for particular classes of values of x.

2. 11. It is very easy to see that, as a rule, the equation (2. 101) must be very far from expressing the utmost that can be asserted about s_n.

It follows from the well known theorem of Riesz-Fischer that the series

(2. 111)
$$\sum \frac{\cos n^2\pi x}{n^{\frac{1}{2}+\delta}}, \quad \sum \frac{\sin n^2\pi x}{n^{\frac{1}{2}+\delta}} \qquad (\delta > 0)$$

are Fourier's series. Hence, by a theorem of W. H. Young[1], it follows that they become convergent almost everywhere after the introduction of a convergence factor $n^{-\delta'}$ $(\delta' > 0)$. As δ and δ' are both arbitrarily small, the series themselves must converge almost everywhere. Hence the equation

(2. 112)
$$s_n^3 = o\left(n^{\frac{1}{2}+\delta}\right)$$

must hold for almost all values of x. It is evident that the same argument may be applied to s_n^2 and s_n^4, and to the analogous sums associated with such series as (2. 102).

If, instead of the series (2. 111), we consider the series

(2. 113)
$$\sum \frac{\cos n^2\pi x}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\delta}}, \quad \sum \frac{\sin n^2\pi x}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\delta}},$$

and use, instead of Young's theorem, the more precise theorem that any Fourier's series becomes convergent almost everywhere after the introduction of a convergence factor $1/\log n$,[2] we find that we can replace (2. 112) by the more precise equation

(2. 114)
$$s_n^3 = o\left\{n^{\frac{1}{2}}(\log n)^{\frac{3}{2}+\delta}\right\};$$

[1] *Comptes Rendus*, 23 Dec. 1912.

[2] Hardy, *Proc. Lond. Math. Soc.*, vol. 12, p. 370. The theorem was also discovered independently by M. Riesz.

and it is evident that we can obtain still more precise equations by the use of repeated logarithmic factors. These we need not state explicitly, for none of them are as precise as those which we shall obtain later in the paper. These latter results have, moreover, a considerable advantage over those enunciated here, in that the exceptional set of measure zero, for which our equations may possibly cease to hold, will be precisely defined instead of being, as here, entirely unspecified. The main interest of the argument sketched here lies in the fact that it can be extended to series such as (2. 102).[1]

2. 120. We proceed now to the analysis on which the principal results of the paper depend. These are contained, first in the equation (2. 101), and secondly in the equation

(2. 1201) $$s_n = O(\sqrt{n}),$$

which we shall prove for extensive classes of values of x.

In Chap. 3 of his *Calcul des Résidus*, Lindelöf gives an extremely elegant proof of the formula

(2. 1202) $$\sum_0^{q-1} e^{n^2 \pi i p/q} = \sqrt{\frac{iq}{p}} \sum_0^{p-1} e^{-n^2 \pi i q/p},$$

where p and q are positive integers of which one is even and the other odd.[2] Our first object will be to obtain, by an appropriate modification of Lindelöf's argument, analogous, though naturally rather less simple, formulae, applicable to the series $\sum e^{n^2 \pi i x}$, where x is irrational, and to the other series which we are considering.

We shall, however, consider sums of a more general form than those of which we have spoken hitherto, viz. the sums

(2. 1203) $$\begin{cases} s_n^2(x, \theta) = \sum\limits_{\nu \leq n} e^{\left(\nu - \frac{1}{2}\right)^2 \pi i x} \cos (2\nu - 1)\pi\theta, \\ s_n^3(x, \theta) = \sum\limits_{\nu \leq n} e^{\nu^2 \pi i x} \cos 2\nu\pi\theta, \\ s_n^4(x, \theta) = \sum\limits_{\nu \leq n} (-1)^\nu e^{\nu^2 \pi i x} \cos 2\nu\pi\theta. \end{cases}$$

[1] The argument may even be extended to series of the type $\Sigma e^{\lambda_n i x}$, where λ_n is not necessarily a multiple of π; but for this we require a whole series of theorems concerning Dirichlet's series.

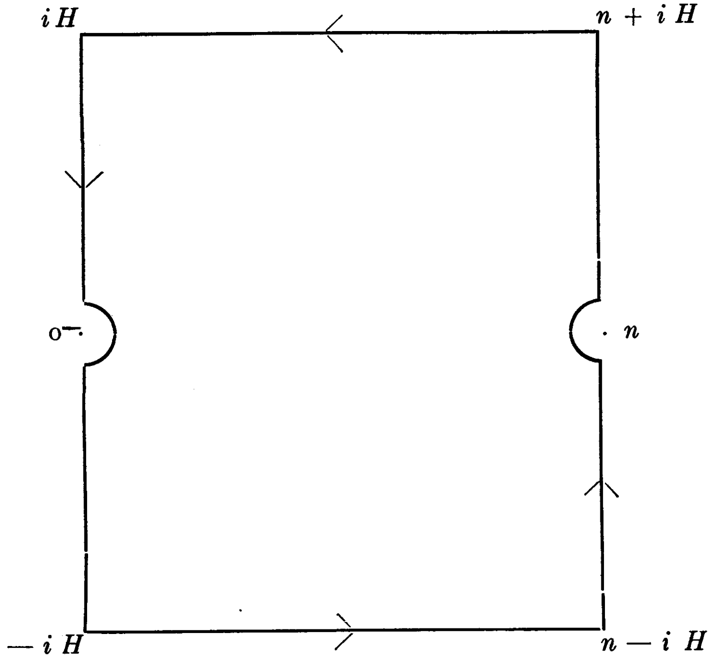
[2] The formula is due to Genocchi and Schaar. See Lindelöf, *l. c.*, p. 75, for references to the history of the formula.

Here x and θ are positive and less than 1, x is irrational, and n is not necessarily an integer. These sums are related to the functions $\vartheta_2(v, \tau), \ldots$ as $s_n^2, \ldots$ are related to $\vartheta_2(0, \tau), \ldots$

2. 121. We consider the complex integral

$$\int e^{z^2 \pi i x} \cos 2 z \pi \theta \; \pi \cot \pi z \, dz$$

taken round the contour C shown in the figure. We suppose that the points 0, n are in the first instance avoided, as in the figure, by small semicircles of



radius ϱ, and that ϱ is then made to tend to zero. An obvious application of Cauchy's Theorem gives the result

$$(2.1211) \qquad \sum_{0}^{n}{}' e^{\nu^2 \pi i x} \cos 2 \nu \pi \theta = \frac{1}{2i} P \int_C e^{z^2 \pi i x} \cos 2 z \pi \theta \cot \pi z \, dz,$$

where P is the sign of Cauchy's principal value, and the dashes affixed to the sign of summation imply that the terms for which $\nu = 0$ and $\nu = n$ are to be divided by 2.

We shall find it convenient to divide the contour C into two parts C_1 and C_2, its upper and lower halves, and to consider the integrals along C_1 and C_2

separately. When we attempt to do this a difficulty arises from the fact that, owing to the poles of the subject of integration at $z=0$ and $z=n$, the two integrals are not separately convergent. This difficulty is, however, trivial and may be avoided by means of a convention.

Suppose that $f(x)$ is a real or complex function of a real variable x which, near $x=\alpha$, is of the form

$$\frac{C}{x-\alpha}+\varphi(x),$$

where $\varphi(x)$ is a function which possesses an absolutely convergent integral across $x=\alpha$; and suppose that, except at $x=\alpha$, $f(x)$ is continuous in the interval (a, A), where $a<\alpha<A$. Then Cauchy's principal value

$$P\int_a^A f(x)\,dx$$

exists; but $f(x)$ has no integral in any established sense from a to α or from α to A. We shall, however, write

$$P\int_a^\alpha f(x)\,dx=\lim_{\varepsilon\to 0}\left\{\int_a^{\alpha-\varepsilon} f(x)\,dx-C\log\varepsilon\right\},$$

$$P\int_\alpha^A f(x)\,dx=\lim_{\varepsilon\to 0}\left\{\int_{\alpha+\varepsilon}^{A} f(x)\,dx+C\log\varepsilon\right\},$$

and it is clear that, with these conventions, we have

$$P\int_a^\alpha f(x)\,dx+P\int_\alpha^A f(x)\,dx=P\int_a^A f(x)\,dx.$$

It is clear, moreover, that a similar convention may be applied to complex integrals such as those which we are considering; thus

$$P\int_0^{iH} e^{z^2\pi i x}\cos 2z\pi\theta\ \pi\cot\pi z\ dz$$

(taken along the line o, iH) is to be interpreted as meaning

$$\lim_{\varepsilon\to 0}\left(\int_{i\varepsilon}^{iH} e^{z^2\pi i x}\cos 2z\pi\theta \quad \pi\cot\pi z \quad dz+\log\varepsilon\right).$$

We may now write (2. 1211) in the form

$$(2.\ 1212) \qquad {\sum_0^n}' e^{\nu^2\pi i x}\cos 2\nu\pi\theta=\frac{1}{2i}\left(P\int_{C_2}-P\int_{C_1}\right)e^{z^2\pi i x}\cos 2z\pi\theta\cot\pi z\,dz,$$

where now C_1 and C_2 are each supposed to be described starting from o. In the first of these two integrals we write

$$\cot\pi z=i+\frac{2i}{e^{2z\pi i}-1},$$

and in the second

$$\cot\pi z=-i-\frac{2i}{e^{-2z\pi i}-1}.$$

The two constant terms in these expressions give rise to integrals which may be taken along the real axis from o to n, instead of along C_2 and C_1; uniting and transposing these terms we obtain

$$(2.\ 1213) \qquad {\sum_0^n}' e^{\nu^2\pi i x}\cos 2\nu\pi\theta-\int_0^n e^{z^2\pi x}\cos 2z\pi\theta\,dz=I_1+I_2,$$

where

$$I_1=P\int_{C_1}\frac{e^{z^2\pi i x}\cos 2z\pi\theta}{e^{-2z\pi i}-1}dz,$$

$$I_2=P\int_{C_2}\frac{e^{z^2\pi i x}\cos 2z\pi\theta}{e^{2z\pi i}-1}dz.$$

We now write

$$\frac{1}{e^{-2z\pi i}-1}=e^{2z\pi i}+e^{4z\pi i}+\cdots+e^{2(k-1)z\pi i}+\frac{e^{2kz\pi i}}{1-e^{2z\pi i}}$$

in I_1, and

$$\frac{1}{e^{2z\pi i}-1}=e^{-2z\pi i}+e^{-4z\pi i}+\cdots+e^{-2(k-1)z\pi i}+\frac{e^{-2kz\pi i}}{1-e^{-2z\pi i}}$$

in I_2. If we observe that

$$\int_{C_1} e^{z^2\pi i x + 2\nu z\pi i} \cos 2z\pi\theta\, dz + \int_{C_2} e^{z^2\pi i x - 2\nu z\pi i} \cos 2z\pi\theta\, dz$$

$$= 2\int_0^n e^{z^2\pi i x} \cos 2\nu z\pi \cos 2z\pi\theta\, dz,$$

we see that (2. 1213) may be transformed into

$$(2.\ 1214)\qquad \sum_0^{n}{}' e^{\nu^2\pi i x} \cos 2\nu\pi\theta - 2\sum_0^{k-1}{}' \int_0^n e^{z^2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta\, dz = K_1 + K_2,$$

where

$$K_1 = P\int_{C_1} e^{z^2\pi i x} \cos 2z\pi\theta \frac{e^{2kz\pi i}}{1 - e^{2z\pi i}}\, dz,$$

$$K_2 = P\int_{C_2} e^{z^2\pi i x} \cos 2z\pi\theta \frac{e^{-2kz\pi i}}{1 - e^{-2z\pi i}}\, dz.$$

2. 122. We shall now suppose that $H \to \infty$, so that the parts of C_1 and C_2 which are parallel to the axis of x go off to infinity. If $z = \xi + i\eta$, and η is large and positive, the modulus of the subject of integration in K_1 is very nearly equal to

$$\frac{1}{2}\exp\left\{-2\pi\eta(k + \xi x - \theta)\right\};$$

while if $z = \xi - i\eta$, and η is again large and positive, the modulus of the subject of integration in K_2 is very nearly equal to

$$\frac{1}{2}\exp\left\{-2\pi\eta(k - \xi x - \theta)\right\}.$$

From this it follows immediately that, if

$$(2.\ 1221)\qquad k > nx + \theta,$$

the contributions to K_1 and K_2 of the parts of C_1 and C_2 which we are causing to tend to infinity will tend to zero.

We are now left with two integrals each of which is composed of two parts taken along rectilinear contours, and we may write

$$K_1 = \left(P\int_0^{i\infty} - P\int_n^{n+i\infty}\right) e^{z^2\pi i x} \cos 2z\pi\theta \frac{e^{2kz\pi i}}{1-e^{2z\pi i}}\, dz$$

$$K_2 = \left(P\int_0^{-i\infty} - P\int_n^{n-i\infty}\right) e^{z^2\pi i x} \cos 2z\pi\theta \frac{e^{-2kz\pi i}}{1-e^{-2z\pi i}}\, dz.$$

Of the four rectilinear integrals thus obtained two, viz. the two taken along the imaginary axis, cancel one another. In the other two we write

$$z = n + it, \quad z = n - it$$

respectively, and then unite the two into a single integral with respect to t; and when we substitute the result in (2. 1214) we obtain

$$\sum_0^{n}{}' e^{\nu^2\pi i x} \cos 2\nu\pi\theta - 2\sum_0^{k-1}{}' \int_0^n e^{z^2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta\, dz = K,$$

where

$$K = i\int_0^\infty e^{\pi i x(n^2-t^2)} \frac{e^{-2k\pi t}}{1-e^{-2\pi t}} \{ e^{2nx\pi t} \cos 2(n-it)\pi\theta - e^{-2nx\pi t} \cos 2(n+it)\pi\theta \}\, dt.$$

2. 123. We now write

$$K = i\int_0^\infty = i\int_0^1 + i\int_1^\infty = K' + K'';$$

and we proceed to show that

(2. 1231) $$K'' = O\sqrt{\frac{1}{x}}$$

uniformly in respect to θ, by which we imply that there is an absolute constant A such that

$$|K''| < \frac{A}{\sqrt{x}}$$

for $0 < x < 1$, $0 \leq \theta \leq 1$, all values of n, and all values of k subject to the inequality (2. 1221).

We may plainly ignore the factor $i e^{n^2 \pi i x}$ in K. The factor in curly brackets is equal to

$$2(\cos 2n\pi\theta \cosh 2t\pi\theta \sinh 2nx\pi t + i \sin 2n\pi\theta \sinh 2t\pi\theta \cosh 2nx\pi t).$$

The factor $e^{-t^2 \pi i x}$ we separate into its real and imaginary parts. When we multiply these two factors together our integral splits up into four, of which the integral

$$\text{(2. 1232)} \qquad \int_1^\infty \cos t^2 \pi x \cosh 2t\pi\theta \sinh 2nx\pi t \frac{e^{-2k\pi t}}{1 - e^{-2\pi t}}\, dt$$

is typical; and it will be sufficient to consider this integral, the same arguments applying to all four.

The function $1/(1 - e^{-2\pi t})$ decreases steadily as t increases from 1 to ∞. Hence, by the second mean value theorem, the integral (2. 1232) may be written in the form

$$\text{(2. 1233)} \qquad A \int_1^T \cos t^2 \pi x \cosh 2t\pi\theta \sinh 2nx\pi t\, e^{-2k\pi t}\, dt,$$

where A (as always in this part of the paper) denotes an absolute numerical constant, and $T > 1$. In (2. 1233) we replace the hyperbolic functions by their expressions in terms of exponentials; and the integral then splits up into four, of which we need only consider

$$\text{(2. 1234)} \qquad A \int_1^T \cos t^2 \pi x\, e^{-2\pi t(k - nx - \theta)}\, dt,$$

the arguments which we apply to this integral applying *a fortiori* to the rest. The integral (2. 1234) may, by another application of the second mean value theorem, be transformed into

$$(2.\ 1235) \qquad A\int_1^{T'} \cos t^2\pi x\, dt,^1 \quad (1 < T' < T).$$

Now, if T and T' are any positive numbers whatever, we have

$$\int_T^{T'} \cos t^2\pi x\, dt = \frac{1}{\sqrt{x}}\int_{T\sqrt{x}}^{T'\sqrt{x}} \cos \pi u^2\, du;$$

and the integral last written is less in absolute value than an absolute constant. We have therefore proved the equation (2. 1231), and it follows that

$$(2.\ 1236) \qquad \sum_0^n{}' e^{\nu^2\pi i x} \cos 2\nu\pi\theta - 2\sum_0^{k-1}{}' \int_0^n e^{z^2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta\, dz = K' + O\sqrt{\frac{1}{x}}.$$

2. 124. The next step in the proof consists in showing that, in the equation (2. 1236), k may be regarded as capable of variation to an extent $O(1)$ on either side, that is to say that we may replace k by any other integer k' lying between $k - A$ and $k + A$, without affecting the truth of the equation. That this is so if k is increased is obvious from what precedes, as the inequality (2. 1221) is still satisfied; but when k is decreased an independent proof is required.

We consider separately the effects of such a variation on the two sides of the equation (2. 1236). As regards the left hand side, it is plain that our assertion will be true if

$$\int_0^n e^{z^2\pi i x} \cos 2z\pi a\, dz = O\sqrt{\frac{1}{x}}$$

uniformly for all values of n and a, and therefore certainly true if

$$\int_0^n e^{z^2\pi i x + 2z\pi i a}\, dz = O\sqrt{\frac{1}{x}}.$$

---

[1] The A in this formula is of course not the *same* numerical constant as before.

But

$$\int_0^n e^{z^2\pi i x+2z\pi i a}\,dz = e^{-\pi i a^2/x}\int_0^n e^{\pi i x(z+a/x)^2}\,dz$$

$$= e^{-\pi i a^2/x}\int_{a/x}^{n+a/x} e^{z^2\pi i x}\,dz\ .$$

$$= \frac{1}{\sqrt{x}}\,e^{-\pi i a^2/x}\int_{a/\sqrt{x}}^{n\sqrt{x}+a/\sqrt{x}} e^{\pi i u^2}\,du;$$

and this expression is evidently of the form desired.

We have now to consider the effect of a variation of k on the right hand side of (2. 1236). The difference produced by such a variation is plainly of the form

$$O\int_0^1 \frac{|e^{-2k\pi t}-e^{-2k'\pi t}|}{1-e^{-2\pi t}}\,e^{2\pi t(nx+\theta)}\,dt$$

$$= O\int_0^1 e^{-2\pi t(k-nx-\theta)}\,dt$$

$$= O(1) = O\sqrt{\frac{1}{x}}.$$

Thus finally we may regard the k which occurs on either side of (2. 1236) as capable of variation to an extent $O(1)$.

2. 125. We proceed now to replace the integrals which occur on the left hand side of (2. 1236) by integrals over the range $(0, \infty)$. We write

$$I_\nu = \int_0^n e^{z^2\pi i x}\cos 2\nu\pi z\cos 2z\pi\theta\,dz = \int_0^\infty - \int_n^\infty = I'_\nu - I''_\nu.$$

Now consider the integral

$$\int e^{z^2\pi i x}\cos 2\nu\pi z\cos 2z\pi\theta\,dz,$$

taken round the rectangular contour whose angular points are n, $n+N$,

$n + N + iH$, $n + iH$. The modulus of the subject of integration is less than a constant multiple of

$$e^{-2\pi\eta(\xi x - \nu - \theta)};$$

and from this it is easily deduced that, if

$$\nu + \theta < nx,$$

the contributions of the sides $(n + N,\ n + N + iH)$ and $(n + N + iH,\ n + iH)$ tend to zero as N and H tend to infinity, and so that the second integral which occurs in our expression for I_ν may be replaced by one taken along the line $(n,\ n + i\infty)$. In order that this transformation may be legitimate for $\nu = 0, 1, \ldots, k' - 1$ we must have

$$k' < nx + 1 - \theta. \tag{2. 1251}$$

It is important to observe that this condition and the condition (2. 1221) cannot always be satisfied with $k = k'$; but that the difference between the least k such that $k > nx + \theta$ and the greatest k' such that $k' < nx + 1 - \theta$ cannot be greater than 1.[1]

On the assumption that (2. 1251) is satisfied, we have

$$\begin{aligned} 2\sum_0^{k'-1}{}' I''_\nu &= \int_n^{n+i\infty} e^{z^2\pi i x} \cos 2z\pi\theta \frac{\sin(2k'-1)\pi z}{\sin \pi z}\, dz \\ &= i\int_0^\infty e^{(n^2-t^2)\pi i x - 2nx\pi t} \cos 2(n+it)\pi\theta \frac{\sinh(2k'-1)\pi t}{\sinh \pi t}\, dt \\ &= L, \end{aligned}$$

say; and so, bearing in mind the results of the analysis of 2. 124,

$$\begin{aligned} \sum_0^n{}' e^{\nu^2\pi i x} \cos 2\nu\pi\theta - 2\sum_0^{k'-1}{}' \int_0^\infty e^{z^2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta\, dz \\ = K' - L + O\sqrt{\frac{1}{x}}. \end{aligned} \tag{2. 1252}$$

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[1] It is these facts which render necessary the analysis of 2. 124.

2. 126. We next write

$$L = \int_0^\infty = \int_0^1 + \int_1^\infty = L' + L'',$$

and we proceed to show that

$$L'' = O\sqrt{\frac{1}{x}},$$

so that L may be replaced by L' in (2. 1252). The argument is practically the same as that of 2. 123. We have to consider a number of integrals of which

$$\text{(2. 1261)} \qquad \int_1^\infty \cos t^2\pi x \cosh 2t\pi\theta\, e^{-2nx\pi t} \frac{\sinh (2k'-1)\pi t}{\sinh \pi t} dt$$

is typical. Writing $2e^{-\pi t}/(1-e^{-2\pi t})$ for $\operatorname{cosech} \pi t$, observing that the factor $1/(1-e^{-2\pi t})$ is monotonic, and using the second mean value theorem as in 2. 123, we arrive at the result desired.

We may accordingly replace L by L' in (2. 1252). And our next step is to show that the k' which occurs in this modified form of (2. 1252) may be regarded as capable of variation to an extent $O(1)$. Here again our analysis is practically the same as some of our previous work (in 2. 124), and there is therefore no need to insist on its details. We may now write (2. 1252) in the form

$$\text{(2. 1262)} \qquad {\sum_0^n}' e^{\nu^2\pi i x} \cos 2\nu\pi\theta - 2{\sum_0^{k-1}}' \int_0^\infty e^{z^2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta\, dz$$

$$= \mathfrak{K} - \mathfrak{L} + O\sqrt{\frac{1}{x}},$$

where

$$\text{(2. 1263)} \begin{cases} \mathfrak{K} = i\displaystyle\int_0^1 e^{\pi i x(n^2-t^2)} \frac{e^{-2k\pi t}}{1-e^{-2\pi t}} \{e^{2nx\pi t} \cos 2(n-it)\pi\theta - e^{-2nx\pi t} \cos 2(n+it)\pi\theta\}\, dt, \\ \mathfrak{L} = i\displaystyle\int_0^1 e^{\pi i x(n^2-t^2)-2nx\pi t} \cos 2(n+it)\pi\theta \frac{\sinh (2k-1)\pi t}{\sinh \pi t} dt; \end{cases}$$

and, as the k's which occur in these equations may all be regarded as capable of variation to an extent $O(1)$, there is no longer any reason to distinguish between k and k'.

2. 127. Again

$$\mathfrak{R} - \mathfrak{L} = \frac{1}{2} i \int_0^1 \frac{e^{\pi i x(n^2-t^2)}}{\sinh \pi t} Q\, dt, \tag{2. 1271}$$

where

$$\begin{aligned} Q &= e^{-(2k-1)\pi t}\{e^{2nx\pi t}\cos 2(n-it)\pi\theta - e^{-2nx\pi t}\cos 2(n+it)\pi\theta\} \\ &\quad - 2e^{-2nx\pi t}\sinh(2k-1)\pi t\cos 2(n+it)\pi\theta \\ &= 2\cos 2n\pi\theta\cosh 2t\pi\theta\sinh(2nx-2k+1)\pi t \\ &\quad + 2i\sin 2n\pi\theta\sinh 2t\pi\theta\cosh(2nx-2k+1)\pi t. \end{aligned}$$

We select the value of k for which

$$-1 < 2nx - 2k + 1 < 1;$$

and the integral (2. 1271) splits up into two, of which it will be sufficient to consider the first, viz.

$$i\cos 2n\pi\theta \int_0^1 e^{\pi i x(n^2-t^2)}\cosh 2t\pi\theta \frac{\sinh(2nx-2k+1)\pi t}{\sinh \pi t} dt. \tag{2. 1272}$$

This is of the form

$$O(1)\int_0^1 e^{-t^2\pi i x}\cosh 2t\pi\theta \frac{\sinh \alpha\pi t}{\sinh \pi t} dt, \tag{2. 1273}$$

where $\alpha = |2nx - 2k + 1|$. It will be enough to consider the real part of this integral, the imaginary part being amenable to similar treatment.

The function

$$\frac{\sinh \alpha\pi t}{\sinh \pi t} \qquad (0 < \alpha < 1)$$

decreases steadily from α as t increases from zero. Hence

$$\int_0^1 \cos \pi x t^2 \cosh 2t\pi\theta \frac{\sinh \alpha\pi t}{\sinh \pi t}\,dt = \alpha \int_0^\tau \cos \pi x t^2 \cosh 2t\pi\theta\,dt$$

$$= \alpha \cosh 2\tau\pi\theta \int_{\tau'}^{\tau} \cos \pi x t^2\,dt,$$

τ and τ' denoting positive numbers less than 1. Since $0<\alpha<1$, $0\leq\theta\leq 1$, the first factor here is of the form $O(1)$; and the second is (cf. 2. 123) of the form $O\sqrt{\frac{1}{x}}$. Hence finally

$$\mathfrak{K}-\mathfrak{L} = O\sqrt{\frac{1}{x}},$$

and so the left hand side of (2. 1262) is itself of the form $O\sqrt{\frac{1}{x}}$.

2. 128. But

$$\int_0^\infty e^{z^2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta\,dz = \frac{1}{2}\sqrt{\frac{i}{x}}\,e^{-\pi i(\theta^2+\nu^2)/x} \cos \frac{2\nu\pi\theta}{x}.[1]$$

Substituting this expression in (2. 1262), and observing that k may now be supposed to be the integral part of nx, we obtain

Theorem 2. 128 *If* $0<x<1$, $0\leq\theta\leq 1$, *then*

$$\sum^{n} e^{\nu^2\pi i x} \cos 2\nu\pi\theta - \sqrt{\frac{i}{x}}\,e^{-\pi i\theta^2/x} \sum^{nx} e^{-\nu^2\pi i/x} \cos \frac{2\nu\pi\theta}{x} = O\sqrt{\frac{1}{x}},$$

where $O\sqrt{\frac{1}{x}}$ *denotes a function of* n, x, *and* θ *which is in absolute value less than a constant multiple of* $\sqrt{\frac{1}{x}}$.

We have omitted the lower limits of summation, and the dashes, which are now plainly irrelevant.

We can also prove, by arguments of the same character as those of §§ 2. 121 *et seq.*,

[1] Lindelöf, *l. c.*, p. 44.

Theorem 2. 1281. *Under similar conditions*

$$\sum^{n} e^{\left(\nu-\frac{1}{2}\right)^2 \pi i x} \cos (2\nu - 1)\pi\theta - \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x} \sum^{nx} (-1)^\nu e^{-\nu^2 \pi i/x} \cos \frac{2\nu\pi\theta}{x} = O\sqrt{\frac{1}{x}},$$

$$\sum^{n} (-1)^\nu e^{\nu^2 \pi i x} \cos 2\nu\pi\theta - \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x} \sum^{nx} e^{\left(\nu-\frac{1}{2}\right)^2 \pi i/x} \cos \frac{(2\nu-1)\pi\theta}{x} = O\sqrt{\frac{1}{x}}.$$

It will hardly be necessary for us to exhibit any details of the proofs, and we will only remark that the integral

$$\int e^{z^2 \pi i x} \cos 2z\pi\theta \cot \pi z \, dz$$

of 2. 121 is replaced by one or other of the integrals

$$\int e^{z^2 \pi i x} \cos 2z\pi\theta \tan \pi z \, dz, \quad \int e^{z^2 \pi i x} \cos 2z\pi\theta \operatorname{cosec} \pi z \, dz.$$

It is on the transformation formulae contained in Theorems **2. 128** and **2. 1281** that all the results of this part of the paper will depend.

2. 13. We have the following system of formulae:

$$\begin{aligned}
s_n^2(x+1, \theta) &= \sqrt{i}\, s_n^2(x, \theta), \\
s_n^3(x+1, \theta) &= s_n^4(x, \theta), \\
s_n^4(x+1, \theta) &= s_n^3(x, \theta), \\
s_n^2(-x, \theta) &= \bar{s}_n^2(x, \theta), \\
s_n^3(-x, \theta) &= \bar{s}_n^3(x, \theta), \\
s_n^4(-x, \theta) &= \bar{s}_n^4(x, \theta),
\end{aligned} \tag{2. 131}$$

$$s_n^2(x, \theta) = \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x} s_{nx}^4\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

$$s_n^3(x, \theta) = \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x} s_{nx}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

$$s_n^4(x, \theta) = \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x} s_{nx}^2\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}}.$$

Here $\bar{s}_n$ denotes the conjugate of s_n. It will be convenient in what follows to write $O\sqrt{\frac{1}{x}}$ in the equivalent form

$$\frac{O(1)}{\sqrt{x}}.$$

Now suppose that x is expressed in the form of a simple continued fraction

(2. 132) $$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\cdots,$$

and write

(2. 133) $$x=\frac{1}{a_1+x_1},\quad x_1=\frac{1}{a_2+x_2},\ \ldots,$$

$$\theta_1=\frac{\theta}{x}-\left[\frac{\theta}{x}\right],\quad \theta_2=\frac{\theta_1}{x_1}-\left[\frac{\theta_1}{x_1}\right],\ \ldots,$$

so that

$$0<x_r<1,\quad 0\leq\theta_r<1$$

for all values of r. Further, let λ_r denote an unspecified index chosen from the numbers 2, 3, 4; and let ω denote a number whose modulus is unity but whose exact value will vary from equation to equation.

This being so, we have

$$\begin{aligned}
s_n^{\lambda}(x,\ \theta) &= \frac{\omega}{\sqrt{x}}s_{nx}^{\lambda_1}\left(-\frac{1}{x},\ \frac{\theta}{x}\right)+\frac{O(1)}{\sqrt{x}} \\
&= \frac{\omega}{\sqrt{x}}s_{nx}^{\lambda_1}(-a_1-x_1,\ \theta_1)+\frac{O(1)}{\sqrt{x}} \\
&= \frac{\omega}{\sqrt{x}}s_{nx}^{\lambda_1}(-x_1,\ \theta_1)+\frac{O(1)}{\sqrt{x}} \\
&= \frac{\omega}{\sqrt{x}}\bar{s}_{nx}^{\lambda_1}(x_1,\ \theta_1)+\frac{O(1)}{\sqrt{x}}.
\end{aligned}$$

Transforming $s_{nx}^{\lambda_1}(x_1,\ \theta_1)$ in the same way, we obtain

$$s_n^{\lambda}(x,\ \theta)=\frac{\omega}{\sqrt{x x_1}}s_{nxx_1}^{\lambda_2}(x_2,\ \theta_2)+O(1)\left\{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{x x_1}}\right\}.$$

Repeating the argument, we find

$$s_n^\lambda(x,\theta)=\frac{\omega}{\sqrt{x\,x_1\ldots x_{\nu-1}}}\,s^{\lambda_\nu}_{nxx_1\ldots x_{\nu-1}}(x_\nu,\theta_\nu)$$

$$+O(1)\left\{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{x\,x_1}}+\cdots+\frac{1}{\sqrt{x\,x_1\ldots x_{\nu-1}}}\right\}$$

(2. 134)

$$=\frac{\omega}{\sqrt{x\,x_1\ldots x_{\nu-1}x_\nu}}\,s^{\lambda_\nu}_{nxx_1\ldots x_{\nu-1}x_\nu}(x_{\nu+1},\theta_{\nu+1})$$

$$+O(1)\left\{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{x\,x_1}}+\cdots+\frac{1}{\sqrt{x\,x_1\ldots x_{\nu-1}x_\nu}}\right\}.$$

Now

$$x_r\leq\frac{1}{1+x_{r+1}},$$

(2. 135)
$$x_r\,x_{r+1}\leq\frac{x_{r+1}}{1+x_{r+1}}<\frac{1}{2},$$

and so $xx_1\ldots x_r\rightarrow 0$ as $r\rightarrow\infty$. We may therefore define ν by the inequalities

(2. 136)
$$n\,x\,x_1\ldots x_{\nu-1}x_\nu<1\leq n\,x\,x_1\ldots x_{\nu-1}.$$

This being so, the first of the equations (2. 134) gives

$$s_n^\lambda(x,\theta)=O(n\sqrt{x\,x_1\ldots x_{\nu-1}})$$

(2. 137)

$$+O(1)\left\{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{x\,x_1}}+\cdots+\frac{1}{\sqrt{x\,x_1\ldots x_{\nu-1}}}\right\},$$

and the second gives

(2. 1371)
$$s_n^\lambda(x,\theta)=O(1)\left\{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{xx_1}}+\cdots+\frac{1}{\sqrt{xx_1\ldots x_{\nu-1}x_\nu}}\right\}.$$

We have thus two inequalities for $s_n^\lambda(x,\theta)$, the further study of which depends merely on an analysis of the continued fraction (2. 132). These inequalities, however, may be simplified. For, by (2. 135), $x_rx_{r+1}<\frac{1}{2}$, and so

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{xx_1}}+\cdots+\frac{1}{\sqrt{x\,x_1\ldots x_{\nu-1}}}$$

$$=\frac{1}{\sqrt{x\,x_1\ldots x_{\nu-1}}}\left(1+\sqrt{x_{\nu-1}}+\sqrt{x_{\nu-2}x_{\nu-1}}+\cdots+\sqrt{x_1x_2\ldots x_{\nu-1}}\right)$$

$$< \frac{1}{\sqrt{x x_1 \ldots x_{\nu-1}}} \left(1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} + \cdots \right)$$

$$< \frac{K}{\sqrt{x x_1 \ldots x_{\nu-1}}}.$$

Hence (2. 137) may be replaced by

$$(2.\ 138) \qquad s_n^\lambda(x,\ \theta) = O\,(n\sqrt{x x_1 \ldots x_{\nu-1}}) + O\frac{1}{\sqrt{x x_1 \ldots x_{\nu-1}}};$$

and similarly (2. 1371) may be replaced by

$$(2.\ 1381) \qquad s_n^\lambda(x,\ \theta) = O\frac{1}{\sqrt{x x_1 \ldots x_{\nu-1} x_\nu}}.$$

2. 14. From (2. 138) and (2. 1381) we can very easily deduce the principal results of this part of the paper.

Theorem 2. 14. *We have*

$$s_n(x,\ \theta) = o(n)$$

for any irrational x, *and uniformly for all values of* θ. *In particular, if* $\theta = 0$, *we have*

$$s_n = o(n)$$

Since $n x x_1 \ldots x_{\nu-1} \geq 1$, the second term on the right hand side of (2. 138) is of the form $O(\sqrt{n})$. And since $x x_1 \ldots x_{\nu-1} \to 0$ as $\nu \to \infty$, the first is of the form $o(n)$. Thus the theorem is proved.

Theorem 2. 141. *If the partial quotients* a_n *in the expression of* x *as a continued fraction are limited, then*

$$s_n(x,\ \theta) = O(\sqrt{n}),$$

uniformly in respect to θ; *and in particular*

$$s_n = O(\sqrt{n}).$$

These results hold, for example, when x *is any quadratic surd, pure or mixed.*

For, if $a_n < K$, x_ν lies between

$$\frac{1}{K}, \quad \frac{K}{K+1}$$

and so

$$x x_1 \ldots x_{\nu-1} x_\nu > x x_1 \ldots x_{\nu-1} / K > 1/(nK).$$

Using (2. 1381), the result of the theorem follows.

Theorem 2. 142. *If* $a_n = O(n^\varrho)$, *then*

$$s_n(x, \theta) = O\left\{n^{\frac{1}{2}} (\log n)^{\frac{1}{2}\varrho}\right\}.$$

Theorem 2. 143. *If* $a_n = O(e^{\varrho n})$, *where* $\varrho < \frac{1}{2} \log 2$, *then*

$$s_n(x, \theta) = O\left(n^{\frac{1}{2} + \frac{\varrho}{\log 2} + \varepsilon}\right),$$

for any positive value of ε.

For

$$x x_1 \ldots x_\nu < \frac{1}{n} \leq x x_1 \ldots x_{\nu-1} < 2^{-\frac{1}{2}\mu},$$

where $\mu = \nu$ or $\mu = \nu - 1$, according as ν is even or odd. Hence

$$n > 2^{\frac{1}{2}\mu},$$

$$\nu < \frac{(2+\varepsilon)\log n}{\log 2}.$$

But

$$x x_1 \ldots x_\nu > H \nu^{-\varrho} x x_1 \ldots x_{\nu-1},$$

where H is a constant, and so

$$\frac{1}{\sqrt{x x_1 \ldots x_\nu}} = O\left(\nu^{\frac{1}{2}\varrho} \sqrt{n}\right) = O\left\{n^{\frac{1}{2}} (\log n)^{\frac{1}{2}\varrho}\right\}.$$

This proves Theorem **2. 142**. Similarly, under the conditions of Theorem **2. 143**, we have

$$\frac{1}{\sqrt{x x_1 \ldots x_\nu}} = O\left(e^{\frac{1}{2}\varrho\nu} \sqrt{n}\right) = O\left(n^{\frac{1}{2} + \frac{\varrho}{\log 2} + \varepsilon}\right).$$

2. 15. Suppose now that $\varphi(n)$ is a logarithmico-exponential function[1] (L-function) of n such that the series

$$\sum \frac{1}{\varphi(n)} \tag{2. 151}$$

is, to put it roughly, near the boundary between convergence and divergence, so that the increase of $\varphi(n)$ is near to that of n. Then, arguing as in 2. 14, we see that, if $a_n = O\{\varphi(n)\}$,

$$x x_1 \ldots x_\nu > \frac{H}{\varphi(\nu)} x x_1 \ldots x_{\nu-1},$$

$$\frac{1}{\sqrt{x x_1 \ldots x_\nu}} = O\sqrt{n\varphi(\nu)} = O\sqrt{n\varphi(\log n)}.$$

Now it has been proved by Borel and Bernstein[2] that the set of values of x for which

$$a_n = O\{\varphi(n)\}$$

is of measure zero when the series (2. 151) is divergent, and of measure unity when the series is convergent. Hence we obtain

Theorem 2. 15. *If $\varphi(n)$ is a logarithmico-exponential function of n such that*

$$\sum \frac{1}{\varphi(n)}$$

is convergent, then

$$s_n = O\sqrt{n\varphi(\log n)}$$

for almost all values of x. In particular, if δ is positive, then

$$s_n = O\left\{n^{\frac{1}{2}} (\log n)^{\frac{1}{2}+\delta}\right\}$$

for almost all values of x.

It was this last result to which reference was made in 2. 11.

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[1] Hardy, *Orders of Infinity*, p. 17.

[2] See Borel, *Rendiconti di Palermo*, Vol. 27, p. 247, and *Math. Annalen*, Vol. 72, p. 578; Bernstein, *Math. Annalen*, Vol. 71, p. 417 and Vol. 72, p. 585.

2. 16. Suppose that a series $\sum u_n$ possesses the property that

$$s_n = u_1 + u_2 + \cdots + u_n = O\{\psi(n)\},$$

ψ being a function which tends steadily to infinity with n; and let φ be a function which tends steadily to zero as $n \to \infty$, and satisfies the condition that

$$\sum \psi(n) \Delta \frac{\varphi(n)}{\psi(n)}$$

is convergent. Then it follows immediately, by an elementary application of ABEL's transformation, that the series

$$\sum \frac{\varphi(n)}{\psi(n)} u_n$$

is convergent. This obvious remark may be utilised to deduce a number of corollaries from some of our theorems. To give one instance only, it follows from Theorem **2. 15** that the series

$$\sum n^{-\alpha} e^{n^2 \pi i x} \cos 2n\pi\theta \qquad \left(\alpha > \frac{1}{2}\right)$$

is convergent for almost all values of x, and, for any particular x, uniformly with respect to θ.

A rather more subtle deduction can be made from Theorem **2. 14**. It does not follow that, because $s_n = o(n)$, the series $\sum \frac{u_n}{n}$ is convergent; and indeed we shall see later that it is not true that (*e. g.*) the series

$$\sum \frac{e^{n^2 \pi i x}}{n} \tag{2. 161}$$

is convergent for all irrational values of x. But it *is* true that, if $s_n = o(n)$, the series $\sum \frac{u_n}{n}$ is either convergent or not summable by any of CESÀRO's means[1]; and this conclusion accordingly holds of the series (2. 161). Similarly, if x is such that $a_n = O(1)$, the series

$$\sum \frac{e^{n^2 \pi i x}}{\sqrt{n}}$$

[1] HARDY and LITTLEWOOD, *Proc. Lond. Math. Soc.*, Vol. 11, p. 433.

possesses the same property. We shall see later that it is the second alternative which is true.

2. 17. So far we have dealt with series in which the parameter θ occurs in a cosine $\cos 2n\pi\theta$ or $\cos(2n-1)\pi\theta$. It is naturally suggested that similar results should hold for the corresponding series involving $\sin 2n\pi\theta$ and $\sin(2n-1)\pi\theta$; and this is in fact the case. These series are, from the point of view of the theory of functions, of a less elementary character: they are not limiting forms of series which occur in the theory of elliptic functions. But it is not difficult to make the necessary modifications in our analysis.

We write

$$(2.171)\quad \begin{cases} \sigma_n^2(x,\theta) = \sum\limits_{\nu \leq n} e^{\left(\nu-\frac{1}{2}\right)^2 \pi i x} \sin(2\nu-1)\pi\theta \\ \sigma_n^3(x,\theta) = \sum\limits_{\nu \leq n} e^{\nu^2 \pi i x} \sin 2\nu\pi\theta \\ \sigma_n^4(x,\theta) = \sum\limits_{\nu \leq n} (-1)^\nu e^{\nu^2 \pi i x} \sin 2\nu\pi\theta \end{cases}$$

Theorem 2. 17. *If* $0 < x < 1$, $0 \leq \theta \leq 1$, *then*

$$\sigma_n^2(x,\theta) = \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x}\, \sigma_{nx}^4\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

$$\sigma_n^3(x,\theta) = \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x}\, \sigma_{nx}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

$$\sigma_n^4(x,\theta) = \sqrt{\frac{i}{x}}\, e^{-\pi i\theta^2/x}\, \sigma_{nx}^2\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

uniformly in respect to θ.

Let us consider, for example, the second of these equations. We start from the integral

$$\int e^{z^2 \pi i x} \sin 2z\pi\theta\ \pi \cot \pi z\, dz,$$

and we arrive, by arguments practically the same as those of 2. 121—2. 127, at the equation

$$(2.172)\quad \sum^{n} e^{\nu^2 \pi i x} \sin 2\nu\pi\theta - 2\sum^{nx} \int_0^\infty e^{z^2 \pi i x} \cos 2\nu\pi z \sin 2z\pi\theta\, dz = O\sqrt{\frac{1}{x}}.$$

The only substantial differences between the reasoning required for the proof of this equation and those which we used before lie in the facts, first that some of the signs of the principal value which we then used are now unnecessary, and secondly that the two integrals along the axis of imaginaries no longer cancel one another. These integrals, however, are of the form

$$\int_0^\infty e^{-t^2\pi i x} \sinh 2t\pi\theta \frac{e^{-2k\pi t}}{1-e^{-2\pi t}}\, dt,$$

and are easily seen to be small when k is large. They are accordingly without importance in our argument.

The integrals which occur in (2. 172), unlike the corresponding cosine integrals, cannot be evaluated in finite form. We have, however,

$$(2.\ 173) \qquad 2\int_0^\infty e^{z^2\pi i x} \cos 2\nu\pi z \sin 2z\pi\theta\, dz = I(\nu+\theta) - I(\nu-\theta),$$

where

$$(2.\ 174) \qquad I(A) = \int_0^\infty e^{z^2\pi i x} \sin 2z\pi A\, dz.$$

Now let us consider the integral

$$\int e^{z^2\pi i x + 2z\pi i A}\, dz \qquad (A > 0)$$

taken round the contour defined by the positive halves of the axes and a circle of radius R. It is easy to show, by a type of argument familiar in the theory of contour integration, that the contribution of the curved part of the contour tends to zero as $R \to \infty$. Hence we deduce

$$\int^\infty e^{z^2\pi i x + 2z\pi i A}\, dz = i\int_0^\infty e^{-t^2\pi i x - 2t\pi A}\, dt;$$

and so

$$I(A) = \frac{1}{i}\int_0^\infty e^{z^2\pi i x}(e^{2z\pi i A} - \cos 2z\pi A)\,dz$$

$$= i\int_0^\infty e^{z^2\pi i x}\cos 2z\pi A\,dz + \int_0^\infty e^{-t^2\pi i x - 2t\pi A}\,dt.$$

Again, it is easy to show that

$$\int_0^\infty e^{-t^2\pi i x - 2t\pi A}\,dt = \frac{\beta}{A} + O\left(\frac{1}{A^3}\right),$$

where $\beta = 1/2\pi$. Hence

$$I(\nu+\theta) - I(\nu-\theta) = i\int_0^\infty e^{z^2\pi i x}\{\cos 2(\nu+\theta)\pi z - \cos 2(\nu-\theta)\pi z\}\,dz$$

$$(2.\ 175)\qquad + \frac{\beta}{\nu+\theta} - \frac{\beta}{\nu-\theta} + O\left(\frac{1}{\nu^3}\right)$$

$$= \sqrt{\frac{i}{x}}\,e^{-(\theta^2+\nu^2)\pi i/x}\sin\frac{2\nu\pi\theta}{x} + O\left(\frac{1}{\nu^2}\right).$$

From (2. 171), (2. 173), and (2. 175) we at once deduce the second equation of Theorem **2. 17**; and the others may be established similarly.

2. 18. From Theorem **2. 17** follow the analogues for the sums σ of those already established for the sums s. Thus we have

Theorems 2. 18, 2. 181—4. *The results established in Theorems* 2. 14, 2. 141—3, 2. 15, *for series involving cosines, are true also for the corresponding series involving sines.*

2. 19. The preceding results have a very interesting application to the theory of TAYLOR's series.

Let

$$f(z) = \sum a_n z^n$$

be a power series whose radius of convergence is unity, and let, as usual, $M(r)$ denote the maximum of $|f|$ along a circle of radius r less than 1. Further, suppose that

$$M(r) = O(1-r)^{-\alpha},$$

and let

$$g(r) = \sum |a_n| r^n.$$

Then it is known that[1]

$$g(r) = O(1-r)^{-a-\frac{1}{2}}.$$

Further, it is known that the number $\frac{1}{2}$ occurring in the last formula cannot be replaced by any smaller number, that is to say that, if δ is any positive number, a function $f(z)$ can be found such that the difference between the orders of $g(r)$ and $M(r)$ is $\frac{1}{2} - \delta$.[2] But so far as we are aware, no example has been given of a function $f(z)$ such that the orders of $g(r)$ and $M(r)$ differ by as much as $\frac{1}{2}$. We are now in a position to supply such an example.

Let

$$f(z) = \sum e^{n^2 \pi i \xi} z^n,$$

where ξ is an irrational of the type considered in Theorem **2. 141**, so that the partial quotients in its expression as a continued fraction are limited. Then, if $z = re^{2\pi i \theta}$, we have, by Theorems **2. 141** and **2. 181**,

$$S_n = \sum^{n} e^{\nu^2 \pi i \xi + 2\nu\pi i \theta} = O(\sqrt{n}),$$

uniformly in θ; and from this it follows that

$$f(z) = f(re^{2\pi i \theta}) = \sum r^n e^{n^2 \pi i \xi + 2n\pi i \theta} = O\sqrt{\frac{1}{1-r}},$$

uniformly in θ. Hence

$$M(r) = O\sqrt{\frac{1}{1-r}},$$

while

$$g(r) = \sum r^n = \frac{1}{1-r}.$$

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[1] Hardy, *Quarterly Journal*, Vol. 44, p. 147.
[2] Hardy, *l c.*, p. 156.

Thus the orders of $g(r)$ and $M(r)$ differ by exactly $\frac{1}{2}$. If we consider, instead of $f(z)$, the function

$$\sum n^{\alpha-\frac{1}{2}} e^{n^2\pi i\xi} z^n \qquad \left(0 < \alpha < \frac{1}{2}\right),$$

we obtain in the same way an example of a function such that

$$M(r) = O(1-r)^{-\alpha},$$

$$g(r) \sim \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{(1-r)^{\alpha+\frac{1}{2}}}.$$

These examples show that the equation

$$M(r) = O(1-r)^{-\alpha} \qquad (\alpha > 0)$$

does *not* involve

$$g(r) = o(1-r)^{-\alpha-\frac{1}{2}};$$

a possibility which had before remained open.[1]

2. 19. Theorems 2. 14 etc. also enable us to make a number of interesting inferences as to the behaviour of the modular functions

$$\sum q^{\left(n-\frac{1}{2}\right)^2}, \quad \sum q^{n^2}, \quad \sum (-1)^n q^{n^2}$$

as q tends along a radius vector[2] to an irrational place $e^{\pi i\xi}$ on the circle of convergence. Thus from Theorem 2. 14 we can easly deduce that, if $f(q)$ denotes any one of these functions, then

$$f(q) = o(1-|q|)^{-\frac{1}{2}};$$

and from Theorem 2. 141 that, if ξ is an irrational of the class there considered, then

$$f(q) = O(1-|q|)^{-\frac{1}{4}}.$$

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[1] Hardy, *l. c.*, p. 150.

[2] Or along any 'regular path' which does not touch the circle of convergence.

These results are, however, more easily proved by a more direct method, which enables us at the same time to assign certain *lower* limits for the magnitude of $|f(q)|$, and to show that Theorems **2. 14** *et seq* are in a certain sense the best possible of their kind. It is to the development of this method, which depends on a direct use of the ordinary formulae for the linear transformation of the ϑ-functions, that the greater part of the rest of the paper will be devoted.

2. 2. — Ω Theorems.

2. 20. We have occupied ourselves, so far, with the determination of certain upper limits for the magnitude of sums of the type s_n. Thus we proved that $s_n = o(n)$ for any irrational x, and that $s_n = O(\sqrt{n})$ for an important class of such irrationals, including for example the class of quadratic surds. But we have done nothing to show that these results are the best of their kind that are true. The theorems which follow will show that this is the case.

We shall begin, however, by proving a theorem of a more elementary character which involves no appeal to the formulae of the transformation theory.

Theorem 2. 20. *Suppose that $\varphi(n)$ is a positive decreasing function of n, such that the series $\sum \varphi(n)$ is divergent. Then it is possible to find irrationals x such that the series*

$$\sum \varphi(n)\, e^{n^2 \pi i x}$$

is not convergent. The same is true of the series

$$\sum \varphi(n)\, e^{\left(n-\frac{1}{2}\right)^2 \pi i x}, \quad \sum (-1)^n \varphi(n)\, e^{n^2 \pi i x},$$

and of the real and imaginary parts of all these series.

Consider, for example, the real part of the first series. We shall suppose that, among the convergents p_ν / q_ν to x, there are infinitely many of the form $2\lambda/(4\mu + 1)$. Let (q_ν) be a subsequence selected from the denominators of these convergents. We are clearly at liberty to suppose that the increase of $a_{\nu+1}$, when compared with that of any number which depends only on q_ν and the function φ, is as rapid as we please.

We shall consider the sum

$$S_\nu = \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n) \cos(n^2 \pi x),$$

where A_ν is an integer large compared with q_ν but small compared with $q_{\nu+1}/q_\nu$. We shall suppose A_ν so chosen that

(2. 201)
$$q_\nu^{-\frac{3}{2}} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) \to \infty$$

(2. 202)
$$a_{\nu+1} / A_\nu^2 q_\nu \to \infty;$$

and we shall show that, in these circumstances, $|S_\nu|$ tends to infinity with ν, and hence that the series

$$\sum \varphi(n) \cos(n^2 \pi x)$$

cannot converge.

We may consider, instead of S_ν, the sum

(2. 203)
$$S'_\nu = \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n) \cos(n^2 \pi p_\nu / q_\nu).$$

For

$$S_\nu - S'_\nu = \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n) \{\cos(n^2 \pi x) - \cos(n^2 \pi p_\nu / q_\nu)\}.$$

Now

$$\left| n^2 \left(x - \frac{p_\nu}{q_\nu} \right) \right| = \frac{n^2}{q_\nu q'_{\nu+1}} < \frac{A_\nu^2 q_\nu}{q'_{\nu+1}},$$

where $a'_{\nu+1}$ is the complete quotient corresponding to the partial quotient $a_{\nu+1}$, and $q'_{\nu+1} = a'_{\nu+1} q_\nu + q_{\nu-1}$; and from this it follows that $|S_\nu - S'_\nu|$ is less than a constant multiple of

$$\frac{A_\nu^2 q_\nu}{q'_{\nu+1}} \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n),$$

and so of

$$A_\nu^3 q_\nu^2 / q'_{\nu+1} < A_\nu^3 q_\nu / a_{\nu+1}.$$

Thus $S_\nu - S'_\nu \to 0$ as $\nu \to \infty$, in virtue of (2. 202).

We may write S'_ν in the form

$$S'_\nu = \sum_{r=1}^{A_\nu - 1} \sum_{s=0}^{q_\nu - 1} \varphi(r q_\nu + s) \cos(s^2 \pi p_\nu / q_\nu).$$

If in this sum we replace $\varphi(r q_\nu + s)$ by $\varphi(r q_\nu)$, the error introduced is not greater than

$$\sum_{r=1}^{A_\nu-1} \sum_{s=0}^{q_\nu-1} \{\varphi(r q_\nu) - \varphi(r q_\nu + s)\} \leq q_\nu \sum_{r=1}^{A_\nu-1} \{\varphi(r q_\nu) - \varphi[(r+1) q_\nu]\}$$

$$\leq q_\nu \varphi(q_\nu).$$

Thus, with an error not greater than $q_\nu \varphi(q_\nu)$, and *a fortiori* not greater than $q_\nu \varphi(1)$, we can replace S'_ν by

$$(2.\ 204) \qquad S''_\nu = \sum_{r=1}^{A_\nu-1} \varphi(r q_\nu) \sum_{s=0}^{q_\nu-1} \cos(s^2 \pi p_\nu / q_\nu) = \pm \sqrt{q_\nu} \sum_{r=1}^{A_\nu-1} \varphi(r q_\nu).$$

Now

$$\varphi(q_\nu) + \varphi(2 q_\nu) + \cdots + \varphi\{(A_\nu - 1) q_\nu\} \geq \frac{1}{q_\nu} \sum_{2q_\nu}^{A_\nu q_\nu} \varphi(n)$$

$$\geq \frac{1}{q_\nu} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) - \varphi(1),$$

and so

$$|S''_\nu| \geq \frac{1}{\sqrt{q_\nu}} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) - \sqrt{q_\nu}\, \varphi(1).$$

Hence

$$\frac{|S''_\nu|}{q_\nu \varphi(1)} \geq \frac{1}{\varphi(1)} q_\nu^{-\frac{3}{2}} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) - \frac{1}{\sqrt{q_\nu}},$$

which tends to infinity with ν, in virtue of (2. 201). Hence S'_ν, and so S_ν, tends to infinity with ν; which proves the theorem.

In particular it is possible to find irrational values of x for which the series

$$\sum \frac{\cos(n^2 \pi x)}{n}, \quad \sum \frac{\cos(n^2 \pi x)}{n \log n}, \ldots$$

are not convergent.

2. 21. We shall find it convenient at this stage to introduce a new notation. We define the equation

$$f = \Omega(\varphi),$$

where φ is a positive function of a variable, which may be integral or continuous but which tends to a limit, as meaning that there exists a constant H and a sequence of values of the variable, themselves tending to the limit in question, such that

$$|f| > H\varphi$$

for each of these values. In other words, $f = \Omega(\varphi)$ *is the negation of* $f = o(\varphi)$. In the notation of Messrs Whitehead and Russell we should write

$$f = \Omega(\varphi) \,.=.\, \sim (f = o(\varphi)). \qquad Df.$$

2. 22. We shall now prove the following theorems.

Theorem 2. 22. *If x is irrational, then*

$$s_n = \Omega(\sqrt{n}).$$

Theorem 2. 221. *If φ is any positive function of n, which tends to zero as $n \to \infty$, then it is possible to find irrationals x such that*

$$s_n = \Omega(n\varphi).$$

These theorems show that the equation

$$s_n = O(\sqrt{n}),$$

established by Theorem **2. 141** for a particular class of values of x, cannot possibly be replaced by any better equation; and that the equation

$$s_n = o(n)$$

of Theorem **2. 14** is the best that is true of *all* irrationals. We shall deduce these theorems from certain results concerning the elliptic modular functions.

2. 23. We write

$$q = e^{\pi i \tau} = e^{\pi i(x+iy)} = e^{-\pi y + \pi i x}$$
$$= re^{\pi i x} \qquad (x > 0,\ y > 0,\ 0 < r < 1).$$

$$\vartheta_2(0, \tau) = 2\sum_1^\infty q^{\left(n-\frac{1}{2}\right)^2},$$

$$\vartheta_3(0, \tau) = 1 + 2\sum_1^\infty q^{n^2},$$

$$\vartheta_4(0, \tau) = 1 + 2\sum_1^\infty (-1)^n q^{n^2}.$$

We suppose that p_n/q_n is a convergent to

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \cdots,$$

and write

$$p_{n-1}q_n - p_n q_{n-1} = \eta_n = \pm 1.$$

We shall consider a linear transformation

$$T = \frac{c + d\tau}{a + b\tau},$$

where

$$\left.\begin{array}{ll} a = p_n, & b = -q_n, \\ c = \eta_n p_{n-1}, & d = -\eta_n q_{n-1}, \end{array}\right\} (p_n \textit{ odd}),$$

$$\left.\begin{array}{ll} a = -p_n, & b = q_n, \\ c = -\eta_n p_{n-1}, & d = \eta_n q_{n-1}, \end{array}\right\} (p_n \textit{ even}).$$

In either case $ad - bc = \eta_n^2 = 1$.

Finally, if a'_{n+1} is the complete quotient corresponding to a_{n+1}, we write

$$q'_{n+1} = a'_{n+1} q_n + q_{n-1},$$

and we take

$$y = 1/(q_n q'_{n+1}).$$

When

$$p_{n-1} \textit{ is even}, \quad p_n \textit{ is odd},$$

$$q_{n-1} \textit{ is odd}, \quad q_n \textit{ is even},$$

we shall say that the convergents p_{n-1}/q_{n-1}, p_n/q_n form a system of type

$$\begin{pmatrix} E & O \\ O & E \end{pmatrix}.$$

There are six possible types of system, viz.

$$\begin{pmatrix} E & O \\ O & E \end{pmatrix}, \quad \begin{pmatrix} O & O \\ O & E \end{pmatrix}, \quad \begin{pmatrix} E & O \\ O & O \end{pmatrix}, \quad \begin{pmatrix} O & O \\ E & O \end{pmatrix}, \quad \begin{pmatrix} O & E \\ E & O \end{pmatrix}, \quad \begin{pmatrix} O & E \\ O & O \end{pmatrix},$$

which we number

$$1^\circ,\ 2^\circ,\ 3^\circ,\ 4^\circ,\ 5^\circ,\ 6^\circ.$$

The following remark is of fundamental importance for our present purpose. *In any continued fraction whatever, one or other of the systems* 1°, 2°, 5°, 6° *must occur infinitely often.* This appears from the fact that the second column in cases 3° and 4° is O, O, and that all cases in which the first column is O, O fall under 1°, 2°, 5°, or 6°.

2. 24. In cases 1°, 2°, 5°, or 6° we have

$$\vartheta_3(0, \tau) = \frac{1}{\omega\sqrt{a + b\tau}}\, \vartheta(0, T),$$

where ω is an 8-th root of unity, and ϑ stands for one or other of ϑ_3 and ϑ_4.[1] Now

$$|a + b\tau| = |p_n - q_n x - q_n i y| = \frac{|\pm 1 - i|}{q'_{n+1}} = \frac{\sqrt{2}}{q'_{n+1}}.$$

Also, if $Q = e^{\pi i T}$, we have

$$|Q| = e^{-\pi\lambda},$$

where

$$\lambda = \mathbf{I}(T) = \mathbf{I}\left(\frac{c + d\tau}{a + b\tau}\right) = \mathbf{I}\left\{\frac{d}{b} - \frac{1}{b(a + b\tau)}\right\}$$

$$= \frac{y}{(1/q'_{n+1})^2 + q_n^2 y^2} = \frac{q'_{n+1}}{2q_n} > \frac{1}{2}.$$

Hence

$$|Q| < e^{-\frac{1}{2}\pi} < 1/(4.8) < .21,$$

$$2|Q| + 2|Q|^4 + \cdots < 2(.21) + 2(.21)^4 + \cdots$$

$$< \frac{1}{2},$$

[1] *T. and M.*, Vol. 2, p. 262 (Table XLII).

$$|\vartheta(0,\ T)| = |1 \pm 2Q + 2Q^4 \pm \cdots| > \frac{1}{2}.$$

Consequently

$$|\vartheta_3(0,\ \tau)| > K\sqrt{q'_{n+1}} > K\sqrt[4]{q_n q'_{n+1}} = K\sqrt[4]{1/y}.$$

From this follows at once

Theorem 2. 24. *If* $q = re^{\pi i x}$, *where* x *is irrational, then*

$$1 + 2\sum_1^\infty q^{n^2} = \Omega\left\{\sqrt[4]{\frac{1}{1-r}}\right\}$$

as $r \to 1$.

From this we can deduce Theorem 2. 22 as a corollary. For if we had

$$s_n = o(\sqrt{n}),$$

the series

$$1 + 2\sum_1^\infty e^{n^2\pi i x} r^{n^2} = \sum u_n r^n$$

would satisfy the condition

$$u_0 + u_1 + \cdots + u_n = o(\sqrt[4]{n}),$$

and so we should have

$$\begin{aligned}\sum u_n r^n &= (1-r)\sum(u_0 + u_1 + \cdots + u_n) r^n \\ &= (1-r)\sum o(\sqrt[4]{n}) r^n \\ &= o\left\{\sqrt[4]{\frac{1}{1-r}}\right\};\end{aligned}$$

an equation which Theorem 2. 24 shows to be untrue.

Again, let $\varphi(1/y)$ be any function which tends to zero with y. We have

$$|\vartheta_3(0,\ \tau)| > K\sqrt{q'_{n+1}} = K\sqrt{1/q_n y}.$$

We choose a value of x such that, for an infinity of values of n corresponding to one of the favourable cases 1°, 2°, 5°, 6°, we have

$$\sqrt{1/q_n} > \varphi(q_n q'_{n+1});$$

this may certainly be secured by supposing that a_{n+1} is sufficiently large. We have then

$$|\vartheta_3(0, \tau)| > K\sqrt{1/y}\,\varphi(1/y).$$

From this we deduce

Theorem 2. 241. *Given any function φ which tends to zero, it is possible to find irrational values of x such that*

$$1 + 2\sum_1^{\infty} q^{n^2} = \Omega\left\{\sqrt{\frac{1}{1-r}}\,\varphi\left(\frac{1}{1-r}\right)\right\}$$

when $q = re^{\pi i x}$ and $r \to 1$.

From this theorem Theorem **2. 221** follows as a corollary just as Theorem **2. 22** followed from Theorem **2. 24.**

2. 25. It is interesting to consider a little more closely the case in which x is an irrational for which $a_n = O(1)$.

Let us, instead of considering only the special value $1/(q_n q'_{n+1})$ of y, consider the range R_n defined by

$$\frac{1}{q'^2_{n+1}} \leq y \leq \frac{1}{q_n^2}$$

or

$$\frac{\eta}{q_n q'_{n+1}} \leq y \leq \frac{1}{\eta q_n q'_{n+1}},$$

where $\eta = q_n / q'_{n+1}$. It is clear that, for different values of n, these ranges cover up the *whole* range of variation of y. If now $y = \zeta/(q_n q'_{n+1})$, so that $\eta \leq \zeta \leq 1/\eta$, we have

$$\lambda = \frac{y}{(1/q'_{n+1})^2 + q_n^2 y^2} = \frac{\zeta}{1+\zeta^2}\,\frac{q'_{n+1}}{q_n}.$$

The least values of λ correspond to $\zeta = \eta,\ 1/\eta$; and then

$$\lambda = \frac{q'^2_{n+1}}{q_n^2 + q'^2_{n+1}} > \frac{1}{2}.$$

Suppose first that n corresponds to a system of one of the types 1°, 2°, 5°, 6°. Then the argument of 2. 24 shows that the absolute value of $\vartheta(0, T)$ lies between $\frac{1}{2}$ and $\frac{3}{2}$. If on the other hand n corresponds to a system of type 3° or 4°, we have

$$\vartheta_3(0, \tau) = \frac{1}{\omega\sqrt{a + b\tau}} \vartheta_2(0, T).$$

Now

$$\vartheta_2(0, T) = 2Q^{\frac{1}{4}}(1 + Q^2 + Q^6 + \cdots),$$

and the absolute value of the second factor lies between $\frac{3}{4}$ and $\frac{5}{4}$. On the other hand λ lies between $q'^2_{n+1}/(q_n^2 + q'^2_{n+1})$ and $q'_{n+1}/2q_n$, and *a fortiori* between $\frac{1}{2}$ and $\frac{1}{2}(K + 1)$, where K is the greatest value of a partial quotient. Hence in this case also $|\vartheta(0, T)|$ lies between fixed positive limits.

Thus, as the ranges R_n fill up the whole range of variation of y, we can determine two constants H_1, H_2 so that

$$\frac{H_1}{\sqrt{|a + b\tau|}} < |\vartheta_3(0, \tau)| < \frac{H_2}{\sqrt{|a + b\tau|}}.$$

But

$$\sqrt{|a + b\tau|} = \sqrt[4]{\left\{y\left(\frac{1}{q'^2_{n+1}y} + q_n^2 y\right)\right\}}$$

and it is easy to see that the second factor under the radical lies between fixed positive limits. Hence we obtain

Theorem 2. 25. *If $q = re^{\pi i x}$, the partial quotients to x being limited, and $r \to 1$, then*

$$\left|1 + 2\sum_1^\infty q^{n^2}\right| \asymp \sqrt[4]{\frac{1}{1 - r}}.^{1}$$

2. 26. In the preceding discussion, the argument which showed that $\lambda > \frac{1}{2}$ was independent of any hypothesis as to the continued fraction. Hence we have in any case

$$|\vartheta_3(0, \tau)| < \frac{H_2}{\sqrt{|a + b\tau|}} = \frac{H_2}{\sqrt[4]{(1/q'_{n+1})^2 + q_n^2 y^2}}$$

$$= O\left\{\frac{1}{\sqrt{q_n y}}\right\} = o\left(\frac{1}{\sqrt{y}}\right),$$

as $q_n \to \infty$. Hence we obtain

Theorem 2. 26. *For any irrational value of x, we have*

[1] The formula $f \asymp \varphi$ implies that $|f|/\varphi$ lies between fixed positive limits: see Hardy, *Orders of Infinity*, pp. 2, 5.

$$1 + 2\sum_1^{\infty} q^{n^2} = o\left\{\sqrt[4]{\frac{1}{1-r}}\right\}.$$

This result may of course also be proved as a corollary of Theorem **2. 14**, by reasoning analogous to that used in 2. 24. But the direct proof is none the less interesting.

2. 27. The argument used in 2. 24, in deducing Theorem **2. 22** from Theorem **2. 24**, may be adapted so as to prove an interesting generalisation of the former theorem. Let us write, as before

$$1 + 2\sum_1^{\infty} e^{n^2\pi i x} r^{n^2} = \sum u_n r^n,$$

and suppose that $k!\, S_n^k / n^k$ is one of CESÀRO's means associated with the series $\sum u_n$. Then

$$S_n^k = \Omega\left(n^{k+\frac{1}{4}}\right). \tag{2. 271}$$

For if this were not so, we should have

$$\sum u_n r^n = (1-r)^{k+1} \sum S_n^k r^n = (1-r)^{k+1} \sum o\left(n^{k+\frac{1}{4}}\right) r^n$$
$$= o\left\{\sqrt[4]{\frac{1}{1-r}}\right\}.$$

From (2. 271) it follows that the series $\sum u_n$ cannot become summable (Ck) on the introduction of a convergence factor $n^{-\frac{1}{4}}$.[1] And from this we deduce

Theorem 2. 27. *The series*

$$\sum n^{-\alpha} e^{n^2\pi i x} \qquad \left(\alpha \leq \frac{1}{2}\right)$$

cannot be convergent, or summable by any of CESÀRO's *means, for any irrational x.*

We need hardly remark that the same is true of

$$\sum n^{-\alpha} e^{\left(n-\frac{1}{2}\right)^2 \pi i x}, \qquad \sum (-1)^n n^{-\alpha} e^{n^2\pi i x}.$$

On the other hand, if $\alpha > \frac{1}{2}$, all these series converge *presque partout* (2. 11, 2. 16).

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[1] HARDY and LITTLEWOOD, *Proc. Lond. Math. Soc.*, Vol. 11, p. 435.

2. 3. — An application to the theory of trigonometrical series.[1]

2. 30. The problem of finding a trigonometrical series whose coefficients tend to zero, and which converges, if ever, only for a set of values of the argument of measure 0, was first formulated by FATOU[2] and first solved by LUSIN.[3] The results of the earlier part of this paper have led us to a solution of FATOU's problem which seems to us to have considerable advantages over LUSIN's.

We can, in fact, prove the following theorem, which is an extension of Theorem **2. 27**.

Theorem 2. 30. *The series*

$$\sum n^{-a} \cos (n^2 \pi x), \quad \sum n^{-a} \sin (n^2 \pi x),$$

where $0 < a < \frac{1}{2}$, *are never convergent, or summable by any of* CESÀRO's *means, for any irrational value of* x.[4]

Considered simply as solutions of FATOU's problem, these series have, as against LUSIN's, two advantages. In the first place, they are series of a simple, natural, and elegant analytical form. In the second place, the problem of convergence is solved completely; there is no exceptional set of values of x for which doubt remains.[5]

2. 31. We proceed to the proof of Theorem **2. 30**. This theorem is a corollary of

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[1] An abstract of the contents of this part of the paper appeared, under the title »Trigonometrical Series which Converge Nowhere or Almost Nowhere», in the *Records of Proceedings of the London Math. Soc.* for 13 Febr. 1913.

[2] *Acta Mathematica*, Vol. 30, p. 398.

[3] *Rendiconti di Palermo*, Vol. 32, p. 386.

[4] The cosine series converges when x is a rational of the form $(2\lambda + 1)/(2\mu + 1)$ or $2\lambda/(4\mu + 3)$, the sine series when x is a rational of the form $(2\lambda + 1)/(2\mu + 1)$ or $2\lambda/(4\mu + 1)$ (see 2. 01). In the abstract referred to above this part of the result (which is of course trivial) was stated incorrectly.

[5] It is only since this paper was written that we have become aware of a different solution given by H. STEINHAUS (*Comptes Rendus de la Société Scientifique de Varsovie*, 1912, p. 223). STEINHAUS also solves the problem of convergence for his series completely; they converge, in fact, for *no* values of x. Thus in this respect our examples have no advantage over his; the advantage, if anywhere, is on his side. In respect of simplicity etc. our examples have the advantage over his as much as over LUSIN's.

Theorem 2. 31. *If $q = re^{\pi i x}$, where x is irrational, then, as $r \to 1$, both the real and the imaginary parts of*

$$f(q) = 1 + 2\sum_{1}^{\infty} q^{n^2}$$

are of the form $\Omega\left\{\sqrt[4]{\frac{1}{1-r}}\right\}$.

In fact, when once this theorem has been established, Theorem **2. 30** follows from it in the same way as Theorem **2. 22** followed from Theorem **2. 24**. And the proof of Theorem **2. 31** is in principle the same as that of Theorem **2. 24**, though naturally more complicated.

Our notation will be the same as in 2. 23. We shall prove first that, *in cases* 1°, 2°, 5°, *and* 6°, *we have*

$$\text{(i)} \qquad |\vartheta_3(0, \tau)| > K y^{-\frac{1}{4}},$$

$$\text{(ii)} \qquad \left|\operatorname{am}\, \vartheta_3(0, \tau) - \frac{1}{2} m\pi\right| > \delta$$

for all integral values of m, K and δ being positive constants, provided either

$$(\alpha) \quad a_{n+1} > 1$$

or

$$(\beta) \quad a_{n+1} = 1, \ a_{n+2} = 1.$$

We shall express this shortly by saying that 1°, 2°, 5°, 6° are *favourable cases*, except possibly when

$$a_{n+1} = 1, \ a_{n+2} > 1;$$

a 'favourable case' being one in which we can prove the inequalities

$$(2.\ 311) \qquad |\mathbf{R}\{\vartheta_3(0, \tau)\}| > K y^{-\frac{1}{4}}, \quad |\mathbf{I}\{\vartheta_3(0, \tau)\}| > K y^{-\frac{1}{4}}.$$

We have

$$(2.\ 312) \qquad \vartheta_3(0, \tau) = \frac{1}{\omega\sqrt{a + b\tau}}\, \vartheta(0, T).$$

If $a_{n+1} > 1$,

$$|Q| = e^{-\pi q'_{n+1}/2q_n} < e^{-\pi} < \frac{1}{23};$$

and if $a_{n+1} = 1$, $a_{n+2} = 1$,

$$\frac{q'_{n+1}}{q_n} = 1 + \frac{1}{a'_{n+2}} + \frac{q_{n-1}}{q_n} > \frac{3}{2},$$

$$|Q| < e^{-\frac{3}{4}\pi} < \frac{1}{10}.$$

In either case

$$2|Q| + 2|Q|^4 + \cdots < \frac{1}{4},$$

and so

(2. 313) $$|\vartheta(0, T)| > \frac{3}{4}, \quad |\text{am}\, \vartheta(0, T)| < \text{arc tan}\frac{1}{4} < \frac{1}{12}\pi.$$

Again

$$a + b\tau = \pm (\eta_n + i)/q'_{n+1},$$

(2. 314) $$|a + b\tau|^{-\frac{1}{2}} = 2^{-\frac{1}{4}}\sqrt{q'_{n+1}} > Ky^{-\frac{1}{4}},$$

(2. 315) $$\text{am}\left\{(a + b\tau)^{-\frac{1}{2}}\right\} \equiv -\frac{1}{8}\eta_n\pi \qquad \left(\text{mod.}\ \frac{1}{2}\pi\right).$$

From (2. 312), (2. 313), (2. 314), and (2. 315) it follows, first that the modulus of $\vartheta_3(0, \tau)$ is greater than a constant multiple of $y^{-\frac{1}{4}}$ (as has been shown already under 2. 24), and secondly that

(2. 316) $$\text{am}\, \vartheta_3(0, \tau) \equiv -\frac{1}{8}\eta_n\pi + \left\{\frac{1}{12}\pi\right\} \qquad \left(\text{mod.}\ \frac{1}{4}\pi\right),$$

where $\left\{\frac{1}{12}\pi\right\}$ denotes a number whose absolute value is less than $\frac{1}{12}\pi$. Hence am $\vartheta_3(0, \tau)$ must differ by at least

$$\frac{\pi}{8} - \frac{\pi}{12} = \frac{\pi}{24}$$

from any multiple of $\frac{1}{2}\pi$; and so the cases which we are considering are all favourable.

2. 32. We shall now prove that, as $n \to \infty$, *favourable cases must recur infinitely often.* This will complete the proof of Theorem **2. 31**.

We represent the state of affairs, as regards the oddness or evenness of p_n and q_n, in a way which will be made most clear by an example. If every

p_n is odd, and q_n is alternately odd and even, we represent the continued fraction diagrammatically in the form

$$\begin{matrix} O & O & O & O & O \dots \\ O & E & O & E & O \dots \end{matrix}$$

— and so in other cases.

Suppose first that $O\ O$ occurs infinitely often above. Then one or other of the systems

$$\begin{pmatrix} O & O \\ O & E \end{pmatrix}, \quad \begin{pmatrix} O & O \\ E & O \end{pmatrix}$$

must occur infinitely often. If the first, which is system 2°, either favourable cases recur infinitely often, or the ensuing partial quotient is always 1. We represent this state of affairs by the symbol

$$\left.\begin{matrix} O & O \\ O & E \end{matrix}\right|.$$

In this case our diagram continues

$$\left.\begin{matrix} O & O \\ O & E \end{matrix}\right|\begin{matrix} E \\ O \end{matrix};$$

and as $\begin{pmatrix} O & E \\ E & O \end{pmatrix}$ is case 5°, either favourable cases recur continually, or the next quotient is also 1, so that we have

$$\left.\begin{matrix} O & O \\ O & E \end{matrix}\right|\left.\begin{matrix} E \\ O \end{matrix}\right|.$$

But then the first four letters represent a system of type 2° followed by two quotients $a_{n+1} = 1$, $a_{n+2} = 1$; and this is a favourable case. Thus if $\begin{pmatrix} O & O \\ O & E \end{pmatrix}$ recurs continually, favourable cases recur continually.

We consider next the result of supposing that $\begin{pmatrix} O & O \\ E & O \end{pmatrix}$ recurs continually. This is case 4°. If the diagram continues with an O above, it must continue in the form

$$\begin{matrix} O & O & O \\ E & O & E \end{matrix}$$

and then we can repeat our previous argument. The only alternative is that it should continue

$$\begin{array}{ccc} O & O & E \\ E & O & O \end{array}$$

— and as the last four letters form a system of type 6°, the next quotient must (in the unfavourable case) be 1. Hence we obtain

$$\begin{array}{ccc|c} O & O & E & O \\ E & O & O & E \end{array}.$$

The next quotient must also be 1; and so the system of type 6° gave in reality a favourable case.

We have thus proved that, whenever the succession OO recurs continually above, we obtain an infinity of favourable cases. It only remains to consider the hypothesis that p_n is alternately odd and even.

If we have OE above, we have one or other of the systems $\begin{pmatrix} O & E \\ O & O \end{pmatrix}$, $\begin{pmatrix} O & E \\ E & O \end{pmatrix}$; systems 5° and 6°. Thus we have a favourable case unless $a_{n+1} = 1$. If the system is of type 5°, we are led to

$$\begin{array}{cc|c|} O & E & O \\ O & O & E \end{array}$$

— so that the system is favourable. On the other hand, if it is of type 6°, we are led to

$$\begin{array}{cc|c} O & E & O \\ E & O & O \end{array}.$$

As the next numerator is even, the next denominator is odd. Hence the next system is $\begin{pmatrix} O & E \\ O & O \end{pmatrix}$, and we have seen that this case must be favourable.

We have now examined all possible hypotheses, and found that they all involve the continual recurrence of favourable cases. Thus Theorem **2. 31** is established.

2. 33. From this theorem we can, as was explained in 2. 31, deduce Theorem **2. 30** as a corollary. The latter theorem has an interesting consequence which we have not seen stated explicitly.

The series

$$\sum n^{-a} \cos (n^2 \pi x), \quad \sum n^{-a} \sin (n^2 \pi x),$$

where $a \leq \frac{1}{2}$, *are not* FOURIER'S *series.*

For if they were they would be summable $(C\,1)$ almost everywhere, by a theorem of LEBESGUE.[1] It follows that trigonometrical series exist, such that

$$\sum (|a_n|^{2+\delta} + |b_n|^{2+\delta})$$

is convergent for every positive δ,[2] which are not FOURIER'S series. This is of interest for the following reason. If $\sum (a_n^2 + b_n^2)$ is convergent, the series is the FOURIER'S series of a function whose square is summable.[3] Further if p is any odd integer, and

$$\sum \left(|a_n|^{1+\frac{1}{p}} + |b_n|^{1+\frac{1}{p}}\right)$$

is convergent, then the function has its $(1 + p)$-th power summable.[4] It would be natural to suppose that the RIESZ-FISCHER Theorem might be capable of extension in the opposite direction. One might expect, for example, to find that a series for which

$$\sum (|a_n|^{1+p} + |b_n|^{1+p})$$

is convergent must be the FOURIER'S series of a function whose $\left(1 + \frac{1}{p}\right)$-th power is summable. That this is not true has been shown by YOUNG, by means of the series

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[1] *Math. Annalen,* Vol. 61, p. 251. See also *Leçons sur les sériès trigonométriques,* p. 94 where however the proof is inaccurate. A FOURIER'S series is in fact summable $(C\,\delta)$, for any positive δ, almost everywhere (HARDY, *Proc. Lond. Math. Soc.,* Vol. 12 p. 365). That our series are not FOURIER'S series when $a < \frac{1}{2}$ can in fact be inferred merely from their non-convergence, since to replace n^{-a} by $n^{-\beta}$, where β is any number greater than a, would, if they were FOURIER'S series, render them convergent almost everywhere (YOUNG, *Comptes Rendus,* 23 Dec. 1912).

[2] Or even for which

$$\sum \frac{|a_n|^2 + |b_n|^2}{(\log n)^{1+\delta}}$$

is convergent.

[3] This is the 'RIESZ-FISCHER Theorem'.

[4] W. H. YOUNG, *Proc. Lond. Math. Soc.,* Vol. 12, p. 71.

$$\sum \frac{\cos nx + \sin nx}{n^{\frac{1}{4}} (\log n)^{\frac{1}{2}}}$$

— here $p = 3$. Our examples however show a good deal more, viz. that as soon as the 2 which occurs in the RIESZ-FISCHER Theorem is replaced by any higher index, the series ceases to be necessarily a FOURIER's series at all.

2. 34. There are other classes of series the theory of which resembles in many respects that of the series studied in this paper. One such class comprises such series as

$$\sum \operatorname{cosec} n\pi x, \quad \sum (-1)^n \operatorname{cosec} n\pi x$$

and the corresponding series in which the cosecant is replaced by a cotangent: these series are limiting forms of q-series such as

$$\sum \frac{q^n}{1 - q^{2n}}.$$

Another class comprises the series

$$\sum \left\{ (nx) - \frac{1}{2} \right\}, \quad \sum (-1)^n \left\{ (nx) - \frac{1}{2} \right\}$$

and the corresponding series in which $(nx) - \frac{1}{2}$ is replaced by $\overline{nx}$. We have proved a considerable number of theorems, relating to these various series, of which we hope to give a systematic account on some future occasion.

CORRECTIONS

p. 211 (2.132) and p. 226 (fourth displayed formula). Read:

$$x = \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots.$$

p. 231. The fourth root in the first displayed formula should be a square root.

COMMENTS

The basic result of the paper, Theorem 2.128, was later named *the approximate functional equation of the theta-function.* It is not, however, of the same character as the approximate functional equation of the Riemann zeta-function, discovered later by Hardy and Littlewood (1923, 5). The former is an approximate transformation of one finite sum into another, the latter is an approximate expression for $\zeta(\sigma+it)$, when $0 \leqslant \sigma \leqslant 1$, as the sum of two finite sums.

The original proof of this result, given in the present paper, is unnecessarily elaborate. Simpler proofs were given later by Hardy and Littlewood themselves (1925, 4), by Mordell (*J. London Math. Soc.* 1 (1926), 68–72), and by Wilton (ibid. 2 (1927), 177–80). Mordell uses contour integration round a parallelogram, but with the vertical sides replaced by sides inclined at an angle $\frac{1}{4}\pi$ to the real axis. Wilton uses Poisson's summation formula

$$\sum_{n=a}^{b} f(n) = \sum_{\nu=-\infty}^{\infty} \int_a^b f(x)e^{2\pi i\nu x}\,dx$$

in much the same way as Dirichlet used it in his evaluation of Gauss's sum. Wilton also gives a numerical estimate for the constant in the error term.

§ 2.10. The transformation formula for a sum of the form

$$\sum e^{n^k \pi i x}$$

(see 2.102) expresses this sum in terms of a similar sum with k replaced by K, where $1/k+1/K = 1$. If k is a positive integer greater than 2, the transformation does not give any useful information about the magnitude of the original sum. A very general transformation formula, applicable to a wide range of sums of the type

$$\sum g(n)e^{2\pi i f(n)},$$

was given by van der Corput (*Math. Annalen*, 87 (1922), 66–77, and 90 (1923), 1–18). For a somewhat simpler treatment, see Wilton, *J. London Math. Soc.* 9 (1934), 194–201 and 247–54. For an account of the part played by such formulae in some problems of analytic number theory, see Rankin, *Quart. J. of Math.* (2) 6, (1955), 147–53.

For references to later work on sums of the form $\sum e^{n^2\pi i x}$, see Koksma, ch. 9, § 3.

§ 2.120. The exact formula (2.1202) is attributed (following Lindelöf) to Genocchi and Schaar. But it is an easy deduction from the value of Gauss's sum, and may well have been known to Gauss or Dirichlet.

§ 2.13. In the estimate (2.138), the last term can be omitted, by virtue of (2.136).

It may be helpful to restate (2.138) and (2.1381) in the usual symbolism of continued fractions. If

$$\frac{p_n}{q_n} = \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n},$$

the choice of ν made in (2.136) is effectively equivalent to

$$q_\nu \leqslant n < q_{\nu+1} \tag{1}$$

(it is actually equivalent to $q'_\nu \leqslant n < q'_{\nu+1}$, where q'_ν is the 'complete quotient', as defined in

§ 2.22) and instead of (2.138) and (2.1381) we get

$$s_n = O(\min(nq_\nu^{-\frac{1}{2}}, q_{\nu+1}^{\frac{1}{2}})). \tag{2}$$

This renders the deduction of Theorems 2.14, 2.141, 2.142 more immediate.

§ 2.14. In Theorem 2.143, the constant $\log 2$ could be replaced, in both occurrences, by $\log \frac{1}{2}(3+\sqrt{5})$. The inequality

$$xx_1 \ldots x_\nu \leqslant 2^{-\frac{1}{2}\nu},$$

used by Hardy and Littlewood, could be replaced by a stronger inequality with $1/u_{\nu-1}$ on the right, where u_ν is the νth Fibonacci number. This leads to the above improvement.

Theorem 2.143 has the defect that it gives no information if ρ is large, and in particular it leaves open the question whether *any* postulated explicit upper bound for a_n as a function of n implies some explicit upper bound for s_n which is better than $o(n)$. This is in fact true. For suppose

$$a_n = O(\exp f(n)),$$

where $f(n)$ is any increasing function. Then

$$q_n = O(\exp C_1 F(n)),$$

where C_1 is a constant and $F(n) = f(1)+\ldots+f(n)$. It follows from (1) above that

$$\nu+1 > G(C_2 \log n),$$

where G is the function inverse to F. Hence, by (2) and the fact that $q_\nu > \exp(C_3 \nu)$ always, we have

$$s_n = O(nq_\nu^{-\frac{1}{2}}) = O(n \exp(-C_4 \nu)) = O(n \exp(-C_5 G(C_2 \log n))).$$

This is the desired explicit upper bound.

Further results of the type of this section were given in 1922, 5.

The dependence of $s_n(x,\theta)$ on θ was investigated by Behnke, who proved the curious result that for any irrational x there is some θ for which

$$s_n(x,\theta) = O(n^{\frac{3}{4}}).$$

See Koksma, p. 111.

§ 2.19. In connexion with this application to the theory of functions, see 1916, 3 and comments.

§ 2.3. Again see 1916, 3 and comments.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: A REMARKABLE TRIGONOMETRICAL SERIES

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1. The title of this note is perhaps not very appropriate: we retain it because the contents of the note form a natural sequel to those of three papers which we have published under same title elsewhere,[1] and in particular those of our second paper in the *Acta Mathematica*. We there discussed in detail the series

$$\sum e^{\alpha\pi i n^2 + 2\theta\pi i n}, \tag{1.1}$$

and other similar series associated with the elliptic Theta-functions, and used our results to elucidate a variety of difficult points in the theory

of Taylor's series and trigonometrical series. We have since discovered that even simpler and more elegant illustrations may be derived from the series

$$\sum e^{\alpha\pi in \log n + 2\theta\pi in} \tag{1.2}$$

This series behaves, for different values of the parameters α and θ, far more regularly than does the series (1.1). To put the matter roughly, the behaviour of the series *does not, in its most essential features, depend upon the arithmetic nature of* α.

2. Our fundamental formula is

$$\sum_0^\infty a^{\rho n} e^{-ya^n} = \frac{1}{\log a}\sum_{-\infty}^{\infty}\Gamma\left(\rho + \frac{2\pi in}{\log a}\right) y^{-\rho-\frac{2\pi in}{\log a}} - \sum_0^\infty \frac{(-y)^n}{n!\,(a^{\rho+n}-1)}. \tag{2.1}$$

Here $a > 1$, ρ is real, and $\mathsf{R}(y) > 0$. The formula becomes illusory when ρ is zero or a negative integer, but the alterations required are of a trivial character. The formula is easily proved by means of Cauchy's Theorem: similar formulae were proved by one of us in a paper published in 1907.[2]

We now write $y = \sigma + it$, where $t > 0$, suppose that $\sigma \to 0$, and approximate to the series of Gamma-functions by means of Stirling's Theorem. We thus obtain

$$\frac{Ht^{-\rho}}{\log a} e^{-\frac{1}{4}\pi i} f(z) = F(\sigma) + \phi(\sigma), \tag{2.2}$$

where

$$f(z) = \sum_1^\infty n^{\rho-\frac{1}{2}} e^{\alpha in \log n} z^n; \tag{2.21}$$

$$\alpha = \frac{2\pi}{\log a},\ H = \frac{(2\pi)^\rho}{(\log a)^{\rho+\frac{1}{2}}},\ z = re^{i\theta},\ r = e^{-\alpha\sigma/t},\ \theta = \alpha \log\left(\frac{\alpha}{et}\right), \tag{2.22}$$

so that $r \to 1$ when $\to \sigma$ 0;

$$F(\sigma) = \sum_0^\infty a^{\rho n} e^{-(\sigma+it)a^n}; \tag{2.23}$$

and $\phi(\sigma)$ is of one or other of the forms

$$A + o(1),\ O\left(\log \frac{1}{\rho}\right),\ O(\sigma^{-\rho+\frac{1}{2}}),$$

according as $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$, or $\rho > \frac{1}{2}$.

3. It is known[3] that, if $\rho > 0$,

$$F(\sigma) = O(\sigma^{-\rho}),\ F(\sigma) = \Omega(\sigma^{-\rho}), \tag{3.1}$$

when $\sigma \rightarrow 0$, the second of these formulae meaning[4] that $F(\sigma)$ is *not* of the form $o(\sigma^{-\rho})$, and the two together that

$$0 < h = \overline{\lim}\, \sigma^{\rho} F(\sigma) < \infty. \tag{3.2}$$

These relations all hold uniformly in t. It follows that, if $\rho > 0$ and $r = |z| \rightarrow 1$, *the function* $f(z)$ *is exactly of the order* $(1 - r)^{-\rho}$, *and this uniformly in* θ. Incidentally, of course, it follows that every point of the unit circle is a singular point: but this is known already.[5]

The series furnishes an example in which the orders in the unit circle of the functions $f(z) = \Sigma\, a_n z^n$ and $g(z) = \Sigma\, |a_n|\, z^n$ differ by exactly $\frac{1}{2}$, the maximum possible.[6]

When $\rho = 0$, $f(z)$ is bounded, but does not tend to a limit when z approaches any point of the unit circle along a radius vector. We know of no other example of a function possessing this property. When $\rho < 0$, $f(z)$ is continuous for $|z| \leq 1$.

4. Let

$$s_n = \sum_{1}^{n} k^{\rho - \frac{1}{2}} e^{\alpha i k \log k + 2\theta\pi i k}; \tag{4.1}$$

and suppose first that $\rho > 0$. Then it is easy to deduce from the results of §3 that s_n is of the form $\Omega(n^{\rho})$ when $n \rightarrow \infty$. The corresponding 'O' result lies a little deeper: all that can be proved in this manner is[7] that $s_n = O(n^{\rho} \log n)$. But a direct investigation, modelled on that of the early part of our second paper in the *Acta Mathematica*, shows that the factor $\log n$ may be omitted. It should be observed that an essential step in our argument depends on an important lemma due to Landau,[8] according to which

$$\left| \int_{1}^{X} x^{\gamma} e^{ix \log (\eta x)}\, dx \right| < 23\, X^{\gamma + \frac{1}{2}} \tag{4.2}$$

for $X \geq 1$, $\gamma \geq 0$, $\eta > 0$. We thus find that s_n *is, for every positive value of* α, *exactly of the order* n^{ρ}, *and this uniformly in* θ. *The series*

$$\sum n^{\rho - \frac{1}{2}} e^{\alpha i n \log n + 2\theta\pi i n} \tag{4.3}$$

is never convergent, or summable by any of Cesàro's means.

When $\rho = 0$, s_n is bounded, but the series is never convergent or summable. When $\rho < 0$ it is convergent; and uniformly in θ.

5. For further applications it is necessary to consider the real and imaginary parts of our function and series separately, and this is most easily effected by introducing some restriction as to the value of α. Suppose that a is an integer, not of the form $4k + 1$. Thus we may take $a = 2$, $\alpha = 2\pi / \log 2$. Then the results of §§3–4 hold for the real and

imaginary parts of the function or the series. In particular *the series*

$$\sum n^{\rho-\frac{1}{2}} \cos (\alpha n \log n + 2\theta\pi n) \quad (\rho \geq 0) \tag{5.1}$$

is never convergent or summable for any value of θ, *and is accordingly not a Fourier's series.* We thus obtain a solution of what, in our former paper, we call Fatou's[9] problem which combines all the advantages of those given previously by Lusin,[9] Steinhaus,[9] and ourselves.

We can also obtain in this manner exceedingly elegant examples of continuous non-differentiable functions. Thus *the function*

$$f(\theta) = \sum \frac{\sin (\alpha n \log n + 2\theta\pi n)}{n^{\beta}} \quad (1 < \beta \leq \tfrac{3}{2}) \tag{5.2}$$

does not possess a finite differential coefficient for any value of θ.

[1] G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation: (i) *Proc. Fifth Int. Congress Math.*, Cambridge, **1**, 223–229 (1912); (ii) *Acta Math.*, **37**, 155–190 (1914); (iii) *Ibid.*, 193–238.

[2] G. H. Hardy, On certain oscillating series, *Quarterly J. Math.*, **38**, 269–288 (1907).

[3] G. H. Hardy, Weierstrass's non-differentiable function, *Trans. Amer. Math. Soc.*, **17**, 301–325, (1916).

[4] *l. c. supra* (1) (iii), p. 225.

[5] G. N. Watson, The singularities of functions defined by Taylor's series, *Quarterly J. Math.*, **42**, 41–53 (1911).

[6] G. H. Hardy: (i) A theorem concerning Taylor's series, *Ibid.*, **44**, 147–160 (1913); (ii) Note in addition to a theorem on Taylor's series, *Ibid.*, **45**, 77–84 (1914).

[7] Cf. E. Landau, Abschätzung der Koeffizientensumme einer Potenzreihe: (i) *Arch. Math. Physik*, ser. 3, **21**, 42–50 (1913); (ii) *Ibid.*, 250–255; (iii) *Ibid.*, **24**, 250–260 (1915).

[8] E. Landau, Uber die Anzahl der Gitterpunkte in gewissen Bereichen, *Göttinger Nachrichten*, 687–771 (p. 707), (1912).

[9] For references see p. 232 of our paper (1) (iii).

CORRECTION

The statements at the end of § 3 and of § 4 about the case $\rho = 0$ are incorrect: see the end of 1916, 9.

COMMENTS

The interest of this paper is almost entirely analytical, and (as the authors remark in the first sentence) its inclusion under the general title 'Some problems of Diophantine approximation' is not entirely appropriate.

The substance of §§ 1–3 is given again, in somewhat more detail, in Littlewood's *Lectures on the theory of functions* (Oxford, 1944), pp. 99–102.

The substance of §§ 4–5, with further results, is given in ch. 5 of Zygmund's *Trigonometric series* (2nd ed., Cambridge, 1959). See also the references on p. 379 of Zygmund.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: THE SERIES $\Sigma e(\lambda_n)$ AND THE DISTRIBUTION OF THE POINTS $(\lambda_n \alpha)$

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Communicated by E. H. Moore, December 5, 1916

1. In our previous writings on the subject of Diophantine approximation, which we refer to in a short note published in the October number of these PROCEEDINGS,[1] we alluded in several places to a series of further results which, we hoped, were to form the material for a third memoir in the *Acta Mathematica*. The prosecution of this work was delayed, in the first instance, by our occupation on a long memoir on the theory of the Riemann Zeta-function, now in type and shortly to appear there, and subsequently by other causes; and there is, under present conditions, little hope of its completion in the immediate future. The subject has since been reopened by the appearance of work by other writers,[2] and in particular of a very beautiful memoir by Weyl in the latest number of the *Mathematische Annalen*.[3] This paper contains allusions to our unpublished work: and it seems desirable that we should make some more definite statement than has appeared hitherto of our results and the relations in which they stand to Weyl's.

The main problems which we considered were three.

2. (*a*) The first problem was that of proving that, *if* $e(x) = e^{2\pi i x}$ *and*

$$\lambda_n = \alpha n^k + \alpha_1 n^{k-1} + \dots + \alpha_k$$

is a polynomial in n with at least one irrational coefficient, then

$$s_n = \sum_1^n e(\lambda_h) = o(n).$$

We may plainly suppose that every α has been reduced to its residue to modulus unity: and there is no substantial loss of generality in supposing the *first* coefficient irrational.

This theorem we enunciated first, in the special case in which $\lambda_n = \alpha n^k$, in our communication to the Cambridge Congress, characterising the proof as 'intricate.' In our second memoir in the *Acta* we discussed in detail the case $k = 2$, using a transcendental method which leads to a whole series of more precise results; and we promised a proof of the more general theorem in the third memoir of the series. Weyl's memoir contains a complete statement and proof, both quite independent of ours, of the theorem in its most general form.

The limitation on the form of λ_n, which appears in the theorem as we stated it, was introduced merely for the sake of compactness of expression and does not correspond to any real simplification of the problem. Our argument indeed depends upon an induction which compels us to consider the problem generally. The most comprehensive result which appears in our analysis is as follows: *given any positive numbers* ϵ *and* η *we can determine* $\nu(\epsilon)$, $N(\epsilon, \eta)$, *and a system of intervals* j, *including all rationals whose denominators are less than* ν, *and of total length less than* η, *so that* $|s_n| < \epsilon n$ *for* $n > N$, *all values of* α *exterior to the intervals* j, *and all values of* $\alpha_1, \alpha_2, \ldots, \alpha^k$. From this result it follows at once that $s_n = o(n)$ for any particular irrational α, and uniformly in $\alpha_1, \alpha_2, \ldots, \alpha^k$.

Weyl's proof and ours are widely different, and each, we hope, may prove to have an interest of its own. The same is true of the deduction of the formula $\zeta(l + it) = o(\log t)$, made by Weyl as well as by ourselves.

3. (*b*) The second principal problem was, to use Weyl's phraseology, that of the 'uniform distribution' (*Gleichverteilung*) of the points (λ_n) where (x) is the residue of x to modulus unity. Suppose that n_j is the number of the first n such points which fall within an interval j of length δ. Then the points are said to be *uniformly distributed* if $n_j \backsim \delta n$ for every such interval j. It is plain that a corresponding definition may be given of uniform distribution of an enumerable sequence of points in space of any number of dimensions.

That the points (λ_n) are uniformly distributed when $k = l$ and α is irrational was proved independently by Bohl, Sierpinski, and Weyl in 1909–10. The general result (with the same unessential limitation as to the form of λ_n) was stated by us in our first paper in the *Acta*. Our proof, which has never been published, proceeded on the same lines as that of the thoerem of §2. But Weyl has now established a 'principle' which renders such a proof entirely unnecessary, and which has led him to results in this direction far more comprehensive than any of ours. This 'principle' is expressed by the theorem: *if*

$$\sum_1^n e(m\lambda_h) = o(n)$$

for every positive integral value of m, *then the points* (λ_n) *are uniformly distributed in* (0, 1). The proof depends on a simple but ingenious use of the theory of approximation to arbitrary functions by finite trigonometrical polynomials; and there is a straightforward generalisation to space of any number of dimensions.

Weyl's 'principle' enables him to deduce, with singular ease and elegance, theorems of 'uniform distribution' from theorems of the character of that of §2, and to generalise them immediately to multidimensional space. It enables him to prove, for example, that *the points whose coordinates are*

$$(np\ \alpha_q)\ (p = 1, 2, \ldots, k;\ q = 1, 2, \ldots, l;\ n = 1, 2, 3, \ldots)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_l$ is any set of linearly independent irrationals, are uniformly distributed in the 'unit cube' of kl dimensions. All that we had been able to prove was that the points were *everywhere dense* in the cube.

4. (*c*) Corresponding questions arise in connection with an arbitrary increasing sequence $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots$ Are the points $(\lambda_n\ \alpha)$, for example, uniformly distributed? The answers to such questions in general involve an unspecified exceptional set of values of α of measure zero, instead of (as when $\lambda_n = n^k$) a specified set such as the rationals; they are, in other words, only 'almost always' true.

In our first paper in the *Acta* we proved quite generally that the set $(\lambda_n\ \alpha)$ is almost always everywhere dense. The corresponding theorem of 'uniform distribution' we discussed only in one especially interesting particular case, that in which $\lambda_n = a^n$, where a is an integer. The theorem is in this case substantially equivalent to results obtained by Borel,[4] from the standpoint of the theory of probabilities, and by Faber,[5] as a corollary of Lebesgue's theorem that a rectifiable curve has a tangent at almost every point. Our analysis however contains the first direct and general discussion of the problem, and leads to results notably more precise than that of mere uniformity of distribution. These results were afterwards made the subject of important generalisations by Fowler,[2] whose investigations covers all cases in which λ_n increases with tolerable regularity and as fast as an exponential of the type e^{n^ρ}. Weyl's 'principle' enables him to reduce this problem to a study of the series $\Sigma e\,(\lambda_n\ \alpha)$, and leads him to the following theorem, so far the most general of its kind. *If $c > 0$, $\delta > 0$, and λ_n increases by at least c whenever n increases from h by as much as h $(\log h)^{-1-\delta}$, then*

$$s_n = \sum_1^n e\,(\lambda_k\alpha) = o\,(n), \tag{1}$$

and the points $(\lambda_n\ \alpha)$ are uniformly distributed, for almost all values of α.

In our second paper in the *Acta* we stated that the equation could, in very many cases, be replaced by the much more precise equation

$$s_n = O\,(n^{\frac{1}{2}+\epsilon}) \tag{2}$$

for every positive ϵ. The publication of Weyl's work had led us to a re-examination of this question and to the following theorems.

A. *If* (i) $\lambda_{n+[n\epsilon]} - \lambda_n \to \infty$,

(ii) $|a_1|^2 + |a_2|^2 + \ldots + |a_n|^2 = O(n^{1+\epsilon})$,

for every positive ϵ, *then*

$$\text{(iii)} \qquad s_n(\alpha) = \sum_1^n a_k e(\lambda_k \alpha) = O(n^{\frac{1}{2}+\epsilon})$$

for almost all α*'s and every positive* ϵ.

B. *If* (i) *is replaced by* (i') $\lambda_{n+[n^{\beta+\epsilon}]} - \lambda_n \to \infty$, *where* $0 < \beta < 1$, *then* (iii) *may be replaced by*

$$\text{(iii')} \qquad s_n(\alpha) = O\left(n^{\frac{1+\beta}{2}+\epsilon}\right).$$

To these two theorems Weyl's forms a completing third. It should be observed that (i) is certainly satisfied if $\lambda_{n+1} - \lambda_n \geq c > 0$, and in particular if λ_n is always integral, and (ii) if $a_n = O(n^\epsilon)$, and in particular if $a_n = 1$.

If λ_n is an integer, and we separate the real and imaginary parts in the equation (iii), we obtain a theorem concerning a particular system of normal orthogonal functions for the interval (0, 1), viz., the functions $\sqrt{2}\cos 2\pi\lambda_n x$, $\sqrt{2}\sin 2\pi\lambda_n x$. Our argument is then directly extensible to a general orthogonal system, and we are led to a new and interesting proof of Hobson's[6] theorem that *if* $\phi_n(x)$ *is any normal orthogonal* system, and $\Sigma n^\delta |c_n|^2$ *is convergent for some positive* δ, *then* $\Sigma c_n\phi_n(x)$ *is convergent almost everywhere.*

Weyl's hypothesis concerning λ_n asserts, roughly, that the increase of λ_n is appreciably more rapid than that of $(\log n)^2$. It is easy to see that this hypothesis cannot be capable of *much* wider generalisation. For, when $\lambda_n = \log n$, s_n is definitely of order n. It seems probable, too, that the index $\frac{1}{2}(1+\beta)$ of Theorem *B* is the correct one.

5. We conclude by correcting an error in our recent note. The results concerning the special case $\rho = 0$ are stated wrongly. It is not true that, when $\rho = 0$, $f(z)$ and s_n are bounded; all that we can assert is that they are of the forms $O\left(\log\frac{1}{1-r}\right)$ and $O(\log n)$ respectively. That $f(z)$ should be bounded would contradict a general theorem of Fatou,[7] in virtue of which a bounded function must tend to a limit, for almost all values of θ, when $z = re^{i\theta}$ tends to the circle of convergence along a radius vector. The error has no bearing on the general case.

[1] Hardy, G. H., and Littlewood, J. E., these PROCEEDINGS, **2**, 1916, (583–586).

[2] Berwick, W. E. H., *Mess. Math., Cambridge,* **45**, 1916, (154–160); Fowler, R. H., *London, Proc. Math. Soc.*, (Ser. 2), **14**, 1915, (189–207); Kakeya, S., *Tôhoku Sci. Rep. Imp. Univ.*, **2**, 1913, (33–54) and *Ibid.*, **4**, 1915, (105–109).

[3] Weyl, H., *Math. Ann., Leipzig,* **77**, 1916, (313–352); see also *Göttingen Nachr. Ges. Wiss.*, 1914, (234–244).

[4] Borel, E., *Palermo, Rend. Circ. Mat.*, **27**, 1909, (247–271); see also notes to Borel, E., *Leçons sur la théorie des fonctions*, 2d. ed., Paris.

[5] Faber, G., *Math. Ann., Leipzig*, **69**, 1910, (372–443), especially p. 400.

[6] Hobson, E. W., *London, Proc. Math. Soc.*, (Ser. 2), **12**, 1912, (297–308).

[7] Fatou, P., *Acta Math., Stockholm,* **30**, 1906, (335–400), especially p. 349.

CORRECTIONS

p. 85. In the last line but one of the second paragraph, read $\zeta(1+it)$.
In the first line of the last paragraph, read $k = 1$.

p. 86. In the first displayed formula, read $n^{\nu}\alpha_q$.

COMMENTS

§ 4. It does not appear that proofs of Theorems A and B were ever published.

For some results, valid for almost all α, when the λ_n are integers, see Koksma, pp. 94–95.

A Problem of Diophantine Approximation

By G. H. Hardy

I have on several occasions found myself faced by the following problem, which appears to be one of considerable interest and difficulty. Suppose that a and θ are positive and $a > 1$, and that (x) denotes the difference between x and the integer nearest to x. Then, *in what circumstances can it be true that*

$$(a^n\theta) \to 0, \tag{1}$$

when $n \to \infty$, *i.e. that*

$$a^n\theta = p_n + \epsilon_n \tag{2}$$

where p_n *is an integer and* $\epsilon_n \to 0$?

The general problem seems, as I said, to be one of great difficulty. There is, however, one case in which the answer is almost immediate, namely, that in which a is an *algebraic* number. We have in fact the following theorem:

Theorem A. *Suppose that a is a real algebraic number greater than 1, the root of an irreducible equation*

$$k_0 a^m + k_1 a^{m-1} + \ldots + k_m = 0, \tag{3}$$

where $k_0, k_1, \ldots, k_m$ are integers. Then, in order that numbers θ should exist which satisfy (1), *it is necessary and sufficient that $k_0 = 1$, so that a is an algebraic integer, and that the moduli of all the roots of* (3), *other than a itself, should be less than* 1. *The numbers θ are then all rational in the corpus of a; and*

$$(a^n\theta) = O(b^n), \tag{4}$$

where b is the numerically largest root of (3), *other than a itself.*

We have

$$\frac{\theta}{1-ax} = \sum p_n x^n + \sum \epsilon_n x^n = p(x) + \epsilon(x),$$

say; and

$$\frac{\theta}{1-ax}(k_0 + k_1 x + \ldots + k_m x^m) = \sum q_n x^n + \sum \zeta_n x^n, \tag{5}$$

where

$$q_n = k_0 p_n + k_1 p_{n-1} + \ldots + k_m p_{n-m},$$
$$\zeta_n = k_0 \epsilon_n + k_1 \epsilon_{n-1} + \ldots + k_m \epsilon_{n-m},$$

p's and ϵ's with negative suffixes being regarded as equal to 0. The

1919, 4, *Journal of the Indian Mathematical Society*, 11, 162–6.

left hand side of (5) is a polynomial of degree $m-1$, so that

$$q_n+\zeta_n = 0 \quad (n \geqslant m).$$

As q_n is an integer, and $\zeta_n \to 0$, we must have

$$q_n = 0, \qquad \zeta_n = 0,$$

from a certain value of n onwards. Thus $p(x)$ and $\epsilon(x)$ are rational functions, with the denominator

$$k_0+k_1 x+\ldots+k_m x^m.$$

It is evident that $p(x)$ can be expressed in the form

$$p(x) = \mathrm{p}(x)+\frac{l_0+l_1 x+\ldots+l_{m-1} x^{m-1}}{k_0+k_1 x+\ldots+k_m x^m},$$

where $\mathrm{p}(x)$ is a polynomial and l_0, $l_1,\ldots$ are integers. From this it follows, in the first place, that $k_0 = 1$, so that a is an algebraic integer.*

Now, let us denote the roots of (3) by

$$a,\ a_1,\ a_2,\ldots,\ b_1,\ b_2,\ldots,$$

the a's having moduli not less than 1, and the b's moduli less than 1. Then

$$p(x) = \mathrm{p}(x)+\frac{A}{1-ax}+\sum \frac{A_k}{1-a_k x}+\sum \frac{B_k}{1-b_k x}, \tag{6}$$

the A's and B's being rational in the corpus of a and none of them being zero. On the other hand $\epsilon(x)$, since its coefficients tend to zero, can have no pole inside or on the circle $|x| = 1$, and so

$$\epsilon(x) = \mathrm{e}(x)+\sum \frac{B'_k}{1-b_k x}, \tag{7}$$

where $\mathrm{e}(x)$ is a polynomial. From (6) and (7) we obtain

$$\frac{\theta}{1-ax} = \mathrm{p}(x)+\mathrm{e}(x)+\frac{A}{1-ax}+\sum \frac{A_k}{1-a_k x}+\sum \frac{B_k+B'_k}{1-b_k x},$$

so that $\qquad A = \theta, \qquad A_k = 0, \qquad B_k+B'_k = 0.$

But we have already seen that no A_k can vanish. It follows that there can be no root of (3) of the type a_k.

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* If $$\frac{l_0+l_1 x+\ldots+l_{m-1} x^{m-1}}{k_0+k_1 x+\ldots+k_m x^m}$$
is expansible in a Taylor's series with integral coefficients, k_0 must be 1. For a proof of this well-known proposition see, e.g., P. Fatou, 'Séries trigonométriques et séries de Taylor', *Acta Mathematica*, vol. 30, 1906, pp. 335–400 (p. 369).

The conditions stated in Theorem A, with reference to the equation (3), are therefore necessary. Also $\theta = A$ is rational in the corpus of a, and ϵ_n, which is, from a certain value of n onwards, equal to $\sum B'_k b_k^n$, is of the form $O(b^n)$, where b is the numerically greatest root of (3), other than a itself.

In order to complete the proof of the theorem, it is only necessary to show that, if (3) satisfies the conditions stated, θ's exist which satisfy (1). And this is obvious; for

$$a^n + \sum b_k^n$$

is integral for all values of n so that

$$(a^n) = O(b^n).$$

Thus (1) is satisfied when $\theta = 1$.

2. Theorem A is a special case of a more general theorem:

THEOREM B. *Suppose that a_j $(j = 1, 2, \ldots, r)$ is an algebraic number, greater than* 1 *in absolute value, and the root of an irreducible equation*

$$k_{0,j}\, a_j^{m_j} + k_{1,j}\, a_j^{m_j-1} + \ldots + k_{m_j,j} = 0; \tag{8}$$

that $P_j(n)$ is a polynomial with integral coefficients; and that

$$\phi(n, \theta) = P_1(n) a_1^n \theta_1 + P_2(n) a_2^n \theta_2 + \ldots + P_r(n) a_r^n \theta_r.$$

Then, in order that it should be possible to find a system of numbers θ_1, θ_2, ..., θ_r, all different from zero, for which

$$(\phi(n, \theta)) \to 0$$

when $n \to \infty$, it is necessary and sufficient that $k_{0,j} = 1$ for each value of j, so that a_j is an algebraic integer, and that, among the complete system of roots of all the equations (8), *all save a_1, a_2,..., a_r themselves should be in absolute value less than* 1. *Each θ_j is then rational in the corpus of the corresponding a_j, and*

$$(\phi(n, \theta)) = O(n^c b^n),$$

where c is a constant and b is the numerically greatest among the roots of the equations (8), *other than a_1, a_2,..., a_r themselves.*

The proof of this theorem does not differ in principle from that of Theorem A, the role of the polynomial

$$k_0 + k_1 x + \ldots + k_m x^m$$

being now played by

$$K(x) = \prod_{j=1}^{r} (k_{0,j} + k_{1,j}\, x + \ldots + k_{m_j,j}\, x^{m_j})^{s_j+1}$$

where s_j is the degree of $P_j(n)$. The special theorem is, from our present point of view, more interesting than the general one, and I shall therefore confine myself to enunciating the latter.

3. With Theorem A should be associated another theorem which is merely a special case of a theorem already proved by Borel.**

THEOREM C. *If a and θ are positive, and $a > 1$, and*

$$(a^n\theta) = O(b^n),$$

where $0 < b < 1$, then a is algebraic.

Combining Theorems A and C we obtain an interesting criterion for the transcendentality of a, viz.

THEOREM D. *If $a > 1$, $\theta > 0$, and*

$$(a^n\theta) \to 0,$$

then a is algebraic or transcendental according as the equation

$$(a^n\theta) = O(e^{-\delta n})$$

is or is not true for some positive value of δ.

4. It is interesting to consider in more detail the two simplest cases of Theorem A.

In the first place, suppose $m = 1$, so that a is rational. If then (1) is satisfied, a must be an integer, and $a^n\theta$ must be an integer for sufficiently large values of n. Thus the only solutions are given by

$$\theta = p/a^m,$$

where p is an integer.†

The next case is that in which $m = 2$. Then the equation satisfied

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** E. Borel, 'Sur une application d'un théorème de M. Hadamard', *Bulletin de la Société Mathématique de France*, ser. 2, vol. 18, 1894, pp. 22–25. Borel's theorem is really (when stated in slightly different language) that which corresponds to B as C corresponds to A.

The criterion is a simple one: to find an application of it is quite another matter; and I know of no example of a transcendental a which satisfies (1) for any value of θ.

† This trivial case of the theorem is of interest in connexion with Weierstrass's function. [See G. H. Hardy, 'Weierstrass's non-differentiable function', *Transactions of the American Mathematical Society*, vol. 17, 1916, pp. 301–25.]

by a is of the form $$x^2-mx-n = 0;$$
and it will easily be verified that, in order that this equation should have two real roots a and b, such that $a > 1$, $-1 < b < 1$, it is necessary and sufficient that
$$m \geqslant 1, \qquad 1-m < n < 1+m.$$
The simplest case is that in which $m = 1$, $n = 1$, and
$$a = \tfrac{1}{2}(\surd 5+1), \qquad b = \tfrac{1}{2}(-\surd 5+1).$$
The determination of the corresponding values of θ presents no difficulty. We have in fact

THEOREM E. *If a is a real quadratic surd greater than* 1, *and values of θ exist which satisfy* (1), *then*
$$a^2-ma-n = 0,$$
where $m \geqslant 1$, $1-m < n < 1+m$. The corresponding values of θ are the numbers
$$\left\{p+\frac{q}{\surd(m^2+4n)}\right\}a^{-r}$$
if m is odd, and the numbers
$$\left\{p+\frac{q}{\surd(\frac{1}{4}m^2+n)}\right\}a^{-r}$$
if m is even: here r is zero or a positive integer, and p and q halves of integers; and, when m is odd, p and q are either both integers or both halves of odd integers.

When $a = \frac{1}{2}(\surd 5+1)$ the simplest values of θ are
$$\theta = 1, \qquad \theta = \surd 5.$$

COMMENTS

This is the only paper on Diophantine approximation which was written by Hardy without Littlewood's collaboration.

In the footnote on p. 163 it is tacitly supposed that $k_0, \ldots, k_m$ have no common factor. For the result quoted there, see also Pólya and Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Section VIII, Problem 156.

The algebraic integers α (or a) occurring in Theorem A, that is, those $\alpha > 1$ whose algebraic conjugates α' all satisfy $|\alpha'| < 1$, have since been called the Pisot–Vijayaraghavan numbers; though it will be seen from this paper that their most important property was discovered by Hardy.

Vijayaraghavan† gave various further properties, the main one being that Theorem A still applies if the sequence $\alpha^n\theta$ has any finite number of limit points (mod 1) instead of just 0.

Theorem C can be regarded as a first step towards the more precise theorem of Pisot (1946) that if

$$\sum_{n=1}^{\infty} (\alpha^n\theta)^2$$

converges, where α, θ are real numbers, $\alpha > 1$ and $\theta \neq 0$, then α is necessarily algebraic and of the type defined in Theorem A. For an account of this and other work, and references, see ch. 8 of Cassels's Tract. For further references, see Pisot, *Journal für Math.* 209 (1962), 82–83.

† For references, see *J. London Math. Soc.* 33 (1958), 252–5.

Some problems of Diophantine approximation: A further note on the trigonometrical series associated with the elliptic theta-functions. By Prof. G. H. HARDY and Mr J. E. LITTLEWOOD.

[*Received* 6 July 1921.]

1. This note contains a short addition to a memoir, with a similar title, published in 1914 in the *Acta Mathematica**. In that memoir we considered the sums

$$s_n^2 = s_n^2(x, \theta) = \sum_{\nu \leqq n} e^{(\nu - \frac{1}{2})^2 \pi i x} \cos(2\nu - 1)\pi\theta,$$

$$s_n^3 = s_n^3(x, \theta) = \sum_{\nu \leqq n} e^{\nu^2 \pi i x} \cos 2\nu\pi\theta,$$

$$s_n^4 = s_n^4(x, \theta) = \sum_{\nu \leqq n} (-1)^\nu e^{\nu^2 \pi i x} \cos 2\nu\pi\theta,$$

where x and θ are real and x irrational†. There is plainly no real loss of generality in supposing either x or θ to be positive and less than unity, if it be understood that θ may be zero.

Our main results may be stated as follows. We denote by $s_n = s_n(x, \theta)$ any one of the sums s_n^2, s_n^3, s_n^4. Then, in the first place,

$$s_n = o(n) \qquad \text{.......................(1·1)},$$

for every irrational x, and uniformly in θ ‡. And this equation is a best possible equation of its kind; there is no function $\phi = \phi(n)$, tending to infinity with n, such that

$$s_n = O\left(\frac{n}{\phi}\right) \qquad \text{.......................(1·2)},$$

for every irrational x §.

* G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine Approximation', *Acta Mathematica*, vol. 37 (1914), pp. 193–238.

† The second and third sums reproduce one another when $\theta + \frac{1}{2}$ is written for θ. They are considered separately for the sake of formal symmetry in the analysis.

‡ p. 213 (Theorem 2·14). It should be observed that we there use s_n in the more restricted sense of $s_n(x, 0)$.

§ p. 225 (Theorem 2·221).

1922, 5 (with J. E. Littlewood) *Proceedings of the Cambridge Philosophical Society*, 21, 1–5.

On the other hand much more than (1·1) is true for special classes of values of x. In particular, if

$$x = \frac{1}{a_1 +} \frac{1}{a_2 + \ldots},$$

and the partial quotients a_n are bounded, then

$$s_n = O(\sqrt{n}) \quad \text{.......................(1·3)},$$

and again uniformly in θ*. And this result too is a best possible of its kind, for

$$s_n = o(\sqrt{n}) \quad \text{.......................(1·4)}$$

is false for $\theta = 0$ and *any* irrational x†.

2. There was one obvious gap in our former results. We did not give any simple criterion for distinguishing the classes of irrationals x for which

$$s_n = O(n^a) \quad \text{.......................(2·1)},$$

where a is an assigned number between $\frac{1}{2}$ and 1. The theorems of this character which we proved‡ were avowedly tentative and unsatisfactory. We did not even prove that *some* equation of the type (2·1) holds for every *algebraic* x. It is this gap which we propose to fill in the present note.

We denote by p_n/q_n a typical convergent to x, taking

$$\frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{a_1}, \quad \frac{p_2}{q_2} = \frac{a_2}{a_1 a_2 + 1}, \ldots;$$

and write, as in our former memoir,

$$x = \frac{1}{a_1 + x_1}, \quad x_1 = \frac{1}{a_2 + x_2}, \ldots,$$

$$a_1' = a_1 + x_1, \quad a_2' = a_2 + x_2, \ldots,$$

$$q_n' = a_n' q_{n-1} + q_{n-2}.$$

We shall say that an irrational x is *of class k* if

$$q_{n+1} < A q_n^k \quad \text{....................(2·2)},$$

where $A = A(x)$ is independent of n. We shall use A generally to denote a number of this kind, not the same in different formulae. If x is of class k, and $k < k'$, then x is of class k'.

If x is of class k,

$$|p_n - q_n x| = \frac{1}{q'_{n+1}} > \frac{A}{q_{n+1}} > \frac{A}{q_n^k}.$$

* p. 213 (Theorem 2·141).

† p. 225 (Theorem 2·22).

‡ p. 214 (Theorems 2·142, 2·143).

Further
$$|p - qx| > \frac{A}{q^k} \quad \text{.......................(2·3)}$$
for all positive integral values of p and q. Thus a number of class k might be defined as one for which (2·3) is true. If a_n is bounded, (2·2) is true with $k=1$, so that x is of class 1. In particular a quadratic surd is of class 1, and every algebraic number is of finite class.

We shall now prove

THEOREM A. *If x is of class k then*
$$s_n = O(n^a), \quad a = \frac{k}{k+1} \quad \text{.................(2·4),}$$
and uniformly in θ.

In particular we have, as a corollary,

THEOREM B. *If x is algebraic then $s_n = O(n^a)$, for some value of a less than* 1, *and uniformly in θ.*

We shall also prove that Theorem A is in a sense the best theorem of its kind. This will follow from

THEOREM C. *It is possible to choose an x of class k and a θ so that*
$$s_n \neq o(n^a), \quad a = \frac{k}{k+1} \quad \text{.................(2·5).}$$

3. We require the following lemmas:

Lemma 1: $$q_{n+1} + x_{n+1} q_n = \frac{1}{x x_1 x_2 \dots x_n}.$$

For
$$q_{n+1} + x_{n+1} q_n = (a_{n+1} + x_{n+1}) q_n + q_{n-1} = \frac{q_n}{x_n} + q_{n-1} = \frac{q_n + x_n q_{n-1}}{x_n}.$$

As
$$q_1 + x_1 q_0 = a_1 + x_1 = \frac{1}{x},$$
the lemma follows. As an obvious corollary we have

Lemma 2: $$q_{n+1} < \frac{1}{x x_1 x_2 \dots x_n}.$$

4. We can now prove Theorem A. If we choose ν so that
$$n x x_1 \dots x_{\nu-1} x_\nu < 1 \leqq n x x_1 \dots x_{\nu-1} \quad \text{...........(4·1),}$$
we have*
$$s_n = O(n\sqrt{x x_1 \dots x_{\nu-1}}) + O\left(\frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}}\right) = O(n\sqrt{x x_1 \dots x_{\nu-1}}), \quad \text{(4·21)}$$

* *L.c.*, p. 213.

and also
$$s_n = O\left(\frac{1}{\sqrt{xx_1 \dots x_\nu}}\right) \quad \text{.................(4·22)}.$$

We write
$$xx_1 x_2 \dots x_{\nu-1} = n^{-j} \quad (0 < j \leqq 1).$$

Then
$$\frac{1}{x_\nu} = a_{\nu+1} + x_{\nu+1} < a_{\nu+1} + 1 < A a_{\nu+1} < A\,\frac{q_{\nu+1}}{q_\nu} < A\, q_\nu^{\,k-1},$$

by (2·2); and so, by Lemma 2,
$$\frac{1}{x_\nu} < A\,(xx_1 \dots x_{\nu-1})^{-(k-1)} < A n^{(k-1)j}.$$

It follows, from (4·21) and (4·22), that
$$s_n = O(n^{1-\frac{1}{2}j}), \quad s_n = O(n^{\frac{1}{2}kj}) \quad \text{...........(4·3)},$$

or
$$s_n = O(n^{\gamma}), \quad \gamma = \operatorname{Min}(1 - \tfrac{1}{2}j, \tfrac{1}{2}kj) \quad \text{........(4·4)}.$$

Now
$$1 - \tfrac{1}{2}j \leqq \frac{k}{k+1} \quad \left(j \geqq \frac{2}{k+1}\right),$$
$$\tfrac{1}{2}kj \leqq \frac{k}{k+1} \quad \left(j \leqq \frac{2}{k+1}\right).$$

Hence in any case
$$s_n = O\left(n^{\frac{k}{k+1}}\right) \quad \text{.......................(4·5)},$$

which proves the theorem.

Theorem B is an immediate corollary, since an algebraic number of degree m is of class $m - 1$*.

5. The proof of Theorem C also requires only a slight modification of our former analysis. We take $\theta = 0$, and write, as before,
$$q = e^{\pi i \tau} = e^{\pi i x - \pi y} = r e^{\pi i x} \quad (x > 0,\ y > 0,\ 0 < r < 1) \text{...(5·1)},$$
$$\vartheta_3 = \vartheta_3(0, \tau) = 1 + 2\sum_1^\infty q^{n^2} \quad \text{..............(5·2)}.$$

Suppose it were true that $s_n = o(n^a)$. Then the series
$$1 + 2\,\Sigma q^{n^2} = 1 + 2\Sigma e^{n^2 \pi i x} r^{n^2} = \Sigma u_m r^m$$

* By the classical theorem of Liouville: see, e.g. Borel, *Leçons sur la théorie des fonctions* (ed. 2, 1914), pp. 26–29. It has indeed been shown by A. Thue ('Über Annäherungswerte algebraischer Zahlen', *Journal für Math.*, vol. 135 (1909), pp. 284–305) that an algebraic number of degree m is of class $\frac{1}{2}m + \epsilon$ for every positive ϵ. See Borel, *Leçons sur la théorie de la croissance* (1910), pp. 164–165. More recently C. Siegel ('Approximation algebraischer Zahlen,' *Math. Zeitschrift*, vol. 10 (1921), pp. 173–213) has shown that an algebraic number of degree m is of class $2\sqrt{m} - 1$.

would satisfy the condition

$$U_m = u_0 + u_1 + \dots + u_m = o\,(m^{\frac{1}{2}a}),$$

and we should have

$$\vartheta_3 = \Sigma u_m r^m = (1-r)\,\Sigma U_m r^m = o\{(1-r)^{-\frac{1}{2}a}\} = o\,(y^{-\frac{1}{2}a}) \quad \dots(5\cdot3).$$

It is therefore sufficient to show that (5·3) is false for an appropriate x; that is to say that

$$|\,\vartheta_3\,(0, \tau)\,| > Ay^{-\frac{1}{2}a} \quad \dots\dots\dots\dots\dots\dots(5\cdot4)$$

for a sequence of values of y whose limit is zero.

We suppose that

$$q_{n+1} > Aq_n{}^k \quad \dots\dots\dots\dots\dots\dots\dots\dots(5\cdot5)$$

for an infinity of values of x, and consider, as on p. 229 of our former memoir, the range R_n of values of y defined by

$$\frac{1}{q'_{n+1}{}^2} \leqq y \leqq \frac{1}{q_n{}^2}.$$

It is sufficient to fix our attention on a single value of y, viz.

$$y = q_n{}^{-k-1},$$

which plainly falls within R_n when $k > 1$.

We employ (as on p. 226 *et seq.*) the linear transformation

$$T = \frac{c + d\tau}{a + b\tau} = \pm\frac{p_{n-1} - q_{n-1}\tau}{p_n - q_n\tau},$$

where the sign is chosen so as to make $ad - bc = 1$; and here we make another assumption, viz. that this transformation is one of the types which (following Tannery and Molk) we denoted by 1°, 2°, 5°, or 6°, and which transform $\vartheta_3\,(0, \tau)$ into one of the functions $\vartheta_3\,(0, T)$ or $\vartheta_4\,(0, T)$. It is plain that this may be secured by an appropriate choice of x*.

This being so we have, as on p. 230†,

$$|\,\vartheta_3\,| > A\left(\frac{1}{q'_{n+1}{}^2} + q_n{}^2 y^2\right)^{-\frac{1}{4}} > A\,(q_n{}^{-2k} + q_n{}^2 . q_n{}^{-2-2k})^{-\frac{1}{4}} > Aq_n{}^{-\frac{1}{2}k}$$
$$> Ay^{-\frac{k}{2(k+1)}} = Ay^{-\frac{1}{2}a},$$

which proves the theorem.

* We cannot prove that $s_n = o\,(n^a)$ is *never* true for an irrational of class k; for it is possible that every n for which (5·5) is true should give rise to a transformation of type 3° or 4°.

† The condition $a_n = O\,(1)$, used there, is only required in connection with cases 3° and 4°, here excluded.

COMMENTS

This is a supplementary note to 1914, 3. The results of that paper, on the order of magnitude of $s_n(x, \theta)$, were proved on various suppositions as to the rate of increase of a_n, in the continued fraction for x, as a function of n. Here irrationals x are classified according to the values of k for which $q_{\nu+1}/q_\nu^k$ is bounded. This classification has been followed by later workers (see Koksma, pp. 27–28).

§ 3. The work of this section can be greatly simplified by using the estimate (2) in the comments on 1914, 3. This gives

$$s_n = O\left((nq_\nu^{-\frac{1}{2}})^{k/(k+1)}(q_{\nu+1}^{\frac{1}{2}})^{1/(k+1)}\right) = O(n^{k/(k+1)})$$

if $q_{\nu+1}/q_\nu^k$ is bounded.

§ 4 (*footnote on p. 4*). Our knowledge concerning approximation to algebraic numbers has been completely transformed by Roth's theorem of 1955. In the language of Hardy and Littlewood, the theorem implies that every algebraic number is of class $1+\epsilon$ for any $\epsilon > 0$. Thus we have

$$s_n(x, \theta) = O(n^{\frac{1}{2}+\epsilon})$$

if x is algebraic.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: THE LATTICE-POINTS OF A RIGHT-ANGLED TRIANGLE

By G. H. Hardy *and* J. E. Littlewood.

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1. *Introduction.*

1.1. The problem considered in this paper may be stated as follows.

Suppose that ω and ω' are two positive numbers whose ratio $\theta = \omega/\omega'$ is irrational; and denote by Δ the triangle whose sides are the coordinate axes and the line

$$\omega x + \omega' y = \eta > 0, \tag{1.11}$$

and by $N(\eta)$ the number of lattice-points* which lie inside Δ. *How accurate an approximation can we find for $N(\eta)$ when η is large? And how does the accuracy of the approximation depend upon the arithmetic character of θ?* We call this problem *Problem A.*

Such "lattice-point" problems are, in general, very difficult. It is enough to recall the two most famous of them, the *problem of the circle* (the problem of Gauss and Sierpinski), and the *problem of the rectangular hyperbola* (Dirichlet's divisor problem), both of which have been the subject of numerous researches during the last ten years. The particular problem which we consider here has not, so far as we know, been stated quite in this form before. It is however easily brought into connection with another problem which has attracted a certain amount of attention, and which has been considered, from varying points of view, by Lerch,† by Weyl,‡ and by ourselves.§ This problem, which we shall call

* A lattice-point (*Gitterpunkt*) is a point whose coordinates x and y are both integral.

† M. Lerch, *l'Intermédiaire des Mathématiciens*, Vol. 11 (1904), pp. 145–146 (Question 1547).

‡ H. Weyl, "Uber die Gleichverteilung von Zahlen mod. Eins", *Math. Annalen*, Vol. 77 (1916), pp. 313–352.

§ G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation", *Proceedings of the fifth international congress of mathematicians*, Cambridge, 1912, Vol 1, pp. 223–229,

1922, 6 (with J. E. Littlewood) *Proceedings of the London Mathematical Society*, (2) 20, 15–36.

Problem B, is as follows. Suppose that, as usual, $[x]$ denotes the integral part of x, and that

$$\{x\} = x-[x]-\tfrac{1}{2}. \tag{1.12}$$

Then *what is the most that can be said as to the order of magnitude of*

$$s(\theta, n) = \sum_{\nu=1}^{n} \{\nu\theta\} \tag{1.13}$$

when n is large?

1.2. We begin, in § 2, by proving the formula which establishes the connection between Problems A and B, and shows that the first problem is a generalised and more symmetrical form of the second. We prove in fact that

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + S(\eta), \tag{1.21}$$

where $S(\eta)$ is a sum very similar to the sum 1.13.

It is trivial that

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} + O(\eta), \tag{1.211}$$

the area of the triangle, together with an error of the order of the perimeter. The second and third terms of (1.21) occur naturally when we consider, instead of Δ, the similar and similarly situated triangle whose vertex is at (1, 1) instead of the origin; for the area of this triangle is

$$\frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \frac{1}{2}.$$

But no closer approximation than (1.211) is in any way trivial; and, when θ is rational, $S(\eta)$ is effectively of order η, so that a universal formula, professing to be more precise than (1.211), would necessarily be false.

In § 3 we deduce transformation formulæ for N and S, which are generalisations of a formula given without proof by Lerch, and which enable us to study these sums by means of the expression of θ as a simple continued fraction. In § 4 we prove (a) that

$$S(\eta) = o(\eta) \tag{1.22}$$

for any irrational θ, and (b) that (1.22) is the most that is true for every such irrational. Incidentally we obtain the corresponding results concerning Problem B: the first of them at any rate is in this case familiar,

In § 5 we consider more closely cases in which the rate of increase of the quotients in the continued fraction is comparatively slow, and in particular the case in which they are bounded; and we prove that in this case

$$S(\eta) = O(\log \eta), \tag{1.23}$$

and that this result too is a best possible result of its kind. There are naturally analogous results for Problem B; that corresponding to (1.23) was stated as a new theorem in our communication to the Cambridge congress, but had, as was pointed out to us by Prof. Landau, been given already by Lerch.

Up to this point our argument is entirely elementary, and both methods and results are of a kind to be found in our previous papers on Diophantine approximation or in those of other writers. We have therefore aimed at the maximum of compression and have omitted a good deal of elementary algebraical calculation. The concluding section (§ 6) is more novel. In it we prove that *if θ is algebraic then*

$$S(\eta) = O(\eta^a), \tag{1.24}$$

where $a < 1$. This result is unlike any which we have been able to prove before, and is obtained by entirely different methods, based on the properties of the analytic function

$$\zeta_2(s, a, \omega, \omega') = \sum_{m,\, n=0}^{\infty} \frac{1}{(a+m\omega+n\omega')^s}. \tag{1.25}$$

This function will be recognised as a degenerate form of the "Double Zeta-function" introduced into analysis by Dr. Barnes.*

2. *Reduction of Problem* A.

2.1. We write

$$\frac{\eta}{\omega} = \left[\frac{\eta}{\omega}\right]+f, \quad \frac{\eta}{\omega'} = \left[\frac{\eta}{\omega'}\right]+f', \tag{2.11}$$

where $$0 \leqslant f < 1, \quad 0 \leqslant f' < 1.$$

* E. W. Barnes, "A memoir on the Double-Gamma-function", *Phil. Trans. Roy. Soc.*, (A), Vol. 196 (1901), pp. 265–387; see in particular pp. 314–349. For a study of some of the properties of the degenerate function (for which the ratio ω/ω' is real) see G. H. Hardy, "On double Fourier series, and in particular those which represent the double Zeta-function with real and incommensurable parameters", *Quarterly Journal*, Vol. 37 (1906), pp. 53–79.

Suppose first that there is no lattice-point on the line (1.11), or AB of the figure. Then the number of lattice-points inside OAB is

$$(2.12)\qquad N(\eta)=\sum_{\mu=1}^{\eta/\omega}\left[\frac{\eta-\mu\omega}{\omega'}\right]=\left[\frac{\eta}{\omega}\right]\left[\frac{\eta}{\omega'}\right]+\sum_{\mu=1}^{\eta/\omega}[f'-\mu\theta],$$

Now $[-x]=-[x]-1+\epsilon_x$, where ϵ_x is 1 or 0 according as x is or is not an integer; and $\mu\theta-f'$ cannot be an integer, since then $\eta-\mu\omega$ would be an integral multiple of ω' and there would be a lattice-point on AB. Thus

$$(2.13)\qquad [f'-\mu\theta]=-[\mu\theta-f']-1=-(\mu\theta-f')+\{\mu\theta-f'\}-\tfrac{1}{2}.$$

Substituting into (2.12), and using (2.11), we obtain, after a little reduction

$$(2.14)\qquad N(\eta)=\frac{\eta^2}{2\omega\omega'}-\frac{\eta}{2\omega}-\frac{\eta}{2\omega'}+\phi+S(\eta),$$

where

$$(2.141)\qquad \phi=\tfrac{1}{2}f+\tfrac{1}{2}\theta f(1-f)$$

and

$$(2.142)\qquad S(\eta)=\sum_{\mu=1}^{\eta/\omega}\{\mu\theta-f'\}.$$

Since ϕ is bounded, the problem is reduced, substantially, to the discussion of $S(\eta)$.

The preceding argument requires a trifling modification when there is a lattice-point on AB; there cannot be more than one, since θ is irrational. In this case the sum (2.12) gives $N(\eta)+1$ instead of $N(\eta)$. There is one value of μ for which $\mu\theta-f'$ is integral, and for this μ the $-\frac{1}{2}$ in (2.13) is changed into $\frac{1}{2}$. The net result is to leave the final formulæ unchanged.

3. *The Transformation Formulæ.*

3.1. In order to obtain a formula for the transformation of $N(\eta)$ or of $S(\eta)$, we employ the familiar device of adding together the number of lattice-points of the triangles OAB, $O'A'B'$ of the figure.

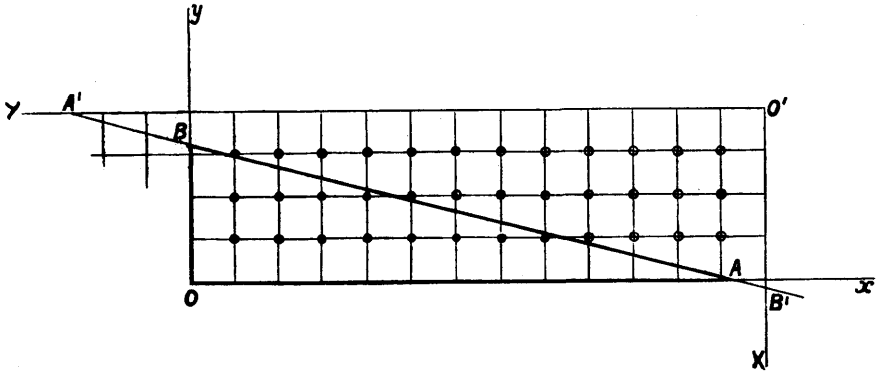
If we take new axes $O'X$, $O'Y$, as shown in the figure, it is plain that

$$x+Y=\left[\frac{\eta}{\omega}\right]+1,\qquad X+y=\left[\frac{\eta}{\omega'}\right]+1;$$

and the equation of AB, referred to the new axes, is

$$\omega' X+\omega Y = \eta+\omega(1-f)+\omega'(1-f') = H, \tag{3.11}$$

say. Repeating the arguments of § 2, we find, for the number $N'(H)$ of



lattice-points of $O'A'B'$,

$$N'(H) = \frac{H^2}{2\omega\omega'}-\frac{H}{2\omega}-\frac{H}{2\omega'}+\Phi+S'(H), \tag{3.12}$$

where

$$\Phi = \tfrac{1}{2}F'+\frac{F'(1-F')}{2\theta} \tag{3.121}$$

and

$$S'(H) = \sum_{\nu=1}^{H/\omega'}\left\{\frac{\nu}{\theta}-F\right\}, \tag{3.122}$$

F and F' being defined by

$$\frac{H}{\omega} = \left[\frac{H}{\omega}\right]+F,\quad \frac{H}{\omega'} = \left[\frac{H}{\omega'}\right]+F',\quad 0\leqslant F<1,\quad 0\leqslant F'<1. \tag{3.123}$$

3.2. We suppose now that $\omega<\omega'$, $\theta<1$. A glance at the figure shows that

$$\left[\frac{H}{\omega'}\right] = \left[\frac{\eta}{\omega'}\right]+1.$$

Substituting for H in terms of η, from (3.11), we find at once that

$$F' = \theta(1-f). \tag{3.21}$$

The same argument shows that

$$F = \frac{1-f'}{\theta}-p, \tag{3.22}$$

where p is an integer; it happens that the value of p is not material to the argument.

It is also clear from the figure that

$$N(\eta)+N'(H) = \left[\frac{\eta}{\omega}\right]\left[\frac{\eta}{\omega'}\right]-\epsilon, \tag{3.23}$$

where ϵ is zero unless there is a lattice point on AB, and then unity. Substituting for $N(\eta)$ and $N'(H)$ from (2.14) and (3.12), using (2.11), (3.11), and (3.21), and reducing, we obtain, finally,

$$S+S'+\epsilon = -\tfrac{1}{2}+\tfrac{1}{2}(f+f')-\tfrac{1}{2}\theta f(1-f)+\frac{f'(1-f')}{2\theta}. \tag{3.24}$$

3.3. It is important, in view of Problem B, to show that this formula includes a formula given by Lerch.* Suppose then in particular that $\omega' = 1$, $\omega = \theta < 1$, and write

$$s = \sum_{1}^{n}\{\mu\theta\}, \quad s' = \sum_{1}^{m}\left\{\frac{\nu}{\theta}\right\}, \tag{3.31}$$

where m is the integral part of $n\theta$.

Starting with an arbitrary positive integral n, we write $n\theta = M+\delta$, where M is an integer and $0<\delta<1$, and take

$$\eta = M+1 = n\theta+1-\delta.$$

Then
$$f' = 0, \quad F \equiv \frac{1}{\theta} \pmod 1,$$

by (2.11) and (3.22); and there is no lattice point on AB, so that $\epsilon = 0$.

Suppose now that q is a positive integer and

$$q < \frac{1-\delta}{\theta} < q+1.\dagger$$

Then
$$\frac{\eta}{\theta} = n+\frac{1-\delta}{f} = n+q+f, \quad f = \frac{1-\delta}{\theta}-q.$$

Also $H = \eta+1+\theta(1-f)$ lies between $M+2$ and $M+3$. Hence

$$S' = \sum_{\nu=1}^{M+2}\left\{\frac{\nu-1}{\theta}\right\} = -\tfrac{1}{2}+\left\{\frac{M+1}{\theta}\right\}+s'; \tag{3.32}$$

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* M. Lerch, *loc. cit.*

† It is easy to see that $(1-\delta)/\theta$ cannot be integral.

and

$$\left\{\frac{M+1}{\theta}\right\} = \left\{\frac{\eta}{\theta}\right\} = \left\{\frac{1-\delta}{\theta}\right\} = \frac{1-\delta}{\theta} - q - \tfrac{1}{2}. \tag{3.321}$$

Also

$$S = \sum_{\mu=1}^{[\eta/\theta]} \{\mu\theta\} = \sum_{1}^{n+q} \{\mu\theta\} = s + \sum_{r=1}^{q} \{(n+r)\theta\} = s + S_0, \tag{3.33}$$

say. And $(n+1)\theta, \ldots, (n+q)\theta$ have all the integral part M, since $q\theta < 1-\delta < (q+1)\theta$. Hence

$$S_0 = \sum_{r=1}^{q} (n\theta + r\theta - M - \tfrac{1}{2}) = \sum_{r=1}^{q} (r\theta + \delta - \tfrac{1}{2}) = \tfrac{1}{2}q(q+1)\theta + q(\delta - \tfrac{1}{2}). \tag{3.34}$$

Substituting from (3.32), (3.321), (3.33), and (3.34) into (3.24), and reducing, it will be found that

$$s + s' = \tfrac{1}{2}\delta - \frac{\delta(1-\delta)}{2\theta}, \tag{3.35}$$

which is the formula of Lerch.

4. *Results concerning an arbitrary irrational θ.*

4.1. THEOREM A1.—*If $\theta = \omega/\omega'$ is irrational, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + o(\eta).$$

We may clearly suppose that $\theta < 1$. Suppose that

$$\theta = \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\frac{1}{a_3 + \ldots}, \tag{4.11}$$

$$\theta = \frac{1}{a_1 + \theta_1}, \quad \theta_1 = \frac{1}{a_2 + \theta_2}, \quad \ldots. \tag{4.12}$$

We have, from (3.24),

$$S + S' = O(1/\theta), \tag{4.13}$$

the constant of the O being independent of both η and θ.

We write $\eta = \omega\xi$, so that

$$\frac{H}{\omega'} = \xi\theta + \theta(1-f) + 1 - f' = \xi\theta + O(1),$$

and we write f_1 and μ_1 in S' instead of F' and ν. Then

$$S' = \sum_{\mu_1=1}^{H/\omega} \left\{\frac{\mu_1}{\theta} - f_1\right\} = O(1) + \sum_{\mu_1=1}^{\xi\theta} \{\mu_1\theta_1 - f_1\} = O(1) + S_1,$$

say; so that

(4.14) $$S = O(1/\theta) - S_1.$$

Similarly, we have

$$S_1 = O(1/\theta_1) - S_2, \quad S_2 = O(1/\theta_2) - S_3, \quad \dots,$$

where $S_2, S_3, \dots$ are sums of the types

$$S_2 = \sum_{\mu_2=1}^{\xi\theta\theta_1} \{\mu_2\theta_2 - f_2\}, \quad S_3 = \sum_{\mu_3=1}^{\xi\theta\theta_1\theta_2} \{\mu_3\theta_3 - f_3\}, \quad \dots,$$

so that $$S_2 = O(\xi\theta\theta_1), \quad S_3 = O(\xi\theta\theta_1\theta_2), \quad \dots.$$

It follows that

(4.151) $$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \dots + O\left(\frac{1}{\theta_{\nu-1}}\right) + O(\xi\theta\theta_1 \dots \theta_{\nu-1})$$

and

(4.152) $$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \dots + O\left(\frac{1}{\theta_\nu}\right) + O(\xi\theta\theta_1 \dots \theta_{\nu-1}\theta_\nu).$$

We shall require both of these equations.

4.2. We choose ν so that

(4.21) $$\xi\theta\theta_1 \dots \theta_{\nu-1}\theta_\nu < 1 \leqslant \xi\theta\theta_1 \dots \theta_{\nu-1}.$$

It may be verified at once* that $\theta_s\theta_{s+1} < \frac{1}{2}$ for every s. Hence on the one hand

(4.22) $$\theta\theta_1 \dots \theta_{\nu-1} = O(2^{-\frac{1}{2}\nu}),$$

and on the other

(4.23) $$\frac{1}{\theta} + \frac{1}{\theta_1} + \dots + \frac{1}{\theta_{\nu-1}} = O\left(\nu \operatorname{Max} \frac{1}{\theta_s}\right) = O\left(\frac{\nu 2^{-\frac{1}{2}\nu}}{\theta\theta_1 \dots \theta_{\nu-1}}\right) = O(\nu 2^{-\frac{1}{2}\nu}\xi).$$

From (4.151), (4.22), and (4.23), we obtain

(4.24) $$S = O(\nu 2^{-\frac{1}{2}\nu}\xi) + O(2^{-\frac{1}{2}\nu}\xi) = o(\xi),$$

since ν tends to infinity with ξ; and the theorem follows from (2.14) and (4.24).

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* See our paper "Some problems of Diophantine approximation (II)" [*Acta Mathematica*, Vol. 37 (1914), pp. 193–238 (p. 212)].

4.3. To Theorem **A1** corresponds, for Problem B, the well known theorem :

THEOREM **B1.**—*If θ is irrational, then*

$$s(\theta, n) = \sum_{\mu=1}^{n} \{\mu\theta\} = o(n).$$

The proof of this theorem is included in that of Theorem **A1.** We have only to take $\eta = k\omega'$, where k is an integer, so that $f' = 0$, and to write $\xi = \eta/\omega = k/\theta$, $n = [\xi]$.

4.4. THEOREM **A2.**—*If $\psi(\eta)$ is any function of η which tends steadily to infinity with η, then there is an irrational θ such that each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} > \frac{\eta}{\psi(\eta)}, \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} < -\frac{\eta}{\psi(\eta)}$$

is satisfied for a sequence of indefinitely increasing values of η.

Thus Theorem **A1** is the best possible theorem of its kind.

Making the transformations indicated in 4.3, we see at once that it is enough to prove

THEOREM **B2.**—*If $\psi(n)$ is any function of n which tends steadily to infinity with n, then there is an irrational θ such that each of the inequalities*

$$s(\theta, n) > \frac{n}{\psi(n)}, \quad s(\theta, n) < -\frac{n}{\psi(n)}$$

is satisfied for an infinity of values of n.

To prove this we use Lerch's formula (3.35). Writing

$$(4.41) \quad n_1 = [n\theta] = n\theta - \delta, \; n_2 = n_1\theta_1 - \delta_1, \; \dots, \; n_{r+1} = n_r\theta_r - \delta_r,$$

$$(4.42) \quad \phi_s = \tfrac{1}{2}\delta_s - \frac{\delta_s(1-\delta_s)}{2\theta_s},$$

we have

$$(4.43) \quad s(\theta, n) = \phi_0 - s\left(\frac{1}{\theta}, n_1\right) = \phi_0 - s(\theta_1, n_1) = \phi_0 - \phi_1 + s(\theta_2, n_2)$$
$$= \dots = \phi_0 - \phi_1 + \dots + (-1)^r \phi_r + s(\theta_{r+1}, n_{r+1}).$$

We suppose a_{r+1} even, and exceedingly large in comparison with the preceding quotients $a_1, a_2, \dots, a_r$, and take $n_r = \frac{1}{2}a_{r+1}$. Then $n_{r+1} = 0$ and

δ_r is practically $\frac{1}{2}$, so that $\frac{1}{2}\delta_r(1-\delta_r)$ is certainly greater than $\frac{1}{9}$. Having fixed n_r, we can determine $n_{r-1}, n_{r-2}, \ldots, n_1, n$ from the equations (4.41); and

$$n \leqslant \frac{2n_1}{\theta} \leqslant \frac{2^2 n_2}{\theta\theta_1} \cdots \leqslant \frac{2^r n_r}{\theta\theta_1 \ldots \theta_{r-1}} = \frac{2^{r-1} a_{r+1}}{\theta\theta_1 \ldots \theta_{r-1}}.$$

It is then plain that, if a_{r+1} is sufficiently large in comparison with the preceding partial quotients, $s(\theta, n)$ will have the sign of $(-1)^r$, and

$$(4.41) \qquad |s(\theta, n)| > \tfrac{1}{2}|\phi_r| > \frac{1}{20\theta_r} > \frac{a_{r+1}}{20} > \frac{n}{\psi(n)}.$$

And, by choosing a θ for which sufficiently violent increments in the order of magnitude of the quotients occur at an infinity of stages in the continued fraction, we can secure the truth of (4 . 41) for an infinity of values of n.

5. *Results concerning special classes of irrationals.*

5 . 1. Theorem **A3**.—*If the quotients a_n in the continued fraction for $\theta = \omega/\omega'$ are bounded, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\log \eta).$$

Theorem **B3**.—*Under the same condition,*

$$s(\theta, n) = O(\log n).$$

To prove Theorem **A3**, we return to the analysis of 4 . 1 and 4 . 2, but use (4.152) instead of (4.151). In this case we have plainly

$$S = O\left(\frac{1}{\theta}\right) + O\left(\frac{1}{\theta_1}\right) + \ldots + O\left(\frac{1}{\theta_\nu}\right) = O(\nu).$$

Since $$2^{\frac{1}{2}\nu} = O\left(\frac{1}{\theta\theta_1 \ldots \theta_{\nu-1}}\right) = O\left(\frac{1}{\theta\theta_1 \ldots \theta_\nu}\right) = O(\xi),$$

we have $\nu = O(\log \xi) = O(\log \eta)$; and the theorem is proved. Theorem **B3** follows *a fortiori*: this is the theorem which, as we explained in 1 . 2, was claimed as a new theorem in our communication to the Cambridge congress, but is really due to Lerch.

It will easily be verified that, if we assume

$$a_n = O(n^\rho) \quad (\rho > 0),$$

we obtain an error term of the order

$$S = O\{(\log \eta)^{\rho+1}\};$$

if we assume $a_n = O(e^{\rho n})$, where ρ lies below a certain limit, we obtain

$$S = O(\eta^{\sigma}) \quad (\sigma < 1).^*$$

As so little is known concerning the order of magnitude of the quotients in the continued fractions which express irrationals of particular types, it is hardly worth while to go into further detail.

5.2. THEOREM **A4**. — *There are values of* $\theta = \omega/\omega'$, *with bounded quotients, such that each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} > K \log \eta, \qquad N(\eta) - \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} < -K \log \eta,$$

where K *is a positive constant, is satisfied for a sequence of indefinitely increasing values of* η.

THEOREM **B4**.—*There are values of* θ, *with bounded quotients, such that each of the inequalities*

$$s(\theta, n) > K \log n, \qquad s(\theta, n) < -K \log n$$

is satisfied for an infinity of values of n.

Thus Theorems **A3** and **B3** are also best possible theorems of their kind. To prove this, it is plainly enough to prove Theorem **B4**; and this we shall do by considering the simplest irrational of all, viz.

$$\theta = \frac{\sqrt{5}-1}{2} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\dots$$

We write

$$\Theta = \frac{1}{\theta} = \frac{\sqrt{5}+1}{2},$$

and take the convergents to θ to be

$$\frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{1}, \quad \frac{p_2}{q_2} = \frac{1}{2}, \quad \dots.$$

Then it is easily verified that

$$q_s = \frac{1}{\sqrt{5}}(\Theta^{s+1} + (-1)^s \theta^{s+1}), \qquad p_s = q_{s-1}.$$

---

* Compare p. 214 of our memoir in the *Acta Mathematica* referred to above (p. 22).

5.3. We first take $n = q_s$ in the formula (3.31). We find without difficulty that
$$[q_s\theta] = q_{s-1}, \quad \delta = q_s\theta - [q_s\theta] = \theta^{s+1},$$
if s is even, and $\quad [q_s\theta] = q_{s-1}-1, \quad \delta = 1-\theta^{s+1},$

if s is odd; and that in either case

(5.31)
$$\sigma_s = \sum_{\nu=1}^{q_s} \{\nu\theta\}$$

satisfies the equation

(5.32)
$$\sigma_s + \sigma_{s-1} = \tfrac{1}{2}\left(\theta^{2s+1} + (-1)^{s+1}\theta^{s+2}\right).$$

Using this recurrence equation to express σ_s in terms of
$$\sigma_0 = \{\theta\} = \tfrac{1}{2}\surd 5 - 1,$$
we find, after reduction, that

(5.33)
$$\sigma_s = \frac{\theta^{2s+2}}{2\surd 5} - \tfrac{1}{2}(-1)^{s+1}\theta^{s+1} + (-1)^{s+1}\frac{\theta}{\surd 5}.$$

Suppose now that

(5.34)
$$s(\theta, n) = \sum_1^n \{\nu\theta\} \quad (q_s \leqslant n < q_{s+1}).$$

We can express n in one and only one way in the form
$$n = q_s + q_{s_1} + q_{s_2} + \ldots + q_{s_k} = q_s + Q_1,$$
where $s, s_1, s_2, \ldots$ are descending integers differing by at least 2; and
$$s(\theta, n) = \sigma_s + \sum_{\mu=1}^{Q_1} \{(q_s+\mu)\theta\}.$$

Now $q_s\theta$ differs from an integer by less than does any $\mu\theta$. Hence
$$[(q_s+\mu)\theta] = q_{s-1} + [\mu\theta]$$
and $\quad \{(q_s+\mu)\theta\} = q_s\theta - q_{s-1} + \mu\theta - [\mu\theta] - \tfrac{1}{2} = (-1)\,\theta^{s+1} + \{\mu\theta\},$
$$s(\theta, n) = \sigma_s + (-1)^s\theta^{s+1}Q_1 + s_{Q_1}.$$

We now write
$$Q_1 = q_{s_1} + q_{s_2} + \ldots + q_{s_k} = q_{s_1} + Q_2, \qquad Q_2 = q_{s_2} + Q_3,$$
and so on, and repeat the argument. We thus obtain

(5.35)
$$s(\theta, n) = \sigma_s + \sigma_{s_1} + \sigma_{s_2} + \ldots + \sigma_{s_k}$$
$$+ (-1)^s\theta^{s+1}Q_1 + (-1)^{s_1}\theta^{s_1+1}Q_2 + \ldots + (-1)^{s_{k-1}}\theta^{s_{k-1}+1}Q_{s_k}.$$

5.4. If in (5.35) we substitute the values of the σ's given by (5.33), the first two terms of (5.33) will plainly give a contribution bounded for all values of s, so that

$$(5.41)\quad \sigma_s+\sigma_{s_1}+\ldots+\sigma_{s_k} = -\frac{\theta}{\sqrt{5}}\Big((-1)^s+(-1)^{s_1}+\ldots+(-1)^{s_k}\Big)+O(1).$$

Again

$$(5.42)\qquad Q_1 = \sum_{r=1}^{k} q_{s_r} = \frac{1}{\sqrt{5}}\sum_{r=1}^{k}\Big(\Theta^{s_r+1}+(-1)^{s_r}\theta^{s_r+1}\Big),$$

and the sum of the second terms is numerically less than k, and *a fortiori* than s. The sum of the contributions of all such terms to (5.35) is therefore less in absolute value than

$$s\theta^{s+1}+s_1\theta^{s_1+1}+\ldots = O(1).$$

These terms, then, may be disregarded. Making this simplification, and substituting from (5.41) and (5.42) into (5.35), we obtain, finally,

$$\begin{aligned}(5.43)\quad s(\theta, n) = O(1) &- \frac{\theta}{\sqrt{5}}\Big((-1)^s+(-1)^{s_1}+\ldots+(-1)^{s_k}\Big)\\ &+\frac{(-1)^s}{\sqrt{5}}(\theta^{s-s_1}+\theta^{s-s_2}+\ldots+\theta^{s-s_k})\\ &+\frac{(-1)^{s_1}}{\sqrt{5}}(\theta^{s_1-s_2}+\theta^{s_1-s_3}+\ldots+\theta^{s_1-s_k})\\ &+\ldots+\frac{(-1)^{s_{k-1}}}{\sqrt{5}}\theta^{s_{k-1}-s_k}.\end{aligned}$$

5.5. This formula enables us to study the behaviour of $s(\theta, n)$ for different forms of n, and in particular to prove our theorem. Let us take, for example,

$$s = 4k+4,\quad s_1 = 4k,\quad s_2 = 4k-4,\quad \ldots,\quad s_k = 4.$$

Then the right-hand side of (5.43) becomes

$$-\frac{s\theta}{4\sqrt{5}}+\frac{1}{\sqrt{5}}\left(\frac{\theta^4-\theta^s+\theta^4-\theta^{s-4}+\ldots+\theta^4-\theta^8}{1-\theta^4}\right)+O(1) = Cs+O(1),$$

where
$$C = \frac{1}{4\sqrt{5}}\left(\frac{\theta^4}{1-\theta^4}-\theta\right) = -\frac{1}{20}\neq 0;$$

and $s(\theta, n)$ is negative and greater than a constant multiple of s. Similarly, if we were to take

$$s = 4k+3,\ s_1 = 4k-1,\ \ldots,\ s_k = 3,$$

we should find $s(\theta, n)$ to be positive and greater than a constant multiple of s. Since s is greater than a constant multiple of $\log n$, this completes the proof of Theorems **A4** and **B4**.

5.6. We should perhaps, before passing to more transcendental investigations, add a word concerning the case, so far excluded, of a *rational* θ. It is easy to see that, when θ is rational, no such results as we have proved in the irrational case are true: $s(\theta, n)$ is effectively of order n, and the oscillatory part of $N(\eta)$ of order η. Thus, to take a simple case, the series $\Sigma\{\frac{2}{3}\mu\}$ is

$$\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{2}+\dots,$$

and

$$s(\tfrac{2}{3}, n) \sim -\tfrac{1}{6}n.$$

In general, for a fixed rational $\theta = p/q$, we have $s(\theta, n) \sim A_q n$, where $A_q \to 0$ when $q \to \infty$.

6. *Transcendental methods: results true for all algebraical values of θ.*

6.1. The substance of our concluding section lies somewhat deeper. Our goal is to prove

THEOREM **A5**.—*If $\theta = \omega/\omega'$ is an algebraic irrational, then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\eta^a),$$

where $a < 1$.

THEOREM **B5**.—*Under the same conditions*

$$s(\theta, n) = O(n^a) \quad (a < 1).$$

We require some preliminary lemmas concerning the function

$$\zeta_2(s, a, \omega, \omega') = \sum_{m,\, n=0}^{\infty} \frac{1}{(a+m\omega+n\omega')^s}, \tag{6.11}$$

where a, ω, and ω' are positive, and $s = \sigma + it$. This function is a degenerate form of the double Zeta-function of Dr. E. W. Barnes. Barnes considers only the case in which (as in the theory of elliptic functions) the ratio $\theta = \omega/\omega'$ is *complex*. The series (6.11) defines the function in the first instance for $\sigma > 2$.

6.21. LEMMA *a*.—*The function $\zeta_2(s, a, \omega, \omega')$ is an analytic function of s, regular all over the plane except for simple poles at the points $s = 2$*

and $s=1$, *where it behaves like*

$$\frac{1}{\omega\omega'}\,\frac{1}{s-2},\qquad \frac{\omega+\omega'-2a}{2\omega\omega'}\,\frac{1}{s-1}$$

respectively.

This is proved by Barnes when θ is complex, and his proof, depending on the formula

$$(6.211)\qquad \zeta_2(s,\,a,\,\omega,\,\omega') = \frac{i\Gamma(1-s)}{2\pi}\int \frac{e^{-au}(-u)^{s-1}}{(1-e^{-\omega u})(1-e^{-\omega' u})}\,du,$$

is equally applicable in the case considered here. We should observe that $(-u)^{s-1} = e^{(s-1)\log(-u)}$, where $\log(-u)$ has its principal value, that the contour of integration is the same as in the well-known Riemann-Hankel formulæ for the ordinary Gamma and Zeta functions, and that the formula is valid for all values of s except positive integral values.

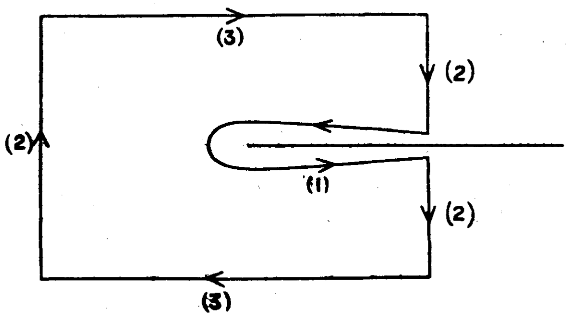
6.22. LEMMA β.—*Suppose that* $0<a\leqslant\omega+\omega'$, *and that* $\theta=\omega/\omega'$ *is an algebraic irrational. Then there is a* K *such that*

$$(6.221)\qquad \frac{\zeta_2(s,\,a,\,\omega,\,\omega')}{(2\pi)^{s-1}\Gamma(1-s)} = \frac{1}{\omega^s}\sum_{m=1}^{\infty}\frac{\sin\left\{\dfrac{2m\pi}{\omega}(\tfrac12\omega'-a)+\tfrac12(1-s)\pi\right\}}{m^{1-s}\sin\dfrac{m\omega'\pi}{\omega}}$$

$$+\frac{1}{\omega'^s}\sum_{m=1}^{\infty}\frac{\sin\left\{\dfrac{2m\pi}{\omega'}(\tfrac12\omega-a)+\tfrac12(1-s)\pi\right\}}{m^{1-s}\sin\dfrac{m\omega\pi}{\omega'}}$$

for $\sigma<-K$.

To prove this formula we start from the integral (6.211) and integrate



round the contour shown in the figure. We suppose, as plainly we may,

that the horizontal lines (3) pass at a distance greater than a constant δ from any pole of the subject of integration, and that the loop (1) passes between the origin and the poles $\pm 2\pi i/\omega$, $\pm 2\pi i/\omega'$ nearest the origin. This being so, it is easy to see that the contributions of the rectilinear parts of the contour tend to zero when the sides of the rectangle move away to infinity, and that

$$\zeta_2 = \Gamma(1-s) \lim \Sigma R,$$

where R is a residue of the integrand. A simple calculation shows that the residues yield the two series required. If $\theta = \omega/\omega'$ is algebraic, we have

$$\left|\sin \frac{m\omega'\pi}{\omega}\right| > m^{-c}, \quad \left|\sin \frac{m\omega\pi}{\omega'}\right| > m^{-c},$$

where c is a constant depending on the degree of the algebraic equation which defines θ. It follows that the two series of the lemma are absolutely convergent if σ is negative and sufficiently large.* We shall suppose in what follows that the series are absolutely convergent for $\sigma < -K$. The formula (6.221) may of course hold in a wider region than this.

6.23. Lemma γ.—*If* $|t| \to \infty$ *then*

$$\zeta_2(s, a, \omega, \omega') = O(e^{\epsilon|t|}),$$

for every positive ϵ, *and uniformly throughout any finite interval of values of* σ.

Suppose that $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. We may suppose the contour of integration in (6.211) deformed in such a manner that

$$|\phi| = |\arg(-u)| \leqslant \tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon$$

at every point of it, and $\qquad |\phi| = \tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon$

at all distant points. We have then

$$|(-u)^{s-1}| < A\,|u|^{A}\,e^{|\phi t|} < A\,|u|^{A}\,e^{(\frac{1}{2}\pi+\frac{1}{2}\epsilon)|t|},$$

where A is a number depending on σ_1 and σ_2,

$$|\Gamma(1-s)| = O(e^{-\frac{1}{2}\pi|t|}\,|t|^{\frac{1}{2}-\sigma}) = O(e^{-(\frac{1}{2}\pi-\frac{1}{2}\epsilon)|t|}),$$

$$\zeta_2 = O\left(e^{\epsilon|t|}\int \frac{|e^{-au}|\,|du|}{|1-e^{-\omega u}|\,|1-e^{-\omega' u}|}\right) = O(e^{\epsilon|t|}).$$

---

* It is hardly necessary to give fuller details of the proof, as the substance of the lemma is contained in the paper of Hardy referred to in the footnote to p. 17.

6.24. Lemma γ is required only in order to prove a somewhat deeper lemma, viz.:

LEMMA δ.*—*The function* $\zeta_2(s, a, \omega, \omega')$ *is of finite order in any half-plane* $\sigma > \sigma_0$, *and its* μ-*function* $\mu(\sigma)$ *satisfies the relations*

$$\mu(\sigma) = 0 \quad (\sigma \geqslant 2), \tag{6.241}$$

$$\mu(\sigma) \leqslant \frac{(\frac{1}{2}+K)(2-\sigma)}{2+K} \quad (-K \leqslant \sigma \leqslant 2), \tag{6.242}$$

$$\mu(\sigma) \leqslant \tfrac{1}{2}-\sigma \qquad (\sigma \leqslant -K). \tag{6.243}$$

Of these relations, (6.241) is obvious, since the series (6.11) is absolutely convergent for $\sigma > 2$; and (6.243) follows from (6.221), since we have

$$(2\pi)^{s-1}\,\Gamma(1-s)\sin\left\{\frac{2m\pi}{\omega}(\tfrac{1}{2}\omega'-a)+\tfrac{1}{2}(1-s)\pi\right\} = O\{e^{\frac{1}{2}\pi|t|}\,|\Gamma(1-s)|\}$$
$$= O(|t|^{\frac{1}{2}-\sigma})$$

uniformly in m, and, of course, a similar result in which ω and ω' are interchanged. Finally, (6.242) follows from (6.241), (6.243), and the well-known theorem of Lindelöf.† Lemma γ is used only to show that the conditions of Lindelöf's theorem are satisfied.

6.25. Our last lemma is of a different character. We write

$$a+m\omega+n\omega' = l_p, \tag{6.251}$$

the numbers l_p (no two of which are equal, since θ is irrational) being arranged in order of magnitude. We suppose that ξ is not equal to any l_p, and we put

$$W(\xi) = \sum_{l_p<\xi} 1.$$

LEMMA ϵ.—*Suppose that* $c > 2$, $T > 1$, *and* $\xi = \sqrt{(l_q l_{q+1})}$. *Then there exists a number* H, *independent of* T *and* ξ, *such that*

$$\left| W(\xi) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta_2(s)\,\frac{\xi^s}{s}\,ds \right| < H\,\frac{\xi^c}{T}.$$

* For explanations concerning the "μ-function" of a function $f(s)$, defined initially by a Dirichlet's series, see G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series," *Cambridge Mathematical Tracts*, no. 18, 1915, pp. 14–18.

† Theorem 14 of the tract referred to above.

We have

(6.252) $$W-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta_2(s)\frac{\xi^s}{s}\,ds = W-\sum_p\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}.$$

Since $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}\begin{array}{ll} =1 & (l_p<\xi) \\ =0 & (l_p>\xi)\end{array}\Bigg\},$$

the right-hand side of (6.252) may be written in the form

$$\sum_p\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c-iT}+\frac{1}{2\pi i}\int_{c+iT}^{c+i\infty}\right)\left(\frac{\xi}{l_p}\right)^s\frac{ds}{s}=\sum_p U_p,$$

say. Now* $$|U_p|\leqslant\frac{2}{T}\frac{(\xi/l_p)^c}{|\log(\xi/l_p)|}.$$

Hence

(6.253) $$\left|W-\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta_2(s)\frac{\xi^s}{s}\,ds\right|\leqslant\frac{2\xi^c}{T}\sum_p\frac{l_p^{-c}}{|\log(\xi/l_p)|}.$$

If we write $l_p=e^{\lambda_p}$, $\xi=e^\rho$, the series becomes

(6.254) $$\sum\frac{e^{-c\lambda_p}}{|\rho-\lambda_p|},$$

and $$\rho=\tfrac{1}{2}(\lambda_q+\lambda_{q+1}).$$

Now Bohr,† generalising a result of Landau,‡ has shown that the series (6.254) is bounded, provided only that

(C) *there is a number l, positive or zero, such that*

$$\frac{1}{\lambda_{p+1}-\lambda_p}=O(e^{(l+\delta)\lambda_p})$$

for every positive δ;

and it is easy to verify that the condition (C) is satisfied by our series $\Sigma l_p^{-s}=\Sigma e^{-s\lambda_p}$. For

$$l_{p+1}-l_p=a+m'\omega+n'\omega'-a-m\omega-n\omega'=h\omega+k\omega'=\omega'(k+h\theta),$$

say, and so, since θ is algebraic and $l_{p+1}<l_p+H$,

$$l_{p+1}-l_p>(|h|+2)^{-H}>H(|m|+|m'|+2)^{-H}>Hl_{p+1}^{-H}>Hl_p^{-H}\ (p>p_0),$$

$$\lambda_{p+1}-\lambda_p=\log\left(1+\frac{l_{p+1}-l_p}{l_p}\right)>Hl_p^{-H};$$

* Landau, *Handbuch*, § 86.

† H. Bohr, "Einige Bemerkungen zum Konvergenzproblem der Dirichletscher Reihen", *Rendiconti del Circolo Matematico di Palermo*, Vol. 37 (1914), pp. 1–16.

‡ *Handbuch*, § 235.

H, wherever it occurs, denoting a positive constant, not of course the same at different occurrences. Thus Bohr's condition is satisfied, and Lemma ϵ follows from (6 . 253).

6 . 3. We can now prove our theorems. We take $T = \xi^\gamma$, where $0 < \gamma < 2$. We choose arbitrary positive numbers δ and ϵ, and take $c = 2+\delta$.

We then apply Cauchy's theorem to the integral

$$\frac{1}{2\pi i}\int \zeta_2(s)\,\frac{\xi^s}{s}\,ds,$$

taken round the rectangle

$$(c-iT,\ c+iT,\ -K+iT,\ -K-iT),$$

the sides of which, taken in order, we denote by (1), (2), (3), and (4). Using Lemma α, we obtain

$$(6.31)\quad \frac{1}{2\pi i}\int \zeta_2(s)\,\frac{\xi^s}{s}\,ds = \int_{(1)}+\int_{(2)}+\int_{(3)}+\int_{(4)} = \frac{\xi^2}{2\omega\omega'}+\frac{\omega+\omega'-2a}{2\omega\omega'}\,\xi+\zeta_2(0).$$

Now

$$(6.32)\qquad \int_{(1)} = W(\xi)+O\left(\frac{\xi^c}{T}\right) = W(\xi)+O(\xi^{2+\delta-\gamma}),$$

by Lemma ϵ; and

$$(6.33)\qquad \int_{(3)} = O\left(\xi^{-K}\int_{-T}^{T}|t|^{K-\frac12+\epsilon}\,dt\right) = O\left(\frac{T^{K+\frac12+\epsilon}}{\xi^K}\right) = O(\xi^{-K+(K+\frac12)\gamma+\epsilon}),$$

by Lemma δ. It remains to estimate the contributions of the horizontal sides; and it is clear, from Lemma δ, that the contribution of either is of the form

$$O\Big(\operatorname*{Max}_{-K\leqslant\sigma\leqslant c}\xi^\sigma T^{\mu(\sigma)-1+\epsilon}\Big) = O(\operatorname{Max}\xi^\eta),$$

where

$$\eta = \sigma+\left\{\frac{(\frac12+K)(2-\sigma)}{2+K}-1\right\}\gamma+\epsilon \quad (-K\leqslant\sigma\leqslant 2),$$

$$\eta = \sigma-\gamma+\epsilon \quad (2\leqslant\sigma\leqslant c).$$

It is clear that η cannot exceed the greater of its values for $\sigma = -K$ and $\sigma = c$, viz.

$$-K+(K-\tfrac12)\gamma+\epsilon,\qquad 2+\delta-\gamma+\epsilon.$$

The possible error-term arising from the first of these values may be absorbed into that already present in (6 . 33). That corresponding to

the second, as well as that in (6.32), may be absorbed in a single term $O(\xi^{2+\delta-\gamma+\epsilon})$. We have therefore, on collecting our results,

$$(6.34) \qquad W(\xi) = \frac{\xi^2}{2\omega\omega'} + \frac{\omega+\omega'-2a}{2\omega\omega'}\xi + O\{\xi^{-K+(K+\frac{1}{2})\gamma+\epsilon}\} + O(\xi^{2+\delta-\gamma+\epsilon}).$$

We have still γ at our disposal. Taking

$$-K+(K+\tfrac{1}{2})\gamma = 2+\delta-\gamma,$$

we obtain

$$\gamma = \frac{2+\delta+K}{\frac{3}{2}+K}$$

(which is, as we supposed, positive and less than 2), and

$$2+\delta-\gamma = \frac{(2+\delta)(\frac{1}{2}+K)-K}{\frac{3}{2}+K}.$$

This is equal to $(1+K)/(\frac{3}{2}+K) < 1$ when $\delta = 0$, and is therefore less than unity if δ is sufficiently small. We have therefore

$$(6.35) \qquad W(\xi) = \frac{\xi^2}{2\omega\omega'} + \frac{\omega+\omega'-2a}{2\omega\omega'}\xi + O(\xi^a),$$

where $a < 1$. In order to obtain Theorem **A5**, it is only necessary to attribute to a the particular value $\omega+\omega'$ and to replace ξ by η, since $W(\xi)$ then becomes $N(\eta)$.

Our argument naturally yields a definite value for a. But it becomes clear, when we consider the particular case of a *quadratic* θ, that the value so obtained is, in the light of Theorem **A2**, not the best value possible. We are therefore content to show that a is in any case less than unity.

Additional Note (*March* 13*th*, 1921).

We have developed the transcendental method of § 6 considerably since this paper was first communicated to the Society.

Suppose that $k \geqslant 0$ and

$$W_k(\xi) = \sum_{l_p<\xi} (\xi-l_p)^k.$$

Then

$$W_k(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta_2(s)\frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)}\xi^{s+k}\,ds$$

if $c > 2$. We transform this equation by (1) moving back the path of integration to the line $\sigma = -q < 0$, with the appropriate corrections for the residues, (2) substituting for $\zeta_2(s)$ from (6.221), and (3) integrating term

by term. This process can be justified if $\theta = \omega/\omega'$ is algebraic and k and q are chosen appropriately, and we obtain an expression for $W_k(\xi)$ in the form of an absolutely convergent series.

We then make use of a lemma which is of some interest in itself, viz.: *if there are constants* $h \geqslant 1$ *and* $H > 0$ *such that*

$$n^h \,|\sin n\theta\pi| > H \tag{1}$$

for all positive integral values of n, *then the series*

$$\Sigma \frac{1}{n^{h+\epsilon}\,|\sin n\theta\pi|}$$

is convergent for every positive ϵ.

Using this lemma and our series for $W_k(\xi)$, we are able to show that *if* (1) *is true for all positive integral values of* n, *then*

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + O(\eta^{a+\epsilon}), \tag{2}$$

where $a = (h-1)/h$, *for every positive* ϵ. This is included in Theorem **A3** if $h = 1$; but is in all other cases considerably more precise than anything proved in the paper.

In (2) the index $a = (h-1)/h$ of the power of η is the best possible one. For we can also show that *if*

$$n^h \,|\sin n\theta\pi| < H \tag{3}$$

for an infinity of values of n, *then each of the inequalities*

$$N(\eta) - \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} > A\eta^a, \quad N(\eta) - \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} < -A\eta^a, \tag{4}$$

where A *is a positive constant depending on* h *and* H, *is true for a sequence of indefinitely increasing values of* η.

We are further able to obtain an "explicit formula" for $N(\eta)$; viz.

$$N(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'} + \frac{\omega^2+\omega'^2+3\omega\omega'}{12\omega\omega'} - \frac{1}{2\pi}\Sigma\left(\frac{\cos\dfrac{2\mu\pi}{\omega}(\eta-\tfrac{1}{2}\omega')}{\mu\sin\dfrac{\mu\omega'\pi}{\omega}} + \frac{\cos\dfrac{2\nu\pi}{\omega'}(\eta-\tfrac{1}{2}\omega)}{\nu\sin\dfrac{\nu\omega\pi}{\omega'}}\right). \tag{5}$$

Here $\theta = \omega/\omega'$ is irrational and algebraic, and the series is to be interpreted as meaning

$$\lim \sum_{\mu<\omega R,\ \nu<\omega' R}$$

when $R \to \infty$ in an appropriate manner.

The most difficult of the remaining problems is that of determining whether there is *any* θ for which the error-term in $N(\eta)$, or the sum $s(\theta, n)$ is *bounded.* The answer is in the negative. We can prove, in fact, that *there exists an* $A > 0$ *such that, for every irrational* θ,

$$|s(\theta, n)| > A \log n$$

for an infinity of values of n. Further, *given* K, *there exists a* $B = B(K) > 0$ *such that, for every* θ *for which* $a_n < K$, *the inequalities*

$$s(\theta, n) > B \log n, \qquad s(\theta, n) < -B \log n,$$

are satisfied each for an infinity of values of n.

The corresponding Cesàro means behave rather differently. It is possible to find θ's for which the first Cesàro mean $\sigma(\theta, n)$ of $s(\theta, n)$ is bounded, and others for which $\sigma(\theta, n)/\log n$ tends to a limit other than zero.

We may take this opportunity of correcting a misstatement in our communication to the Cambridge Congress referred to on p. 15. It was stated there that

$$\sum_{\nu=1}^{n} \{\nu\theta\}^2 = \tfrac{1}{12}n + O(1)$$

for *every* irrational θ. This is untrue; but the equation holds for very general classes of values of θ, and in particular for any θ whose partial quotients are bounded.

CORRECTIONS

p. 32, line above (6.254). Read: $l_p = e^{\lambda_p}$.

p. 34, last line before Additional Note. Read: Theorem A 3.

COMMENTS

This paper was communicated to the Society at its meeting on 22 April, 1920 (see *Proc.* 19, xxiii–xxiv).

§ 1.2. The reference, in the sentence following (1.211), to the triangle with vertex at (1, 1) does not seem to be appropriate. In comparing the number of integer points in a region with the area, the simplest procedure is to put a square of side 1 with its centre at each integer point in the region. As an approximation to the number of points, we then get the area of the triangle

$$x > \tfrac{1}{2}, \qquad y > \tfrac{1}{2}, \qquad \omega x + \omega' y < \eta.$$

This agrees with the main term in (1.21) except for an additive constant.

§ 3. For the proofs of the O results (Theorems A1, B1, A3, B3) one could use in place of the transformation formula (3.24) the simpler formula obtained by counting the integer points horizontally instead of vertically. In the notation of the paper, this is

$$S + \tfrac{1}{2}f + \tfrac{1}{2}\theta f(1-f) = S^* + \tfrac{1}{2}f' + \tfrac{1}{2}\theta^{-1}f'(1-f'), \tag{1}$$

where

$$S^* = \sum_{\nu \leqslant \eta/\omega'} \{\nu\theta^{-1} - f\}.$$

Thus S^* takes the place of S', defined in (3.122). But for the Ω results (Theorems A2, B2, A4, B4) it seems that this formula is not adequate.

The preceding remark is relevant to the tetrahedron problem, mentioned in the introduction. There are still two formulae similar to (1) above, but there is no obvious analogue of (3.24), since it is not possible to put two tetrahedra together to make a rectangular box, in the manner of the figure on p. 19.

§§ 5.2, 5.3. The numbers $q_0, q_1, q_2, \ldots$ are the Fibonacci numbers. The representation of n in the form stated after (5.34) is obtained by taking q_s to be the largest Fibonacci number not exceeding n, then q_{s_1} to be the largest not exceeding $n - q_s$, and so on.

§ 5.6. The behaviour of $\sum \{\mu\theta\}$ for rational θ depends on the convention adopted for the value of $\{t\}$ when t is an integer. In the present paper the value $-\tfrac{1}{2}$ is used, in accordance with (1.12). But in general a more appropriate value is 0, so that $\{t\}$ is an odd function of t. With this convention, $\sum \{\mu\theta\}$ oscillates finitely for rational θ.

§ 6. Although the analytical method of § 6, which was developed further in 1922, 9, is a remarkable triumph of technique, it was not necessary for the proofs of Theorems A5 and B5. More precise results than these were proved elementarily in the second half of 1922, 9, and had in the meantime been found by Ostrowski, using a different elementary method.

As regards algebraic values of θ, see the comments on 1922, 5. The effect of Roth's theorem is that one can take α arbitrarily small in the results of Theorems A5 and B5.

Some problems of Diophantine approximation: The lattice-points of a right-angled triangle

(Second memoir)

By G. H. HARDY in Oxford and J. E. LITTLEWOOD in Cambridge

1. Introduction.

1.1. This memoir is a sequel to one published recently in the *Proceedings of the London Mathematical Society*[1]). It contains the proofs of a number of theorems enunciated in an appendix to our former memoir, together with a considerable amount of additional matter.

The problems which we consider have occupied us at intervals since 1912, when we referred to them briefly in a communication to the Cambridge Congress[2]), and indicated certain questions which we were then unable to answer. In the meantime they have attracted the attention of Herr HECKE[3]) and Herr OSTROWSKI[4]), who have dealt with them in two very beautiful memoirs published recently in this journal, and to whom we are indebted for this opportunity of publishing our own.

The very remarkable analysis of HECKE is mainly transcendental, while OSTROWSKI's is entirely elementary, and we use both elementary and transcendental methods. Our transcendental method is entirely unlike HECKE's, and little need be said as regards the relations between his results and ours. The relations of our elementary work to OSTROWSKI's are a good deal closer. Our method, depending as it does on formulae like those of SYLVESTER and LERCH[5]), is fundamentally different, but the results are to a considerable extent the same. A detailed analysis

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[1]) G. H. HARDY and J. E. LITTLEWOOD, "Some problems of Diophantine approximation: The lattice-points of a right-angled triangle", *Proc. London Math. Soc.* (2), 20 (1921), 15—36. We refer to this memoir as *1*.

[2]) G. H. HARDY and J. E. LITTLEWOOD, "Some problems of Diophantine approximation", *Proceedings of the fifth international congress of mathematicians*, 1912, 1, 223—229.

[3]) E. HECKE, "Über analytische Funktionen und die Verteilung von Zahlen mod. Eins", *Hamburg. Math. Abh.* 1 (1921), 54—76. We refer to this as *H*.

[4]) A. OSTROWSKI, "Bemerkungen zur Theorie der Diophantischen Approximationen", *ibid.*, 77—98. We refer to this as *O*.

[5]) See 3.1.

1922, 9 (with J. E. Littlewood) *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, 1, 212–49.

of the points of resemblance and difference would occupy a good deal of space and seems to us unnecessary, though we indicate the theorems which have been proved by OSTROWSKI as they occur. We should add one word, however, as to the relative advantages of OSTROWSKI's method and our own. In some parts of the theory the advantage of OSTROWSKI's method seems to us incontestable; in others there is little between them; and in others the advantage seems to lie with ours. It seems to us desirable to develop the whole theory systematically from our own point of view; but where OSTROWSKI's method is clearly simpler, we content ourselves with an outline of our demonstrations, suppressing the algebraical details of our work and condensing our argument to the limit of intelligibility. In particular we have followed this course in 3.4.

1.2. All our theorems involve an irrational number θ, which we generally suppose positive, less than 1, and expressed as a simple continued fraction

(1.21) $$\theta = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots.$$

We write

(1.22) $$\theta = \frac{1}{a_1+\theta_1},\quad \theta_1 = \frac{1}{a_2+\theta_2},\quad \ldots,$$

and denote the convergents to (1.21) by

$$\frac{p_1}{q_1} = \frac{1}{a_1},\quad \frac{p_2}{q_2} = \frac{a_2}{a_1 a_2+1},\quad \ldots.$$

We shall make continual use of the two lemmas which follow, which are trivial, but very useful, and which seem to have escaped attention.

Lemma 1. *We have*

(1.23) $$\theta_r\theta_{r+1} < \frac{1}{2}.$$

More generally

(1.231) $$\theta_r\theta_{r+1}\ldots\theta_{r+s-1} < \frac{1}{u_s},$$

where u_s is the s-th term of FIBONACCI's series 1, 2, 3, 5, 8, 13, ...

We deduce this from

Lemma 2. *We have*

(1.24.) $$\frac{1}{2\theta\theta_1\ldots\theta_{r-1}} < q_r < \frac{1}{\theta\theta_1\ldots\theta_{r-1}}.$$

For

$$q_r+\theta_r q_{r-1} = (a_r+\theta_r)\,q_{r-1}+q_{r-2} = \frac{q_{r-1}+\theta_{r-1}q_{r-2}}{\theta_{r-1}},$$

and so

$$q_r + \theta_r q_{r-1} = \frac{1}{\theta\,\theta_1 \ldots \theta_{r-1}},$$

which proves the lemma.

To deduce Lemma 1 we observe that (1.24) gives

$$\theta\,\theta_1 \ldots \theta_{s-1} < \frac{1}{q_s},$$

and that, for given s, q_s is a minimum when $a_1 = a_2 = \ldots = a_s = 1$, in which case $q_s = u_s$. Taking now θ_r for θ we obtain the desired result.

1.3. We write, as usual, $[x]$ for the integral part of x, and

(1.31) $$(x) = x - [x], \quad \{x\} = x - [x] - \frac{1}{2}.$$

Thus $\{x\}$ is the arithmetical function denoted, by HECKE and OSTROWSKI, by $R(x) - \frac{1}{2}$. Further we write

(1.32) $$\overline{x} = x - X$$

where X is the integer nearest to x. If x is of the form $n + \frac{1}{2}$, we take $\overline{x} = \frac{1}{2}$.

Throughout our argument the letter A (or occasionally B, C, ...) denotes a positive constant. This constant may be absolute, or may depend upon the parameters involved in the theorem in question; it will not generally be the *same* constant in successive inequalities. The O's and o's which occur involve constants implicitly. It will generally be obvious on what, if any, parameters these constants depend.

We shall frequently be concerned with conditions of the type

(1.331) $$n^h \,|\sin n\theta\pi\,| > A \qquad (n \geqq 1),$$

or

(1.332) $$n^h \,|\sin n\theta\pi\,| < A \qquad (n = n_j),$$

where $h \geqq 1$, and the notation implies that the second inequality is satisfied for an infinite sequence $n_1, n_2, \ldots, n_j$ of values of n. These conditions are obviously equivalent to the corresponding conditions in which $\sin n\theta\pi$ is replaced by $\overline{n\theta}$. Further, (1.331) and (1.332) are equivalent to

(1.341) $$q_{r+1} < A q_r^h \qquad (r \geqq 1),$$

(1.342) $$q_{r+1} > A q_r^h \qquad (r = r_j),$$

and these again, by Lemma 2, to

$$\frac{1}{\theta_r} < \frac{A}{(\theta\,\theta_1 \ldots . \theta_{r-1})^{h-1}} \qquad (r \geqq 1), \tag{1.351}$$

$$\frac{1}{\theta_r} > \frac{A}{2^h(\theta\,\theta_1 \ldots . \theta_{r-1})^{h-1}} \qquad (r = r_j). \tag{1.352}$$

It is well-known that a condition of the type (1.331) is satisfied by every algebraical θ.

The A's of these inequalities may be absolute or may depend on θ and h. If absolute in (1.331) and (1.332), they are absolute in the other inequalities.

2. The analytic treatment of the triangle problem.

2.1. In this section we continue the study of the "triangle" problem (Problem A of *1*) by analytic methods. We denote by $N(\eta)$ the number of lattice-points inside the triangle whose sides are

$$x = 0, \quad y = 0, \quad wx + w'y = \eta > 0,$$

where w and w' are two positive numbers whose ratio $\theta = \frac{w}{w'}$ is irrational. We proved in *1* that

$$N(\eta) = R(\eta) + U(\eta) = R(\eta) + \Phi(\eta) + S(\eta) \tag{2.11}$$

where

$$R(\eta) = \frac{\eta^2}{2\,w\,w'} - \frac{\eta}{2\,w} - \frac{\eta}{2\,w'}, \tag{2.111}$$

$$\frac{\eta}{w} = \left[\frac{\eta}{w}\right] + f, \quad \frac{\eta}{w'} = \left[\frac{\eta}{w'}\right] + f', \tag{2.112}$$

$$\Phi(\eta) = \frac{1}{2}f + \frac{1}{2}\theta f(1-f), \tag{2.113}$$

$$S(\eta) = \sum_{1 \leqq \mu \leqq \frac{\eta}{w}} \{\mu\,\theta - f'\}. \tag{2.114}$$

Our problem is the study of $U(\eta)$, or, since $\Phi(\eta) = O(1)$, of $S(\eta)$. We proved in *1*, (*a*) that

$$U(\eta) = o(\eta) \tag{2.12}$$

for every irrational θ, (*b*) that this result is the most that is universally true, (*c*) that

$$U(\eta) = O(\log \eta) \tag{2.13}$$

when θ has bounded quotients, (*d*) that there are θ's, with bounded quotients, for which each of the inequalities

$$U(\eta) > A \log \eta, \quad U(\eta) < -A \log \eta \tag{2.14}$$

is satisfied for arbitrarily large values of η, and (*e*) that

$$U(\eta) = O(\eta^{\alpha}), \tag{2.15}$$

where $\alpha = \alpha(\theta) < 1$, whenever θ satisfies an inequality of the type (1.331), and in particular whenever θ is algebraic. Of these results (*a*)—(*d*) were proved by elementary reasoning, and (*e*) analytically. Our immediate object is to prove more precise results in place of (*e*).

We denote by $\zeta_2(s) = \zeta_2(s, a, w, w')$ the analytic function defined, when the real part σ of $s = \sigma + it$ is greater than 2, by the series

$$\zeta_2(s) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(a + mw + nw')^s} = \sum_p l_p^{-s}$$

where a is positive and l_p^{-s} has its principal value. This function is a degenerate case of the "double Zeta-function" of BARNES[1]). Its principal properties, so far as they are relevant to our investigations, are summarised in *1*.

In this section the A's, B's, ... are in general not absolute but functions of the parameters θ, h, ...

2.2. Lemma 3. *If* $h \geqq k \geqq 1$ *and*

$$n^h |\sin n\theta\pi|^k > A \tag{2.21}$$

then

$$S_m = \sum_1^m \frac{1}{n^h |\sin n\theta\pi|^k} = O(\log m)^2. \tag{2.22}$$

It is plain that neither hypothesis nor conclusion is affected if we replace $\sin n\theta\pi$ by $\overline{n\theta}$. We have therefore

$$n^h |\overline{n\theta}|^k > B\text{[2])}, \tag{2.231}$$

and, if we define h_n by

$$n^{h_n} |\overline{n\theta}|^k = B, \tag{2.232}$$

we have $h_n < h$. Consider now the sum

$$T_m = \sum_m^{2m} \frac{1}{n^h |\overline{n\theta}|^k} = \sum_m^{2m} u_n. \tag{2.224}$$

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[1]) E. W. BARNES, "A memoir on the double Gamma-function", *Phil. Trans. Roy. Soc.* (A), 196 (1901), 265—387.

[2]) B is the same constant throughout this sub-section.

The terms of T_m for which $h_n \leqq h-1$ contribute

$$O\left(\sum_m^{2m} \frac{1}{n}\right) = O(1).$$

We classify the remaining terms as follows. We choose a positive integer η, write

$$\mathfrak{h}_r = h - 1 + \frac{r}{\eta} \qquad (r = 0, 1, 2, \ldots, \eta - 1)$$

and call a typical term u_n of T_m a *term of class* r if $\mathfrak{h}_r \leqq h_n < \mathfrak{h}_{r+1}$. If u_n is of class r,

$$|\overline{n\theta}|^k \leqq B n^{-\mathfrak{h}_r}.$$

But

$$|\overline{s\theta}|^k > B s^{-h} > 2^k B n^{-\mathfrak{h}_r}$$

if

$$0 < s < C n^{\frac{\mathfrak{h}_r}{h}} \tag{2.25}$$

and then

$$|\overline{(n+s)\theta}|^k > B n^{-\mathfrak{h}_r} > B(n+s)^{-\mathfrak{h}_r}$$

for all values of s which satisfy (2.25). Hence no term u_{n+s} corresponding to such a value of s is a term of class as high as r.

The number of terms of class r is therefore $O\left(m^{1-\frac{\mathfrak{h}_r}{h}}\right)$, and their contribution to T_m is

$$O\left(m^{1-\frac{\mathfrak{h}_r}{h}-h+\mathfrak{h}_{r+1}}\right) = O\left(m^{\frac{1}{\eta}}\right),$$

since

$$1 - \frac{\mathfrak{h}_r}{h} - h + \mathfrak{h}_{r+1} = \frac{1}{\eta} - \left(1 - \frac{1}{h}\right)\left(1 - \frac{r}{\eta}\right) < \frac{1}{\eta}.$$

It follows that

$$T_m = O\left(\eta\, m^{\frac{1}{\eta}}\right) = O(\log m), \tag{2.26}$$

since we may take $\eta = [\log m]$.

If now we define ν by $D \leqq \frac{m}{2^\nu} \leqq 2D$, we have

$$U_m = \sum_1^m \frac{1}{n^h |\overline{n\theta}|^k} = O(1) + O\left(\sum_{\mu=1}^{\nu} T_{2^{-\mu} m}\right)$$
$$= O(1) + O\left(\sum_{\mu=1}^{\nu} \log \frac{m}{2^\mu}\right) = O(\log m)^2,$$

which is equivalent to (2.22).

As a corollary we have

Lemma 4. *If* (2.21) *is satisfied, the series*

$$\sum \frac{1}{n^{h+\varepsilon}|\sin n\theta\pi|^k}$$

is convergent for every positive ε.

The case of most importance is that in which $k = 1$, when (2.21) reduces to (1.331).

2.31. We proceed to establish an analytical formula for the sum

(2.311) $$W_k(\xi) = \sum_{l_p \leq \xi} (\xi - l_p)^k.$$

The k here has no connection with that of 2.2.

In order to abbreviate our formulae we adopt the following convention. We are often concerned with associated pairs of series, of the forms $\sum \Phi(m, w, w')$ and $\sum \Phi(m, w', w)$; and we write generally

(2.312) $$X(w, w')\sum \Phi(m, w, w') + X(w', w)\sum \Phi(m, w', w) = (X(w, w')\sum \Phi(m, w, w'))^*.$$

Such associated pairs of series have been considered by various writers, and in particular by LERCH[1]. We shall also sometimes use a similar notation when there is no summation.

We recall the formula[2])

(2.313) $$\frac{\zeta_2(s, a, w, w')}{(2\pi)^{s-1}\Gamma(1-s)} = \left(\frac{1}{w^s}\sum_{m=1}^{\infty} \frac{\sin\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' - a\right) + \frac{1}{2}(1-s)\pi\right)}{m^{1-s}\sin\frac{mw'\pi}{w}}\right)^*.$$

This formula is valid whenever $0 < a \leq w + w'$ and the two series on the right are absolutely convergent.

2.32. Theorem 1. *Suppose that* (1.331) *is satisfied and that* $k > h - 1$. *Then*

(2.321) $$W_k(\xi) = V_k(\xi) -$$
$$(2\pi)^{-k-1}\Gamma(k+1)\left(w^k \sum_{m=1}^{\infty} \frac{\cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' + \xi - a\right) - \frac{1}{2}k\pi\right)}{m^{k+1}\sin\frac{mw'\pi}{w}}\right)^* + O(\xi^{k+1-q}),$$

where

(2.3211) $$V_k(\xi) = \sum_{\mu=0}^{q} C_\mu \xi^{k+2-\mu},$$

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[1]) See, for example, M. LERCH, "Sur une série analogue aux fonctions modulaires", *Comptes Rendus*, 18 April 1904; G. H. HARDY, "On certain series of discontinuous functions connected with the modular functions", *Quarterly Journal* (1904), 93—123; and writings of RIEMANN and H. J. S. SMITH there referred to.

[2]) (6.221) of *1*.

the C's being constants, of which

$$(2.3212)\qquad C_0 = \frac{1}{(k+1)(k+2)\,w w'},\quad C_1 = \frac{w+w'-2a}{2(k+1)\,w w'},$$

and q is the integer such that $k+1<q\leqq k+2$.

We suppose for the present that $k>h-\frac{1}{2}$. Since $k>0$, we have[1])

$$(2.322)\qquad W_k(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta_2(s)\frac{\Gamma(k+1)\,\Gamma(s)}{\Gamma(k+1+s)}\,\xi^{k+s}\,ds,$$

if $c>2$. We choose δ so that $\frac{1}{2}<\frac{1}{2}+\delta<1$, $k>h-\frac{1}{2}+\delta$, and $\gamma=\frac{1}{2}-k+\delta$ is not an integer. We have then

$$(2.323)\qquad \frac{1}{2}-k<\gamma<1-h$$

and

$$(2.324)\qquad \zeta_2(s) = O\left(|t|^{\frac{1}{2}-\gamma}\right) = O(|t|^k)$$

uniformly for $\sigma\geqq\gamma$[2]).

We may therefore apply CAUCHY's Theorem to the strip $\gamma\leqq\sigma\leqq c$, and we obtain

$$(2.325)\qquad W_k(\xi) = U_k(\xi) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \zeta_2(s)\frac{\Gamma(k+1)\,\Gamma(s)}{\Gamma(k+1+s)}\,\xi^{k+s}\,ds,$$

where

$$(2.326)\qquad U_k(\xi) = C_0\,\xi^{k+2} + C_1\,\xi^{k+1} + \sum_{\nu=0}^{p}\frac{(-1)^\nu\zeta_2(-\nu)}{\nu!}\,\frac{\Gamma(k+1)}{\Gamma(k+1-\nu)}\,\xi^{k-\nu},$$

p being the largest integer such that $-p>\gamma$. We may write

$$(2.327)\qquad U_k(\xi) = \sum_{\mu=0}^{r} C_\mu\,\xi^{k+2-\mu},$$

where $r=p+2$ is the integer such that $k+\frac{1}{2}-\delta<r<k+\frac{3}{2}-\delta$. The index of the last power in $U_k(\xi)$ lies between $\frac{1}{2}+\delta$ and $\frac{3}{2}+\delta$, whereas that in $V_k(\xi)$ lies between 0 (inclusive) and 1; the form of the two sums is otherwise the same.

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[1]) G. H. HARDY and M. RIESZ, "The general theory of Dirichlet's series", *Camb. Math. Tracts*, 18 (1915), 51 (Theorem 40).

[2]) See *1*, Lemmas β, δ. The series which occur in (2.313) (or (6.221) of *1*) are absolutely convergent for $\sigma=\gamma$, in virtue of (2.323) and Lemma 4; and the conclusion then follows from Lemma δ.

2.33. In (2.325) we substitute for $\zeta_2(s)$ from the formula (2.313), valid since $1-\gamma>h$, and integrate term-by-term. This term-by-term integration is legitimate because

$$(2\pi)^{s-1}\Gamma(1-s)\left(\frac{1}{w^s}\;\frac{\sin\left(\frac{2m\pi}{w}\left(\frac{1}{2}w'-a\right)+\frac{1}{2}(1-s)\pi\right)}{m^{1-s}\sin\frac{mw'\pi}{w}}\right)^*$$

$$=O\left(|t|^{\frac{1}{2}-\gamma}\left(\frac{1}{m^{1-\gamma}\left|\sin\frac{mw'\pi}{w}\right|}\right)^*\right)$$

and

$$\int^{\infty}O\left(|t|^{-k-\frac{1}{2}-\gamma}\right)dt,\quad \sum\left(\frac{1}{m^{1-\gamma}\left|\sin\frac{mw'\pi}{w}\right|}\right)^*$$

are convergent. We thus obtain

(2.331) $$W_k(\xi)=U_k(\xi)+J_1+J_2+J_1'+J_2',$$

where

(2.3311) $$J_1=\frac{\xi^k\,\Gamma(k+1)}{4\pi}\sum_1^{\infty}\frac{e^{\frac{2m\pi i}{w}\left(\frac{1}{2}w'-a\right)}}{m\sin\frac{mw'\pi}{w}}\,\Phi\left(\frac{2m\pi i\xi}{w}\right),$$

J_2 is conjugate to J_1, J_1' and J_2' are obtained from J_1 and J_2 by exchanging w and w',

(2.3312) $$\Phi(u)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\pi}{\sin s\pi}\;\frac{(-u)^s}{\Gamma(k+1+s)}\,ds,$$

and $(-u)^s$ has its principal value, real when u is negative.

The function $\Phi(u)$ is one of a type whose asymptotic expansions have been considered by various writers. We have

(2.332) $$\Phi(u)=\frac{u^{-p-1}}{\Gamma(k-p)}+O(|u|^{-p-2})-u^{-k}e^u+O(|u^{-k-1}e^u|)$$

$$=\Phi_1+\Phi_2+\Phi_3+\Phi_4,$$

say, with similar formulae for the derivatives of $\Phi(u)$, which may be written down by formal differentiation[1]).

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[1]) The function may be expressed in the form

$$-\sum_{m=0}^{\infty}\frac{u^{m-p}}{\Gamma(k+1+m-p)};$$

here p is the integer such that $-p-1<\gamma<-p$. As regards the asymptotic expansions of such functions, see, for example, E. W. Barnes, "On functions defined by simple types of hypergeometric series", *Trans. Camb. Phil. Soc.*, 20 (1906), 253—279. The actual result required here is easily proved in a variety of ways.

We denote by $J_{1,1}, \ldots$ the results of replacing Φ, in $J_1, \ldots$ by $\Phi_1 + \Phi_2$; by $J_{1,2}, \ldots$ the results of replacing Φ by $\Phi_3 + \Phi_4$: so that we have four equations of the type

$$J_1 = J_{1,1} + J_{1,2}. \tag{2.333}$$

Consider first the sums $J_{1,1}, \ldots$ with 1 as second suffix. If we substitute Φ_1 for Φ in $J_1, \ldots$ and combine the results, we obtain, after a straightforward calculation,

$$(-1)^{p+1}(2\pi)^{-p-2}\frac{\Gamma(k+1)}{\Gamma(k-p)}\xi^{k-p-1}$$
$$\cdot\left(w^{p+1}\sum\frac{\sin\left(\frac{2m\pi}{w}\left(\frac{1}{2}w'-a\right)+\frac{1}{2}(p+2)\pi\right)}{m^{p+2}\sin\frac{mw'\pi}{w}}\right)^*,$$

which is equal, by (2.313), to

$$\frac{(-1)^{p+1}\zeta_2(-p-1)}{(p+1)!}\;\frac{\Gamma(k+1)}{\Gamma(k-p)}\xi^{k-p-1}.$$

This is of the same form as the general term in (2.326), and the contribution of Φ_1 may be accounted for by replacing p, in (2.326), by $p+1$. If we do this, the last index in $U_k(\xi)$ will lie between $-\frac{1}{2}+\delta$ and $\frac{1}{2}+\delta$, and $U_k(\xi)$ may become identical with $V_k(\xi)$ or may contain one extra term; it is in any case of the form $V_k(\xi)+O(\xi^{k+1-q})$.

There is also Φ_2 to be considered, but Φ_2 is of lower order than Φ_1 to the extent of a factor $\frac{1}{m\xi}$, and its contribution is accordingly trivial. We therefore obtain

$$U_k(\xi)+J_{1,1}+J_{2,1}+J'_{1,1}+J'_{2,1} = V_k(\xi)+O(\xi^{k+1-q}). \tag{2.334}$$

Next we consider the sums $J_{1,2}, \ldots$. Substituting first Φ_3 for Φ, we obtain, after reduction,

$$-(2\pi)^{-k-1}\Gamma(k+1)\left(w^k\sum\frac{\cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w'+\xi-a\right)-\frac{1}{2}k\pi\right)}{m^{k+1}\sin\frac{mw'\pi}{w}}\right)^*.$$

And as Φ_4 is of lower order than Φ_3, by a factor $\frac{1}{m\xi}$, its contribution is $O\left(\frac{1}{\xi}\right) = O(\xi^{k+1-q})$. Thus

(2.335) $$J_{1,2}+J_{2,2}+J'_{1,2}+J'_{2,2}=S+O(\xi^{k+1-q}),$$
where S is the second term on the right of (2.321).

Collecting our results from (2.331), (2.333), (2.334), and (2.335), we obtain the result of Theorem 1. At present, however, the theorem is proved only when $k>h-\frac{1}{2}$, and it is necessary to extend this range to $k>h-1$.

2.34. Suppose then that $k=h+\eta>h>h-\frac{1}{2}$, so that (2.321) is proved; and let us differentiate formally with respect to ξ, and divide by k. We have
$$\frac{1}{k}\frac{dW_k}{d\xi}=W_{k-1},$$
and
$$\frac{1}{k}\frac{dV_k}{d\xi}=V_{k-1}+O(\xi^{k+1-q})=V_{k-1}+O(\xi^{k-q'}),$$
where $q'=q-1$ is the integer such that $k<q'\leqq k+1$. Finally the same process, applied to the infinite series, yields the corresponding series for $k-1$, a series which is, by Lemma 4, absolutely and uniformly convergent. It appears then that we are led back to our original formula, with $k-1$ in place of k, and this is just what we require. The proof is however insufficient, since we are not entitled to differentiate the error term $O(\xi^{k+1-q})$.

There is no difficulty of principle in completing the proof, but it is necessary to go back to (2.331). We differentiate this equation, and substitute for $\Phi(u)$ and its derivative $\Phi'(u)$ the approximations given by (2.332) and the corresponding derived equation. The result is an absolutely and uniformly convergent series, and the term-by-term differentiation is thereby justified. We have then only to repeat our previous calculations, in a slightly more complicated form, the formulae which we use being in substance the formal derivatives of those which we have used already. The final result is the same as before, except that k is replaced by $k-1$, and that the result holds whenever $k-1=h-1+\eta>h-1$. When we restore k in the place of $k-1$, the proof of Theorem 1 is completed.

2.4. From Theorem 1 we can deduce a proof of the equation numbered (2) in the appendix to our memoir *1*. This equation is not quite so precise as one which we shall obtain later in an elementary manner, but it is of some interest to show how it follows from the analytic theory.

Theorem 2. *If* (1.331) *is satisfied then*

(2.41) $$U(\eta)=O\left(\eta^{1-\frac{1}{h}+\varepsilon}\right)$$
for every positive ε.

Suppose, in (2.321), that k is the integer such that $k \leqq h < k+1$. The q of Theorem 1 is now $k+2$, and

(2.42)
$$W_k(\xi) = V_k(\xi) - A\left(w^k \sum_1^\infty \frac{\cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' + \xi - a\right) - \frac{1}{2}k\pi\right)}{m^{k+1}\sin\frac{m w'\pi}{w}}\right)^* + O\left(\frac{1}{\xi}\right)$$
$$= V_k(\xi) + S(\xi) + O\left(\frac{1}{\xi}\right)$$

say. If now we suppose $0 < \mathfrak{z} < 1$, and write generally.

$$\varDelta f(\xi) = \varDelta_{\mathfrak{z}}^k f(\xi) = f(\xi + k\mathfrak{z}) - \binom{k}{1} f(\xi + (k-1)\mathfrak{z}) + \binom{k}{2} f(\xi + (k-2)\mathfrak{z}) - \ldots,$$

we have

(2.43)
$$\varDelta W_k = \varDelta V_k + \varDelta S + O\left(\frac{1}{\xi}\right).$$

We consider first $\varDelta S$. Since

$$\varDelta \cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' + \xi - a\right) - \frac{1}{2}k\pi\right) = O(\mathrm{Min}(m^k \mathfrak{z}^k, 1)),$$

we have

(2.441)
$$\varDelta S = O\left(\mathfrak{z}^k \sum_{m\mathfrak{z} \leqq 1} \frac{1}{m\left|\sin\frac{m w'\pi}{w}\right|}\right)^* + O\left(\sum_{m\mathfrak{z} > 1} \frac{1}{m^{k+1}\left|\sin\frac{m w'\pi}{w}\right|}\right)^*$$
$$= O\left(\mathfrak{z}^k \left(\frac{1}{\mathfrak{z}}\right)^{h-1+\varepsilon}\right) + O\left(\left(\frac{1}{\mathfrak{z}}\right)^{-k-1+h+\varepsilon}\right) = O(\mathfrak{z}^{k+1-h-\varepsilon}).$$

Next

$$\varDelta W_k(\xi) = k! \int_\xi^{\xi+\mathfrak{z}} d\xi_1 \int_{\xi_1}^{\xi_1+\mathfrak{z}} d\xi_2 \ldots \int_{\xi_{k-1}}^{\xi_{k-1}+\mathfrak{z}} W(\xi_k)\, d\xi_k,$$

and so

(2.442)
$$k!\,\mathfrak{z}^k W(\xi) \leqq \varDelta W_k(\xi) \leqq k!\,\mathfrak{z}^k W(\xi + k\mathfrak{z}).$$

Finally

(2.443)
$$\varDelta V_k(\xi) = \mathfrak{z}^k V_k^{(k)}(\xi) + O(\mathfrak{z}^{k+1}\xi) = k!\,\mathfrak{z}^k V(\xi) + O(\mathfrak{z}^{k+1}\xi),$$

where

(2.4431)
$$V(\xi) = \frac{\xi^2}{2ww'} + \frac{w + w' - 2a}{2ww'}\xi + M,$$

M being a constant.

From (2.43), (2.441), (2.442), (2.443), and (2.4431) we deduce, in the first place,

(2.45)
$$W(\xi) \leqq V(\xi) + O(\mathfrak{z}\xi) + O\left(\frac{1}{\mathfrak{z}^k \xi}\right) + O(\mathfrak{z}^{1-h-\varepsilon}).$$

In (2.45) we take $\mathfrak{z} = \xi^{-\frac{1}{h}}$, and we obtain

$$W(\xi) < V(\xi) + O\left(\xi^{1-\frac{1}{h}+\epsilon}\right). \tag{2.46}$$

Similarly we have

$$W(\xi + k\mathfrak{z}) > V(\xi) + O\left(\xi^{1-\frac{1}{h}+\epsilon}\right), \tag{2.47}$$

or, on replacing $\xi + k\mathfrak{z}$ by ξ

$$W(\xi) > V(\xi) + O\left(\xi^{1-\frac{1}{h}+\epsilon}\right). \tag{2.48}$$

Finally, from (2.46) and (2.48) we deduce

$$W(\xi) = V(\xi) + O\left(\xi^{1-\frac{1}{h}+\epsilon}\right). \tag{2.49}$$

Attributing to a the special value $w + w'$ and replacing ξ by η, we obtain (2.41)[1]).

2.51. We next prove a theorem which shows that the index $1 - \dfrac{1}{h}$ of Theorem 2 is the "correct" one.

Theorem 3. *If $h > 1$ and (1.332) is satisfied for an infinity of values of n, then each of the inequalities*

$$U(\eta) > A\eta^{1-\frac{1}{h}}, \quad U(\eta) < -A\eta^{1-\frac{1}{h}} \tag{2.511}$$

is satisfied for arbitrarily large values of η.

Let $f(x)$ be the function defined, when $\Re(x) > 0$, by the equations

$$f(x) = \sum_{m,n=1}^{\infty} e^{-x(mw+nw')} = \sum_{p=1}^{\infty} e^{-xl_p} = \frac{e^{-x(w+w')}}{(1-e^{-xw})(1-e^{-xw'})}. \tag{2.512}$$

We have

$$f(x) = \frac{1}{ww'x^2} - \frac{w+w'}{2ww'x} + \frac{w^2+3ww'+w'^2}{12ww'} + O(x), \tag{2.513}$$

when x is small.

Suppose next that

$$x = \frac{2n\pi i}{w'} + \delta, \tag{2.514}$$

where δ is small and positive, and n has one of the values for which (1.332) is true. Then

$$|e^{-(w+w')x}| > A, \quad |1 - e^{-w'x}| < A\delta,$$

[1]) The proof of the theorem is modelled on the argument used by Landau, „Über Dirichlets Teiler-Problem", *Münchener Sitzungsberichte*, 1915, 317—328.

and

$$|1-e^{-wx}| = \sqrt{(1-e^{-w\delta})^2+4e^{-w\delta}\sin^2 n\pi\theta} < A\sqrt{\delta^2+n^{-2h}},$$

so that

$$|f(x)| > \frac{A}{\delta\sqrt{\delta^2+n^{-2h}}}. \tag{2.515}$$

On the other hand, we have

$$f(x) = \sum_1^\infty e^{-xl_p} = \sum_1^\infty p\int_{l_p}^{l_{p+1}} xe^{-xu}\,du = x\int_0^\infty N(u)e^{-xu}\,du, \tag{2.516}$$

since $N(u)$ is the number of l's which do not exceed u; or

$$f(x) = \frac{1}{ww'x^2} - \frac{w+w'}{2ww'x} + \Phi(x), \tag{2.517}$$

where

$$\Phi(x) = x\int_0^\infty U(u)e^{-xu}\,du. \tag{2.5171}$$

Comparing with (2.513), we see that

$$\Phi(x) \to M = \frac{w^2+3ww'+w'^2}{12ww'} \tag{2.518}$$

when $x\to 0$, and in particular if $x=\delta$ and $\delta\to 0$.

2.52. Now suppose that $\alpha > 0$ and

$$\chi(u) = U(u) + Bu^\alpha \geqq 0\ ^{1)} \tag{2.521}$$

for all sufficiently large values of u, say for $u \geqq u_0$. It follows from (2.521), (2.5171), and (2.518) that

$$\int_0^\infty \chi(u)e^{-\delta u}\,du \sim B\int_0^\infty u^\alpha e^{-\delta u}\,du = B\Gamma(1+\alpha)\delta^{-1-\alpha} = C\delta^{-1-\alpha}, \tag{2.522}$$

say, when $\delta\to 0$. On the other hand, if x is given by (2.514), we have, by (2.5171) and (2.521),

$$\Phi(x) = x\int_0^\infty \chi(u)e^{-xu}\,du - Bx\int_0^\infty u^\alpha e^{-xu}\,du = x\int_0^\infty \chi(u)e^{-xu}\,du + O(1),$$

[1]) B and C (unlike A) retain the same values throughout the argument which follows.

$$(2.523)\qquad |\Phi(x)| < |x|\left(\left|\int_0^{u_0}\chi(u)e^{-\bar{x}u}\,du\right| + \int_{u_0}^{\infty}\chi(u)e^{-\delta u}\,du\right) + O(1)$$
$$< |x|\left(\left|\int_0^{\infty}\chi(u)e^{-\delta u}\,du\right| + O(1)\right) + O(1)$$
$$< 2\cdot\frac{2n\pi}{w'}\cdot C\delta^{-1-\alpha} < ACn\,\delta^{-1-\alpha}$$

by (2.522). Comparing (2.515), (2.517) and (2.523), we see that

$$ACn\,\delta^{-1-\alpha} > \frac{A}{\delta\sqrt{\delta^2+n^{-2h}}},\quad C^2n^2(\delta^2+n^{-2h}) > A\,\delta^{2\alpha}.$$

Taking in particular $\delta = n^{-h}$, we have

$$(2.524)\qquad C > A\,n^{(1-\alpha)h-1}.$$

From (2.524) it follows that $\alpha \geqq 1-\frac{1}{h}$; and if we take $\alpha = 1-\frac{1}{h}$, then $C > A$ or $B > A$. Unless these conditions are satisfied, (2.521) cannot be true for all sufficiently large values of u; and therefore the second of the inequalities (2.511) must be true for arbitrarily large values of η and *some* value of B. The first inequality can naturally be proved in a similar manner.

We have supposed $h > 1$, so that the critical value of α is positive. In this case a less precise form of (2.518), *viz.* $\Phi(x) = o(x^{-\alpha})$, would have been sufficient for our argument. When $h = 1$, the critical value of α is zero. In this case the value of M becomes relevant to the argument. We must take $\chi(u) = U(u) + M + B$, and the final conclusion is that $B > A$, *i. e.* that *each of*

$$U(\eta) > M + A,\qquad U(\eta) < M - A$$

is true for arbitrarily large values of η. The conclusion is not entirely trivial, but it is certainly much less interesting, and is no longer in any sense a best possible result.

2.61. We proceed next to the proof of the exact formula for $N(\eta)$ enunciated at the end of our former memoir[1]). This is the analogue of VORONOÏ's formula for the number of lattice-points in the area $x > 0$, $y > 0$, $xy \leqq \eta$.

It is now necessary to consider the exact definition of $N(\eta)$ when η is of the form $pw + qw'$ and there is a lattice-point on the boundary of the triangle. We agree that such a point is to be counted as one-half.

[1]) *1*, p. 35, formula (5).

Lemma 5. *If λ is positive and ν real, then the integral*

$$(2.611)\qquad \int_0^\infty \frac{\sin \nu x}{(\cosh w\lambda - \cos w x)(\cosh w'\lambda - \cos w' x)} \frac{dx}{x}$$

is convergent; and it may be evaluated by expanding the subject of integration as a double power-series in $e^{-w\lambda}$ and $e^{-w'\lambda}$, and integrating term-by-term.

The formal result of this process is

$$(2.612)\qquad \frac{4}{\sinh w\lambda \sinh w'\lambda} \sum_{p,q=0}^{\infty} \varepsilon_p \varepsilon_q e^{-\lambda(pw+qw')} \int_0^\infty \frac{\sin \nu x \cos p w x \cos q w' x}{x} dx,$$

where $\varepsilon_p = \frac{1}{2}\ (p = 0)$, $\varepsilon_p = 1\ (p > 0)$. There is at most one term for which $\nu \pm p w \pm q w' = 0$. This term, if it exists, we remove from the double series and consider independently, and we denote the modified series by $\sum'$. Since term-by-term integration is certainly permissible over any finite range $(0, X)$, it is sufficient to show that

$$(2.613)\qquad \sum{}' \varepsilon_p \varepsilon_q e^{-\lambda(pw+qw')} \int_X^\infty$$

is convergent and tends to zero when $X \to \infty$; and this will be so if

$$(2.614)\qquad \sigma = \sum{}' \varepsilon_p \varepsilon_q e^{-\lambda(pw+qw')} \left| \int_X^\infty \frac{\sin \varrho x}{x} dx \right| < \varepsilon,$$

where

$$(2.615)\qquad \varrho = \nu \pm p w \pm q w',$$

is less than ε for every positive ε and sufficiently large values of X.

We write

$$(2.616)\qquad \sigma = \sum{}' = \sum_{|\varrho| X \geqq H} + \sum_{|\varrho| X < H} = \sigma_1 + \sigma_2,$$

say, where $H > 0$. Since

$$\left| \int_X^\infty \frac{\sin \varrho x}{x} dx \right| = \left| \int_{|\varrho| X}^\infty \frac{\sin y}{y} dy \right| < \operatorname{Min}\left(\frac{A}{|\varrho| X}, A\right),$$

we have

$$(2.617)\qquad \sigma_1 < \frac{A}{H} \sum_{p,q=0}^{\infty} e^{-\lambda(pw+qw')} < \frac{A}{H} < \frac{1}{2}\varepsilon$$

if H is sufficiently large. It is therefore sufficient for our purpose to prove that *when H is fixed we can so choose X_0 that*

(2.618) $$A\sum{}' e^{-\lambda(pw+qw')} < \frac{1}{2}\epsilon \qquad (X \geqq X_0),$$

the summation extending over those values of p and q for which

(2.619) $$-\frac{H}{X} < \nu \pm pw \pm qw' < \frac{H}{X}.$$

2.62. We divide the terms in question into four blocks corresponding to the four choices of sign, and establish a corresponding conclusion for each of them, with $\frac{1}{8}\epsilon$ in place of $\frac{1}{2}\epsilon$. If the signs attached to p and q are the same, there is nothing to prove; for, as ν is not of the form $pw+qw'$ or $-pw-qw'$, there is no term which satisfies (2.619) when X is sufficiently large. It is therefore sufficient to consider the case in which, for example, $\varrho = \nu - pw + qw'$. But the inequalities

$$-\frac{H}{X} < \nu - pw + qw' < \frac{H}{X},$$

where H is fixed, are only possible when $p > \xi$, $q > \xi$, where $\xi \to \infty$ when $X \to \infty$, and the number of values of q, corresponding to a given p, is 1 at most. Hence the sum extended over such values of p and q does not exceed

$$\sum_{p>\xi} e^{-Ap} < Ae^{-A\xi},$$

which tends to zero when $X \to \infty$. This establishes our conclusion and completes the proof of the lemma.

2.63. Lemma 6. *If λ is positive and ζ real, then the integrals*

(2.631) $$\int_{i\lambda}^{i\lambda+\infty} \Im\left(\frac{e^{\zeta i x}}{\sin\frac{1}{2}wx \sin\frac{1}{2}w'x}\right)\frac{dx}{x}, \quad \int_{-i\lambda}^{-i\lambda+\infty} \Im\left(\frac{e^{\zeta i x}}{\sin\frac{1}{2}wx \sin\frac{1}{2}w'x}\right)\frac{dx}{x},$$

in which the path of integration is a line parallel to the real axis, are convergent; and their values may be calculated by expanding the cosecants in powers of $e^{-w\lambda}$ and $e^{-w'\lambda}$, and integrating term by term.

The series to be used are different in the two integrals; for

$$\frac{1}{\sin\frac{1}{2}wx} = -2i\sum_0^\infty e^{\left(p+\frac{1}{2}\right)wix} = -2i\sum_0^\infty e^{\left(p+\frac{1}{2}\right)wi\xi-\left(p+\frac{1}{2}\right)w\lambda}$$

or

$$\frac{1}{\sin\frac{1}{2}wx} = 2i\sum_0^\infty e^{-\left(p+\frac{1}{2}\right)wix} = 2i\sum_0^\infty e^{-\left(p+\frac{1}{2}\right)wi\xi-\left(p+\frac{1}{2}\right)w\lambda},$$

according as $x = \xi + i\lambda$ or $x = \xi - i\lambda$. Apart from this, the argument is the same for the two integrals, and it will be sufficient to consider the second.

Consider first the analogous integral in which $\frac{1}{x}$ is replaced by $\frac{1}{x} - \frac{1}{x+i\lambda}$. The method of evaluation contemplated is certainly legitimate for *this* integral, since

$$\int^{\infty} O\left(\frac{1}{x^2}\right) \sum_{p,q} O\left(e^{-(pw+qw')\lambda}\right) dx$$

is convergent. We may therefore replace $\frac{1}{x}$ by $\frac{1}{x+i\lambda}$ and then $\frac{dx}{x+i\lambda} = \frac{d\xi}{\xi}$ is real.

Next

$$\Im\left(\frac{e^{\zeta i x}}{\sin\frac{1}{2}wx\sin\frac{1}{2}w'x}\right) = 4e^{\zeta\lambda}\,\Im\frac{e^{\zeta i\xi}\sin\frac{1}{2}w(\xi+i\lambda)\sin\frac{1}{2}w'(\xi+i\lambda)}{(\cosh w\lambda - \cos w\xi)(\cosh w'\lambda - \cos w'\xi)}.$$

Working out the imaginary part of the numerator, we find that it is a sum of constant multiples of terms of the type $\sin\left(\zeta \pm \frac{1}{2}w + \frac{1}{2}w'\right)\xi$. Thus Lemma 6 is reduced to Lemma 5.

2.64. We define a sequence (R_j) as a sequence of values R_j of R which tends to infinity and all of whose members differ, by more than A, from any of the numbers $\frac{2m\pi}{w}$, $\frac{2n\pi}{w'}$ $(m, n = 0, 1, 2, \ldots)$.

Theorem 4. *If* $\eta > w + w'$ *(so that* $N(\eta) > 0$*), then*

(2.641)
$$N(\eta) = R(\eta) + \frac{w^2 + 3ww' + w'^2}{12ww'} - \frac{1}{2\pi}\sum\left(\frac{\cos\frac{2m\pi}{w}\left(\eta - \frac{1}{2}w'\right)}{m\sin\frac{mw'\pi}{w}}\right)^{*}.$$

The sign of summation is to be interpreted as follows: we form the sum of all those terms of the two associated series for which $m < \frac{wR}{2\pi}$, $n < \frac{w'R}{2\pi}$ *respectively, and then make* R *tend to infinity through a sequence* (R_j).

We write $\zeta = \eta - \frac{1}{2}(w + w')$ and

(2.642)
$$J(R_j) = \Re\left(\frac{1}{2\pi i}\int \frac{e^{\zeta i x} - 1}{\sin\frac{1}{2}wx\sin\frac{1}{2}w'x}\,\frac{dx}{x}\right)$$

$$= \frac{1}{2\pi}\int \Im\left(\frac{e^{\zeta i x} - 1}{\sin\frac{1}{2}wx\sin\frac{1}{2}w'x}\,\frac{dx}{x}\right),$$

the contour of integration being the rectangle $(-i\lambda,\ R_j-i\lambda,\ R_j+i\lambda,\ i\lambda)$, where $\lambda>0$, except that the origin is excluded by a small semi-circle of radius ϱ, described about it as centre. We make $j\to\infty$, $\varrho\to 0$.

The integrals along the imaginary axis vanish identically. The integral along the side parallel to the imaginary axis tends to zero. That along the semi-circle tends to the limit

$$\frac{\zeta^2}{ww'}.$$

The integrals along the sides parallel to the real axis also tend to limits, by Lemma 6. It follows that the sum of the real parts of the residues of the integrand, at poles within the contour, tends to a limit S; and we have

(2.643) $$S = \Re\int_{-i\lambda}^{\infty-i\lambda} - \Re\int_{i\lambda}^{\infty+i\lambda} + \frac{\zeta^2}{ww'} = J_1 - J_2 + \frac{\zeta^2}{ww'},$$

say.

We evaluate J_1 by integration term-by-term, which is shown to be legitimate by Lemma 6; and we obtain

(2.644) $$J_1 = \sum_{p,q=1}^{\infty} j_{p,q},$$

where

(2.6441) $$j_{p,q} = \Re\left(-\frac{2}{\pi i}\int_{-i\lambda}^{-i\lambda+\infty}(e^{\zeta i x}-1)\,e^{-\left(\Omega-\frac{1}{2}(w+w')\right)ix}\frac{dx}{x}\right)$$

and $\Omega = pw+qw'$. We may add to $j_{p,q}$ the corresponding integral along the line $(0, -i\lambda)$, since this vanishes identically, and we may then deform the path of integration into the real axis. This gives

$$j_{p,q} = -\frac{2}{\pi}\int_0^\infty\left(\sin(\eta-\Omega)x + \sin\left(\Omega-\frac{w+w'}{2}\right)x\right)\frac{dx}{x},$$

which is zero if $\Omega>\eta$ and -2 if $\Omega<\eta$. Thus we obtain from (2.644)

(2.645) $$J_1 = -2\sum_{\Omega<\eta}1 = -2N(\eta).$$

This equation still holds when η is of the form $pw+qw'$, if we adopt the convention stated in 2.61.

Similarly we obtain a series for J_2, in which the typical term involves the integral

$$\frac{2}{\pi}\int_0^\infty\left(\sin(\eta+\Omega-w-w')x-\sin\left(\Omega-\frac{w+w'}{2}\right)x\right)\frac{dx}{x}=0.$$

Thus

(2.646) $$J_2=0,$$

as is evident *a priori*, since the value of the integral is independent of λ and tends to 0 when $\lambda\to\infty$.

From (2.643), (2.645), and (2.646) we deduce

(2.647) $$N(\eta)=\frac{\left(\eta-\frac{1}{2}w-\frac{1}{2}w'\right)^2}{2ww'}-\frac{1}{2}S.$$

A straightforward calculation shows that

(2.648) $$S=\frac{1}{\pi}\sum\left(\frac{\cos\frac{2m\pi}{w}\left(\eta-\frac{1}{2}w'\right)}{m\sin\frac{mw'\pi}{w}}-\frac{(-1)^m}{m\sin\frac{mw'\pi}{w}}\right)^*.$$

But[1])

$$\frac{1}{2\pi}\sum\left(\frac{(-1)^m}{m\sin\frac{mw'\pi}{w}}\right)^*=-\frac{w^2+w'^2}{24ww'}=\frac{w^2+3ww'+w'^2}{12ww'}-\frac{(w+w')^2}{8ww'}.$$

Thus we obtain the result of the theorem.

3. The sum $s(n,\theta)$.

3.1. In this section we use elementary methods. We are concerned primarily with what, in our former memoir, we called Problem B, that of the order of magnitude of the sum

(3.11) $$s(n,\theta)=\sum_{m=1}^{n}\{m\theta\},$$

though sometimes we return to Problem A.

Lemma 7. *If θ is positive and irrational, $x\geqq 0$, $y=\theta x$, and $f(0)=g(0)=0$, then*

(3.12)
$$\sum_{1\leqq m\leqq x}f([m\theta])(g(m)-g(m-1))+\sum_{1\leqq n\leqq y}g\left(\left[\frac{n}{\theta}\right]\right)(f(n)-f(n-1))=f([y])g([x])$$

---

[1]) This is easily proved directly by contour integration. See the second memoir quoted in 2.31.

The sum with respect to m is

$$\sum_{n=0}^{[y]} f(n) \sum_{[m\theta]=n} (g(m)-g(m-1)) - f([y]) \sum_{[m\theta]=[y],\, m>x} (g(m)-g(m-1))$$

$$=\sum_{n=0}^{[y]} f(n)\left(g\left(\left[\frac{n+1}{\theta}\right]\right)-g\left(\left[\frac{n}{\theta}\right]\right)\right)-f([y])\left(g\left(\left[\frac{[y]+1}{\theta}\right]\right)-g([x])\right)$$

and the first term here is

$$f([y])\, g\left(\left[\frac{[y]+1}{\theta}\right]\right)-\sum_{n=1}^{[y]} g\left(\left[\frac{n}{\theta}\right]\right)(f(n)-f(n-1)),$$

which gives the result.

In (3.12) take $f(u)=g(u)=u$, and write

$$\alpha_m=\alpha_m(\theta)=\{m\theta\},\quad \beta_n=\beta_n(\theta)=\alpha_n\left(\frac{1}{\theta}\right)=\left\{\frac{n}{\theta}\right\}, \tag{3.13}$$

$$\mu=[x],\quad \nu=[y]=[\theta x],\quad \delta=\theta_{\mu-\nu}=\theta[x]-[\theta x]. \tag{3.14}$$

We obtain

$$\sum_1^{\mu}[m\theta]+\sum_1^{\nu}\left[\frac{n}{\theta}\right]=\mu\nu, \tag{3.15}$$

or, on expressing $[m\theta]$ and $\left[\frac{n}{\theta}\right]$ in terms of α_m and β_n, and reducing,

$$\sum_1^{x}\alpha_m+\sum_1^{\theta x}\beta_n=\frac{\delta}{2}-\frac{\delta(1-\delta)}{2\theta}. \tag{3.16}$$

Lemma 8. *If n, n_1, n_1' are positive or zero integers such that*

$$n\theta=n_1+d=n_1'+e,\quad 0<d<1,\quad -1<e<0, \tag{3.17}$$

then

$$s(n,\theta)+s(n_1,\theta_1)=\frac{d}{2}-\frac{d(1-d)}{2\theta}, \tag{3.18}$$

$$s(n,\theta)+s(n_1',\theta_1)=\frac{e}{2}-\frac{e(1-e)}{2\theta}-\left[\frac{-e}{\theta}\right]. \tag{3.19}$$

The first of these formulae is the special form of (3.16) obtained by supposing that x is an integer n. The second is a simple variant. Since $n_1'=n_1+1$, the left hand sides of (3.18) and (3.19) differ only by $\{n_1'\theta_1\}$, and (3.19) follows from (3.18) by simple algebra.

The formula (3.18) is that which, in our former memoir, we attributed to LERCH[1]). Herr OSTROWSKI has pointed out to us that it had (in substance

[1]) See *1*, p. 20.

at any rate) been found before by SYLVESTER[1]). SYLVESTER's formula is indeed equivalent to the more general formulae (3.15) and (3.16). General formulae of the type (3.12) appear to originate with DIRICHLET, and the actual formula (3.12) was given, in the special case in which x is an integer, by HACKS[2]).

With these formulae should be associated the formula (3.24) of our memoir *1*.

3.2. Theorem 5. *If θ satisfies* (1.331), *and* $h>1$, *then*

$$s(n,\theta)=O\left(n^{1-\frac{1}{h}}\right). \tag{3.21}$$

This theorem is due to OSTROWSKI[3]). In the less precise form in which $1-\dfrac{1}{h}$ is replaced by $1-\dfrac{1}{h}+\epsilon$, it is included in Theorem 2, which was enunciated without proof in our memoir *1*[4]). The reading of OSTROWSKI's memoir suggested to us the following theorem, in which Theorem 5 is included[5]).

Theorem 6. *If θ satisfies* (1.331), *and* $h>1$, *then*

$$U(\eta)=O\left(\eta^{1-\frac{1}{h}}\right). \tag{3.22}$$

This theorem includes both Theorem 5 and Theorem 2. It is easily proved by a combination of Lemma 2 with formulae taken from *1*.

If (1.331) is true, (1.341) is also true. As in *1*, we choose r so that

$$\xi\theta\theta_1\ldots\ldots\theta_{r-1}\theta_r<1\leqq\xi\theta\theta_1\ldots\ldots\theta_{r-1}, \tag{3.23}$$

where $\xi=\dfrac{\eta}{w'}$. We have then[6])

$$U(\eta)=O\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\ldots+\frac{1}{\theta_{r-1}}\right)+O(\xi\theta\theta_1\ldots\ldots\theta_{r-1}). \tag{3.24}$$

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1) J. J. SYLVESTER, "Sur la fonction $E(x)$", *Comptes Rendus*, 50 (1860), 732—734 (*Collected math. papers*, 2, 179—180). See also pp. 176, 177, 179 of the same volume of the collected papers.

2) „Über Summen von größten Ganzen“, *Acta Mathematica*, 10 (1887), 1—52. See also J. W. L. GLAISHER, "On certain transformations of Lejeune-Dirichlet's in the theory of numbers, and similar theorems", *Quarterly Journal*, 43 (1912), 123—142.

3) *l. c.* p. 82.

4) *l. c.* p. 35, equation (2).

5) Generally, an "O" theorem relating to Problem B is included in the corresponding theorem relating to Problem A, as is explained in *1*. An "Ω" theorem, that is to say, a theorem which, like Theorem 3, tends in the opposite direction, is on the other hand more difficult than the corresponding theorem concerning Problem A.

6) p. 22, equation (4.151).

Write now

(3.25) $$\theta\,\theta_1\ldots\theta_{r-1}=\xi^{-j},$$

where $0<j\leqq 1$. Then, by (1.351), $\xi^{-(h-1)j}<A\theta_r$, and so

$$\xi^{1-hj}=\xi^{1-j-(h-1)j}<A\,\xi\,\theta\,\theta_1\ldots\theta_{r-1}\theta_r<A.$$

It follows that $j\geqq\frac{1}{h}$, and

(3.26) $$\xi\,\theta\,\theta_1\ldots\theta_{r-1}=\xi^{1-j}=O\left(\xi^{1-\frac{1}{h}}\right)=O\left(\eta^{1-\frac{1}{h}}\right).$$

Again, if $1\leqq s\leqq r$, we have

$$\frac{1}{\theta\,\theta_1\ldots\theta_{s-1}}\leqq\xi\,\theta_s\ldots\theta_{r-1},$$

by (3.23). Using (1.351), we obtain

(3.27) $$\theta_{s-1}^{-1-\frac{1}{h-1}}<A\,\xi\,\theta_s\ldots\theta_{r-1},\quad \frac{1}{\theta_{s-1}}<A(\xi\,\theta_s\ldots\theta_{r-1})^{1-\frac{1}{h}}.$$

From (3.27) it follows that

(3.28)
$$\frac{1}{\theta_{r-1}}+\frac{1}{\theta_{r-2}}+\ldots+\frac{1}{\theta}<A\,\xi^{1-\frac{1}{h}}\left(1+\theta_{r-1}^{1-\frac{1}{h}}+(\theta_{r-2}\theta_{r-1})^{1-\frac{1}{h}}+\ldots\right)$$
$$<A\,\xi^{1-\frac{1}{h}}\sum_{k=0}^{\infty}2^{-k\left(1-\frac{1}{h}\right)}<A\,\xi^{1-\frac{1}{h}}<A\,\eta^{1-\frac{1}{h}},$$

by Lemma 1. Finally, from (3.24), (3.26), and (3.28) the theorem follows.

The constants of the argument are not absolute: the theorem is not true uniformly in h.

3.31. Theorem 7. *If* $h>1$ *and* (1.332) *is true, then*

(3.311) $$|s(n,\theta)|>A\,n^{1-\frac{1}{h}}$$

for an infinity of values of n.

It should be observed that this theorem has different interpretations, according as the A's of (1.332) and (3.311) depend upon θ and h or are absolute constants. It is true on either interpretation, but is a little harder to prove on the second.

Taking the second interpretation, we may restate what we have to prove as follows:—*If* $h>1$ *and*

(3.3121) $$\underline{\lim}\, n^h|\sin n\,\theta\,\pi|\leqq B$$

then

(3.3122) $$|s(n,\theta)| > Cn^{1-\frac{1}{h}},$$

where C depends only on B. In what follows A's denote absolute constants, C's constants depending only on B; the O's are absolute.

Let H be the upper bound of the numbers x for which

(3.313) $$\underline{\lim}\, n^x |\sin n\theta\pi| \leqq B,$$

Clearly we have $H \geqq h$. We proceed to show that *there exists an $h_1 \geqq h$ such that*

(3.3141) $$\underline{\lim}\, n^{h_1} |\sin n\theta\pi| \leqq B$$

and

(3.3142) $$\lim (\theta\theta_1 \dots \theta_{n-1})^{h_1-1}\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\dots+\frac{1}{\theta_{n-1}}\right) = 0$$

when $n \to \infty$. We must distinguish two cases.

Case (i): $H > 2$. In this case we have only to take

$$h_1 = \mathrm{Max}(h, 2).$$

For, since (3.313) holds both when $x = h$ and when $x = 2 < H$, (3.3141) is satisfied. Also

$$(\theta\theta_1 \dots \theta_{n-1})^{h_1-1}\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\dots+\frac{1}{\theta_{n-1}}\right)$$
$$\leqq (\theta\theta_1 \dots \theta_{n-1})\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\dots+\frac{1}{\theta_{n-1}}\right) < Ane^{-An} = o(1).$$

Case (ii): $H \leqq 2$. We begin the discussion of this case by showing that numbers h_1 and H_1 exist for which

(3.315) $$h \leqq h_1 \leqq H < H_1,$$

(3.316) $$K_1 = (h_1-1)(H_1-1)-(H_1-h_1) > 0,$$

(3.317) $$\underline{\lim}\, n^{h_1} |\sin n\theta\pi| \leqq B,$$

(3.318) $$\underline{\lim}\, n^{H_1} |\sin n\theta\pi| = \infty.$$

The last of these is an immediate consequence of $H < H_1$, and we need only consider the first three. If $h = H$ we choose $h_1 = h$, and H_1 greater than H by so little that (3.316) is satisfied. If $h < H$ we choose h_1 and H_1 on either side of H, and differing from it by so little

that $h \leqq h_1$ and (3.316) is satisfied. It is clear that (3.317), or (3.3141), is satisfied in either case.

3.32. We denote by K's positive constants depending only on B, h_1 and H_1. It follows from (1.352) and (3.317) that

$$\frac{1}{\theta_\nu} > \frac{C}{2^{h_1}(\theta\theta_1 \ldots \theta_{\nu-1})^{h_1-1}}, \tag{3.321}$$

for an infinity of values of ν; and from (1.351) and (3.318) that

$$\frac{1}{\theta_n} < \frac{K}{(\theta\theta_1 \ldots \theta_{n-1})^{H_1-1}}, \tag{3.322}$$

for all values of n. Hence, observing that $h_1 \leqq 2$, and using Lemma 1, we have

(3.323)
$$\frac{(\theta\theta_1 \ldots \theta_{n-1})^{h_1-1}}{\theta_r} < Ke^{-K(n-r)} \frac{(\theta\theta_1 \ldots \theta_r)^{h_1-1}}{\theta_r} \quad (0 \leqq r \leqq n-1),$$

(3.324)
$$\frac{(\theta\theta_1 \ldots \theta_r)^{h_1-1}}{\theta_r} = \left(\frac{(\theta\theta_1 \ldots \theta_{r-1})^{H_1-1}}{\theta_r}\right)^{2-h_1} (\theta\theta_1 \ldots \theta_{r-1})^{h_1-1-(H_1-1)(2-h_1)}$$
$$< K^{2-h_1}(\theta\theta_1 \ldots \theta_{r-1})^{K_1} < Ke^{-Kr}.$$

From (3.323) and (3.324) it follows that

$$(\theta\theta_1 \ldots \theta_{n-1})^{h_1-1} \sum_{r=0}^{n-1} \frac{1}{\theta_r} < K \sum_{r=0}^{n-1} e^{-Kn} < Kne^{-Kn} = o(1),$$

which is (3.3142). This completes the discussion of case (ii).

3.33. It is now not difficult to prove Theorem 7. We suppose ν selected so that (3.321) is true, and we write

$$n_\nu = \left[\frac{1}{3\theta_\nu}\right], \quad n_r = \left[\frac{n_{r+1}}{\theta_r}\right] \; (0 \leqq r < \nu), \quad n_0 = n, \tag{3.331}$$

so that

$$n < \frac{n_\nu}{\theta\theta_1 \ldots \theta_{\nu-1}} < \frac{A}{\theta\theta_1 \ldots \theta_\nu} < \left(\frac{2^{h_1}}{C}\right)^{\frac{1}{h_1-1}} \left(\frac{1}{\theta_\nu}\right)^{\frac{h_1}{h_1-1}}, \tag{3.332}$$

by (3.321), or

$$\frac{1}{\theta_\nu} > C^{\frac{1}{h_1}} n^{1-\frac{1}{h_1}} > Cn^{1-\frac{1}{h_1}}. \tag{3.333}$$

On the other hand we have, by (3.19)[1]),

[1]) We have $n_{r+1} = n_r\theta_r + g$, where $0 < g < 1$. It is therefore the second of the transformation formulae (3.18) and (3.19) to which we appeal.

$$s(n_r, \theta_r) + s(n_{r+1}, \theta_{r+1}) = O\left(\frac{1}{\theta_r}\right)$$

for $0 \leqq r < \nu$; and so

$$s(n, \theta) = (-1)^\nu s(n_\nu, \theta_\nu) + O\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right),$$

$$|s(n, \theta)| > |s(n_\nu, \theta_\nu)| - A\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right).$$

But

$$s(n_\nu, \theta_\nu) = \sum_1^{n_\nu} \{n\theta_\nu\} < \sum_1^{n_\nu} \left(\frac{1}{3} - \frac{1}{2}\right) < -An_\nu < -\frac{A}{\theta_\nu},$$

and so

(3.334) $$|s(n, \theta)| > \frac{A}{\theta_\nu} - A\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right).$$

The ratio of the second term on the right to the first is less than

$$K(\theta\theta_1 \ldots \theta_{\nu-1})^{h_1-1}\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right)$$

by (3.321), and this tends to zero as $\nu \to \infty$, by (3.3142). Hence

$$|s(\theta, n)| > \frac{A}{\theta_\nu} > Cn^{1-\frac{1}{h_1}} > Cn^{1-\frac{1}{h}},$$

since $h \leqq h_1$.

3.34. It will be useful to observe that the n of our argument satisfies inequalities

(3.341) $$Aq_{\nu+1} < n < Aq_{\nu+1} < q_{\nu+1}$$

when ν is large. The second and third of these are immediate consequences of (3.331), (3.332), and Lemma 2. To prove the first we observe that

$$n_r \geqq n_\nu > \frac{A}{\theta_\nu} > \frac{K}{(\theta\theta_1 \ldots \theta_{\nu-1})^{h-1}} > K2^{\frac{1}{2}(h-1)\nu} > \nu \quad (0 \leqq r < \nu)$$

for all sufficiently large values of ν. Hence

$$n_r > \frac{n_{r+1}}{\theta_r} - 1 > \frac{n_{r+1}}{\theta_r}\left(1 - \frac{1}{n_{r+1}}\right) > \frac{n_{r+1}}{\theta_r}\left(1 - \frac{1}{\nu}\right),$$

and so

$$n > \left(1 - \frac{1}{\nu}\right)^\nu \frac{A}{\theta\theta_1 \ldots \theta_\nu} > Aq_{\nu+1}.$$

3.41. The proofs of our next two theorems are the most difficult in the memoir. The results were enunciated without proof in our former memoir[1]), and discovered and proved independently by OSTROWSKI[2]).

We give a proof here based on the formulae (3.18) and (3.19). We have abbreviated this proof in every possible way, and present it almost in the form of a sketch; for we recognise that the quite different proof of OSTROWSKI is simpler. It is indeed here that OSTROWSKI's method shows to the greatest relative advantage. At the same time our proof seems to us interesting in itself, and it is essential, if we are to develop the theory systematically from our own point of view, that this crucial theorem should appear in its proper place.

Theorem 8. *There is a positive A such that*

$$|s(n,\theta)| > A\log n \tag{3.411}$$

for every irrational θ and an infinity of values of n.

Theorem 9. *There is a $B = B(K)$ such that each of the inequalities*

$$s(n,\theta) > B\log n, \quad s(n,\theta) < -B\log n \tag{3.412}$$

is true for every θ for which $a_n < K$ and for an infinity of values of n.

In proving Theorem 8, we may suppose that θ satisfies (1.331) for some h: we may take, for example $h = 2$, in which case

$$\frac{1}{\theta_\nu} < \frac{A}{\theta\,\theta_1 \ldots \theta_{\nu-1}}. \tag{3.413}$$

For, if the condition is not satisfied for $h = 2$, we have, by Theorem 7,

$$|s(n,\theta)| > A\sqrt{n} > A\log n$$

for an infinity of values n[3]).

3.42. Let

$$\alpha_r = 1\,(a_r \leqq 3), \quad \alpha_r = \left[\tfrac{1}{2}a_r\right](a_r > 3), \tag{3.421}$$

$$\gamma = \delta = \alpha_4\theta\,\theta_1\theta_2\theta_3 + \alpha_8\theta\,\theta_1 \ldots \theta_7 + \alpha_{12}\theta\,\theta_1 \ldots \theta_{11} + \ldots, \tag{3.422}$$

$$\gamma_1 = -\delta_1 = \frac{\delta}{\theta}, \quad \gamma_2 = \delta_2 = -\frac{\delta_1}{\theta_1}, \quad \gamma_3 = -\delta_3 = \frac{\delta_2}{\theta_2}, \tag{3.423}$$

$$\gamma_4 = \delta_4 = -\frac{\delta_3}{\theta_3} - \alpha_4 = \alpha_8\theta_4\theta_5\theta_6\theta_7 + \alpha_{12}\theta_4\theta_5 \ldots \theta_{11} + \ldots, \tag{3.424}$$

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[1]) p. 36. We raised the question which they answer in our note of 1912.

[2]) *O.*, pp. 85—92.

[3]) One of the inherent advantages of OSTROWSKI's method is that it enables him to avoid making this distinction.

with corresponding equations in which every suffix is increased by 4, 8, 12, ...

Further, let m be a large positive integer, and

(3.425) $\zeta_{4m+2} = \theta_{4m+2}(1-\gamma_{4m+3}),\quad \zeta_{r-1} = -\theta_{r-1}\zeta_r (0 \leqq r \leqq 4m+2),$

(3.426) $$N_r\theta_r = N_{r+1} + \delta_r + \zeta_r \quad (0 \leqq r \leqq 4m+2),$$

(3.427) $$N_{4m+3} = 0,$$

it being understood that a zero suffix may always be omitted, so that, *e. g.*, $\gamma_0 = \gamma$. The ζ's are defined by (3.425), and the equations (3.426) and (3.427) then define N_{4m+2}, N_{4m+1}, $\ldots$, $N_0 = N$ in turn. It is not obvious from the definitions that the N's are integers, but it follows immediately from them that $N_{4m+2} = 1$. If now N_r is an integer, and we consider congruences to modulus 1, we have

$$N_{r+1} = N_r\theta_r - \delta_r - \zeta_r \equiv N_r(a_r+\theta_r) - \delta_r - \zeta_r = \frac{N_r}{\theta_{r-1}} - \delta_r - \zeta_r$$
$$= \frac{N_{r-1}\theta_{r-1} - \delta_{r-1} - \zeta_{r-1}}{\theta_{r-1}} - \delta_r - \zeta_r = N_{r-1},$$

by (3.427) and (3.426). It follows by induction that every N_r is integral.

We write

(3.428) $$s_r = s(N_r, \theta_r).$$

3.43. We use the following properties of the numbers γ, δ, ζ, N:—

(3.431) $$0 < \gamma < \gamma_1 < \gamma_2 < \gamma_3 < 1,$$

(3.432) $$\gamma_3 > A,$$

(3.433) $$1-\gamma_i > A\frac{a_4 - \alpha_4}{\alpha_4} \quad (0 \leqq i \leqq 3),$$

(3.434) $$\alpha_4 = \left[\frac{\gamma_3}{\theta_3}\right]$$

(with similar results in which every suffix is increased by 4, 8, 12, ...),

(3.435) $$|\zeta_r| < 1-\gamma_r \quad (0 \leqq r \leqq 4m+2),$$

(3.436) $$0 < \delta_{2r} + \zeta_{2r} < 1 \quad (2r \leqq 4m+2),$$
$$-1 < \delta_{2r+1} + \zeta_{2r+1} < 0 \quad (2r+1 \leqq 4m+1),$$

(3.437) $$N_r \geqq 1 \quad (0 \leqq r \leqq 4m+2),$$

(3.438) $$N_r \to \infty \quad (m \to \infty),$$

(3.439) $$N < \frac{2^{4m+2}}{\theta\theta_1 \ldots \theta_{4m+1}}.$$

Of these results, (3.431), (3.432), (3.433) and (3.434) follow from Lemma 1 and the definitions of the γ's; (3.435) is obvious from the definitions when $r = 4m+2$, and is easily proved generally by induction; and (3.436) is an immediate consequence of (3.435).

The results (3.437) and (3.438) follow at once from (3.426) and (3.436):—we find in fact that $N_{r-1} \geqq N_r$, and that the sign of equality is impossible if r is even. Finally, (3.426) now gives $N_r < 2N_{r+1}$, which proves (3.439).

3.44. Lemma 9. *If*

(3.441) $$u_r = \delta_r - \frac{\delta_r(1-\delta_r)}{\theta_r},$$

(3.442) $$U_t = u_{4t} - u_{4t+1} + u_{4t+2} - u_{4t+3} + 2\alpha_{4t+4},$$

then

(3.443) $$2s(n,\theta) = \sum_{t=0}^{m-1} U_t + u_{4m} - u_{4m+1} + O(1).$$

Here, and in the arguments which follow, the O's and A's are absolute.

Let $\varphi_r = (N_r\theta_r) = (N_{r+1} + \delta_r + \zeta_r)$. Then, by (3.426) and (3.436), we have

$$\varphi_{2r} = \delta_{2r} + \zeta_{2r}, \quad \varphi_{2r+1} = 1 + \delta_{2r+1} + \zeta_{2r+1}.$$

By (3.18)

(3.4441) $$2(s+s_1) = \delta + \zeta - \frac{(\delta+\zeta)(1-\delta-\zeta)}{\theta}$$

$$= \delta - \frac{\delta(1-\delta)}{\theta} + O\left(\frac{\zeta}{\theta}\right) = u + O(\zeta_1),$$

and by (3.19)

(3.4442)
$$2(s_1+s_2) = \delta_1 + \zeta_1 - \frac{(\delta_1+\zeta_1)(1-\delta_1-\zeta_1)}{\theta_1} - 2\left[-\frac{\delta_1+\zeta_1}{\theta}\right] = u_1 + O(\zeta_2),$$

since

$$\left[-\frac{\delta_1+\zeta_1}{\theta_1}\right] = \left[\frac{\gamma_1}{\theta_1}\right] + O\left(\frac{\zeta_1}{\theta_1}\right) = O(\zeta_2).$$

Similarly we find

(3.4443) $$2(s_2+s_3) = u_2 + O(\zeta_3),$$

(3.4444) $$2(s_3+s_4) = u_3 - 2\alpha_4 + O(\zeta_4).$$

From (3.4441)—(3.4444) it follows that

(3.445) $$2(s_0 - s_4) = U_0 + O(\zeta_4).$$

We have similar equations in which every suffix is increased by 4, 8, ... Adding them, and using (3.425) and Lemma 1, we obtain

(3.446) $$2(s_0 - s_{4m}) = \sum_0^{m-1} U_t + O(1).$$

We have also

(3.447) $$2(s_{4m} - s_{4m+2}) = u_{4m} - u_{4m+1} + O(\zeta_{4m+2}),$$

and (3.443) follows from (3.446) and (3.447), since $s_{4m+2} = O(1)$ and $\zeta_{4m+2} = O(1)$.

3.45. Lemma 10. *We have*

(3.451) $$s(N, \theta) > -A + A \sum_{t=0}^{m-1} (a_{4t+4} - \alpha_{4t+4}) + A \sum_{t=0}^{m-1} \theta_{4t+2}.$$

An elementary reduction shows that

(3.452) $$\begin{aligned} U_t &= 2(\gamma_{4t} - \gamma_{4t+4}) + \frac{\gamma_{4t+1}}{\theta_{4t+1}} (1 - \theta_{4t}\theta_{4t+1})(1 - \gamma_{4t+1}) \\ &\quad + \frac{\gamma_{4t+3}}{\theta_{4t+3}} (1 - \theta_{4t+2}\theta_{4t+3})(1 - \gamma_{4t+3}) > 2(\gamma_{4t} - \gamma_{4t+4}) \\ &\quad + \frac{1}{2}\theta_{4t+2}\gamma_{4t+3}(1 - \theta_{4t+1}\theta_{4t+2}\gamma_{4t+3}) + \frac{1}{2}\alpha_{4t+4}(1 - \gamma_{4t+3}) \\ &\quad > 2(\gamma_{4t} - \gamma_{4t+4}) + A\theta_{4t+2} + A(a_{4t+4} - \alpha_{4t+4}), \end{aligned}$$

by Lemma 1, (3.423), (3.432), and (3.433). Also

(3.453) $$u_{4m} - u_{4m-1} = O(1) + O\left(\frac{\gamma_{4m}}{\theta_{4m}}\right) + O\left(\frac{\gamma_{4m+1}}{\theta_{4m+1}}\right) = O(1).$$

The result follows from (3.452) and (3.453).

3.46. Theorems 8 and 9 follow easily from Lemma 10.

The number N is a function $N(m, \theta)$ of m and θ alone. We write

(3.461) $$N_i = N(m, \theta_i) \qquad (i = 0, 1, 2, 3).$$

By Lemma 10,

(3.462) $$\sum_{i=0}^{3} s(N_i, \theta_i) > -A + A \sum_{r=0}^{4m+3} (a_r - \alpha_r) + A \sum_{r=1}^{4m+1} \theta_r.$$

If $a_r \geqq 4$, $a_r - \alpha_r > A\,a_r > \dfrac{A}{\theta_{r-1}}$; and if $a_r \leqq 3$, $\theta_{r-1} > \dfrac{A}{\theta_{r-1}}$. Hence

(3.462) involves

$$\sum_{i=0}^{8} s(N_i, \theta_i) > -A + A \sum_{r=1}^{4m+2} \frac{2}{\theta_{r-1}} \tag{3.463}$$

$$> -A + A(4m+2)\,\mathfrak{N}^{\frac{1}{4m+2}} > -A + A \log \mathfrak{N} > A \log \mathfrak{N},$$

where

$$\mathfrak{N} = \frac{2^{4m+2}}{\theta\theta_1 \ldots \theta_{4m+1}}. \tag{3.4631}$$

If now $\overline{N}$ is the greatest of N, N_1, N_2, N_3, we have, by (3.439) and (3.413),

$$\overline{N} < \frac{2^{4m+5}}{\theta\theta_1 \ldots \theta_{4m+4}} < \frac{8\,\mathfrak{N}}{\theta_{4m+2}\,\theta_{4m+3}\,\theta_{4m+4}} < \mathfrak{N}^A. \tag{3.464}$$

Hence, if $s(\overline{n}, \overline{\theta})$ is that one of the sums $s(N_i, \theta_i)$ whose modulus is greatest, we have

$$|s(\overline{n}, \overline{\theta})| > A \log \mathfrak{N} > A \log \overline{N} > A \log \overline{n};$$

and $\overline{n} \to \infty$. Theorem 8 is therefore true for one of the three numbers $\theta, \theta_1, \theta_2, \theta_3$ and therefore, by the fundamental formula (3.18), it is true for θ.

The deduction of Theorem 9 is more immediate. In this case $\theta_r > B$[1]), and so

$$s(N, \theta) > -A + A \sum_{0}^{m-1} \theta_{4t+2} > Bm,$$

while, by (3.439), $N < B^{4m+2}$. It follows that

$$s(N, \theta) > B \log N,$$

which is one of the desired inequalities. The formula (3.18) then shows at once that

$$s(N', \theta) < -B \log N'$$

for arbitrarily large values of N'.

It is not possible, in the general case, to prove two-sided inequalities of the type $s > A \log n$, $s < -A \log n$. Herr OSTROWSKI has gone further in this direction; he has shown that s may in fact be bounded on one side, and has investigated the conditions under which this is possible[2]). We have not attempted to apply our own method to this problem, as we had not considered it before the publication of OSTROWSKI's memoir, and it is clear that a proof on these lines could not be so simple as his.

[1]) B denotes throughout a positive number depending only on K.
[2]) The question is left open in O. (S. 92).

3.51. We return for a moment to the triangle problem. In this problem the analogue of Theorem 9 holds without restriction on θ. To prove this we require the lemma which follows, which is of course included in OSTROWSKI's work, where it occupies a more central position.

Lemma 11. *We have*

$$s(q_\nu, \theta) = O(1). \tag{3.511}$$

It is evident that, if $\frac{p_\nu}{q_\nu}, \frac{p_{\nu,1}}{q_{\nu,1}}, \frac{p_{\nu,2}}{q_{\nu,2}}, \ldots$ are typical convergents to $\theta, \theta_1, \theta_2, \ldots$, we have

$$p_\nu = q_{\nu-1,1}, \; p_{\nu-1,1} = q_{\nu-2,2}, \ldots .$$

If now we take $n = q_\nu$ in (3.18) or (3.19), we have $n_1 = p_\nu$ or $n_1' = p_\nu$, while d or e, whichever is relevant, is $O\left(\frac{1}{q_{\nu+1}}\right)$. Hence

$$s(q_\nu, \theta) + s(q_{\nu-1,1}, \theta_1) = O\left(\frac{1}{q_{\nu+1}\theta}\right),$$

$$s(q_{\nu-1,1}, \theta_1) + s(q_{\nu-2,2}, \theta_2) = O\left(\frac{1}{q_{\nu,1}\theta_1}\right),$$

and so on. Consequently

$$\begin{aligned} s(q_\nu, \theta) &= O\left(\frac{1}{q_{\nu+1}\theta} + \frac{1}{q_{\nu,1}\theta_1} + \ldots + \frac{1}{q_{1,\nu}\theta_\nu}\right) \\ &= O(\theta_1\theta_2 \ldots \theta_\nu + \theta_2 \ldots \theta_\nu + \ldots + \theta_\nu + 1) = O(1), \end{aligned}$$

by Lemma 1.

3.52. Theorem 10. *There is an A such that each of the inequalities*

$$U(\eta) > A \log \eta, \quad U(\eta) < -A \log \eta \tag{3.521}$$

is satisfied for every θ and arbitrarily large values of η.

It is sufficient[1]) to prove that

$$S(\eta) = \sum_{1 \leq \mu \leq \frac{\eta}{w}} \{\mu\theta - f'\} > A \log \eta \tag{3.522}$$

for arbitrarily large values of η, together with an analogous inequality involving $-A \log \eta$. By a change in η, of magnitude $O(1)$, we can make f' anything we please between 0 and 1[2]). It is therefore enough

[1]) *1*, p. 18.

[2]) *1*, p. 17.

to establish the existence of a $g = g(n)$ such that $0 \leqq g < 1$ and

$$S_1(n) = \sum_1^n \{\mu\theta - g\} > A \log n \tag{3.523}$$

for arbitrarily large values of n, with a corresponding result for $-A \log n$.

We suppose first that (1.331) is satisfied for some particular h, say $h = 2$. Then

$$\log q_{\nu+1} < A \log q_\nu. \tag{3.524}$$

There are, by Theorem 8, large values of n for which *one* of the inequalities

$$\sum_1^n \{\mu\theta\} > A \log n, \quad \sum_1^n \{\mu\theta\} < -A \log n$$

is true, say the first. Then (3.523) is true, with $g = 0$, for such values of n. Let N be one of these values, as determined in 3.42, and let

$$q_{\nu-1} \leqq N < q_\nu; \tag{3.525}$$

then, by Lemma 11,

$$\sum_{N+1}^{q_{\nu+1}} \{\mu\theta\} = \sum_1^{q_{\nu+1}} - \sum_1^N < -A \log N,$$

or

$$\sum_1^{q_{\nu+1}-N} \{m\theta + (N\theta)\} < -A \log N. \tag{3.526}$$

If now p is the least positive integer for which

$$0 < g = p\theta - (N\theta) < 1$$

and

$$n = q_{\nu+1} - N + p, \tag{3.527}$$

then $p < \dfrac{A}{\theta}$ and

$$\sum_1^n \{\mu\theta - g\} = \sum_1^{q_{\nu+1}-N} \{m\theta + (N\theta)\} + O\left(\frac{1}{\theta}\right) < -A \log N, \tag{3.528}$$

if N is large enough. But

$$\log N \geqq \log q_{\nu-1} > A \log q_{\nu+1} > A \log n,$$

by (3.525) and (3.524), and so

$$\sum_1^n \{\mu\theta - g\} < -A \log n$$

for an infinity of values of n.

3.53. This proves the theorem when (1.331) is satisfied for $h=2$. When it is not satisfied, much more is true; for then (1.332) is satisfied for $h=2$, and in this case, by Theorem 3, we have

(3.531) $$N(\eta) > A\sqrt{\eta}, \quad N(\eta) < -A\sqrt{\eta},$$

each for arbitrarily large values of η.

The proof of Theorem 3, given in 2.5, was transcendental; and it is worth while to observe that it may also be proved by an argument like that of 3.52. We have only to suppose that N is the n of 3.3 so that (say)

$$\sum_1^N \{\mu\theta\} > AN^{1-\frac{1}{h}}.$$

Arguing just as in (3.52), and making use of (3.341), we establish the existence of an infinity of values of n for which

$$\sum_1^n \{\mu\theta - g\} < -An^{1-\frac{1}{h}},$$

and the theorem then follows in the same way.

The Cesàro means of the series $\sum\{n\theta\}$.

3.61. A good deal of additional light is thrown on the behaviour of $s(n,\theta)$ by the study of the corresponding Cesàro mean

(3.611) $$\sigma(x,\theta) = \frac{1}{x}\sum_{m=1}^{x}(x-m)\{m\theta\}.$$

The study of $\sigma(x,\theta)$ leads us naturally to consider also the sum

(3.612) $$t(x,\theta) = \sum_{m=1}^{x}\left(\{m\theta\}^2 - \frac{1}{12}\right).$$

Lemma 12. *We have, in the notation of 3.1,*

(3.613) $$\sum_1^x\left(\alpha_m^2 - \frac{1}{12}\right) - \theta\sum_1^y\left(\beta_n^2 - \frac{1}{12}\right) = \frac{\Phi(\theta,\delta)}{\theta},$$

where

(3.614) $$\Phi(\theta,\delta) = \frac{1}{6}\delta(2\delta^2 - 3(1-\theta)\delta + 1 - 3\theta + \theta^2).$$

If in (3.12) we take $f(u)=u^2$, $g(u)=u$, express the summands in terms of α_m and β_n, and reduce, we obtain

(3.615)
$$\sum_1^x\left(\alpha_m^2 - \frac{1}{12}\right) - 2\left(\theta\sum_1^x m\,\alpha_m + \sum_1^y n\,\beta_n\right) + \left(\sum_1^x \alpha_m + \sum_1^y \beta_n\right) = \frac{P(\mu,\theta,\delta)}{\theta},$$

where P is a polynomial whose coefficients are absolute constants. If on the other hand we take $f(u)=u$, $g(u)=u^2$, we obtain a similar formula in which x, y; $\theta, \frac{1}{\theta}$; α_m, β_n are interchanged. When we multiply this by θ, and subtract from (3.615), the terms in $\sum m\alpha_m$ and $\sum n\beta_n$ disappear; and when finally we substitute for $\sum\alpha_m+\sum\beta_n$ from (3.16), we obtain the result of the lemma.

3.62. We say that θ belongs to class $\Gamma(H)$ if

(3.621) $$\sum_{r=0}^{\infty}\frac{\theta\theta_1\ldots\theta_{r-1}}{\theta_r}\leqq H.$$

The convergence of the series is equivalent to that of the series $\sum\frac{q_{r+1}}{q_r^2}$.

Theorem 11. *If θ is of class $\Gamma(H)$ then*

(3.622) $$t(x,\theta)=O(H).$$

Write

(3.623) $$x_1=\theta x,\quad x_2=\theta_1 x_1,\ldots,\quad \alpha_m^2-\frac{1}{12}=a_m.$$

By (3.613), we have

$$\sum_1^x a_m(\theta)-\theta\sum_1^{x_1}a_m(\theta_1)=O\left(\frac{1}{\theta}\right).$$

Write x_1, θ_1 for x, θ, and multiply by θ; x_2, θ_2 for x, θ and multiply by $\theta\theta_1$; and so on until the second sum disappears. Adding the resulting equations, and using (3.621), we obtain the result[1]).

3.63. Lemma 13. *If θ belongs to class $\Gamma(H)$, then*

(3.631) $$\sigma(x,\theta)+\sigma(y,\theta_1)=c(\theta)+O\left(\frac{H}{\theta x}\right)+O\left(\frac{1}{\theta^2 x}\right),$$

where

(3.632) $$c(\theta)=\frac{1}{4}-\frac{1}{12}\left(\theta+\frac{1}{\theta}\right).$$

We have, from (3.611) and (3.615),

$$2y\left(\sigma(x,\theta)+\sigma(y,\theta_1)\right)=\frac{P}{\theta}+(2y-1)\left(\sum_1^x\alpha_m+\sum_1^y\beta_n\right)-\sum_1^x\left(\alpha_m^2-\frac{1}{12}\right).$$

The last term is $O(H)$, by Theorem 11. We write $\mu\theta+O(1)$ for y, substitute for $\sum\alpha_m+\sum\beta_n$ from (3.16), and divide by $2y$; and a simple calculation gives the result.

3.64. We say that θ belongs to class $C(K)$ if $a_n<K$, where $K>2$. In this case all of θ, θ_1, θ_2, ... belong to a class $\Gamma(H)$ for which $H<AK$.

[1]) This is the revised form of the theorem stated incorrectly on p. 229 of our Cambridge communication, and noted as incorrect on p. 36 of *1*.

Lemma 14. *If θ is of class $C(K)$, and $\nu = \nu(x)$ is defined by*

$$x_{\nu+2} = \theta\theta_1 \dots \theta_{\nu+1} x < 1 \leqq \theta\theta_1 \dots \theta_\nu x = x_{\nu+1}, \tag{3.641}$$

then

$$\sigma(x, \theta) = \sum_{r=0}^{\nu-1} (-1)^r c(\theta_r) + O(K^2). \tag{3.642}$$

We have, by repeated use of Lemma 13,

$$\begin{aligned}\sigma(x, \theta) = (-1)^\nu \sigma(x_\nu, \theta_\nu) &+ \sum_{r=0}^{\nu-1} (-1)^r c(\theta_r) \\ &+ O\left(\frac{H}{x}\left(\frac{1}{\theta} + \frac{1}{\theta\theta_1} + \dots + \frac{1}{\theta\theta_1 \dots \theta_{\nu-1}}\right)\right) \\ &+ O\left(\frac{1}{x}\left(\frac{1}{\theta^2} + \frac{1}{\theta\theta_1^2} + \dots + \frac{1}{\theta\theta_1 \dots \theta_{\nu-2}\theta_{\nu-1}^2}\right)\right).\end{aligned} \tag{3.643}$$

But, since $a_n < K$, we have $\dfrac{1}{\theta_r} = O(K)$, and the second line of (3.643) is

$$O\left(\frac{K}{\theta\theta_1 \dots \theta_{\nu-1} x}(1 + \theta_{\nu-1} + \theta_{\nu-1}\theta_{\nu-2} + \dots)\right) = O\left(\frac{K}{\theta_\nu}\right) = O(K^2),$$

by (3.641) and Lemma 1. Similarly the third line of (3.643) is $O(K^2)$. Finally

$$\sigma(x_\nu, \theta_\nu) = O(x_\nu) = O\left(\frac{x_{\nu+2}}{\theta_\nu \theta_{\nu+1}}\right) = O(K^2),$$

whence the result.

3.65. Theorem 12. *There are θ's of class $C(K)$ for which*

$$\sigma(x, \theta) = O(1), \tag{3.651}$$

and others for which

$$\sigma(x, \theta) \sim L \log x, \tag{3.652}$$

where $L > \dfrac{AK}{\log K}$ or $L < -\dfrac{AK}{\log K}$.

(i) If

$$\theta = \frac{1}{1+}\,\frac{1}{1+\dots}$$

then $\theta_r = \theta$ for every r, and (3.651) follows directly from (3.642).

(ii) If $k = [K] \geqq 2$ and

$$\theta = \frac{1}{k+}\,\frac{1}{1+}\,\frac{1}{k+}\,\frac{1}{1+\dots},$$

that so

$$\theta + \frac{1}{\theta} - \theta_1 - \frac{1}{\theta_1} = k - 1,$$

then it follows from (3.642) that

$$\sigma(x,\theta) = (k-1)\left[\frac{1}{2}\nu\right] + O(K^2) \sim \frac{1}{2}(k-1)\nu. \tag{3.653}$$

Also

$$\log x = \sum_{r=0}^{\nu} \log\frac{1}{\theta_r} + O(\log K) = \sum_{r=0}^{\left[\frac{1}{2}\nu\right]} \log\frac{1}{\theta_{2r}\theta_{2r+1}} + O(\log K)$$

$$= \left(\left[\frac{1}{2}\nu\right] + 1\right)\log\frac{1}{\theta\,\theta_1} + O(\log K) \sim \frac{1}{2}\nu\log\frac{k+2+\sqrt{k^2+4k}}{2}.$$

Thus we obtain (3.652) with

$$L = \frac{k-1}{\log\frac{1}{2}\left(k+2+\sqrt{k^2+4k}\right)} > \frac{AK}{\log K}.$$

If we exchange θ and θ_1, we obtain an example in which L has the opposite sign.

4. Conclusion.

4.1. The proof of Lemma 13 indicates clearly that, if we were to attempt the construction of a *complete* theory of the series $\sum \alpha_n$, it would be necessary to construct at the same time a theory of the series $\sum\left(\alpha_n^2 - \frac{1}{12}\right)$. A little further investigation shows that we must also consider the series $\frac{1}{12}\sum(4\alpha_n^3 - \alpha_n)$, ..., the n^{th} term of the p^{th} series being substantially the p^{th} Bernoullian function of $\alpha_n + \frac{1}{2}$. There are many curious theorems connected with these series: we content ourselves with mentioning one.

Theorem 11. *If θ belongs to a class $\Gamma(H)$ (and in particular if a_n is bounded) then the series $\sum\left(\alpha_n^2 - \frac{1}{12}\right)$ is summable $(C, 1)$, or by any Cesàro mean of positive order, to sum $-\frac{1}{12}$.*

4.2. We conclude by a brief reference to a different matter. It is of considerable interest to determine the largest half-planes in which the functions

$$f_1(s) = \sum\frac{\alpha_n}{n^s},\quad f_2(s) = \sum\frac{\alpha_n^2 - \frac{1}{12}}{n^s},\quad f_3(s) = \frac{1}{12}\sum\frac{4\alpha_n^3 - \alpha_n}{n^s}, \ldots \tag{4.21}$$

are regular. Hecke has shown that, when θ is a quadratic irrational, $f_1(s)$ is *meromorphic* all over the plane, and has at most a doubly infinite system of simple poles, at the points

$$-2k \pm 2m\gamma\pi i \qquad (k, m = 0, 1, 2, \ldots)$$

where γ is a constant depending on θ.

Our more elementary methods are applicable to problems of this kind also. Let λ be defined as the least number for which

$$\frac{(\theta\,\theta_1 \ldots \theta_{n-1})^{\lambda+\varepsilon}}{\theta_n} \to 0$$

for every positive ε, so that (1.331) is satisfied with $h = 1+\lambda+\varepsilon$ but not with $h = 1+\lambda-\varepsilon$. Then we can prove that

(a) *$f_p(s)$ is regular for*

$$\sigma > \sigma_p = 1 - \frac{p}{1+\lambda};$$

(b) *$\sigma = \sigma_p$ is a barrier of singularities for $f_p(s)$, except possibly when $\lambda = 0$, so that*

(c) *$f_p(s)$ is either regular for $\sigma > 1-p$, or has a barrier of singularities to the right of $\sigma = 1-p$, and in particular*

(d) *$f_1(s)$ is either regular for $\sigma > 0$ or has a barrier to the right of $\sigma = 0$;*

(e) *the series for $f_p(s)$ is summable by Cesàro's means for $\sigma > \sigma_p$ and in particular*

(f) *the series $\sum \alpha_n n^{-s}$ is convergent, when $\lambda > 0$, throughout the region of existence of the function $f_1(s)$.*

The propositions (a) and (e) have been proved independently, and in a different manner, by Herr BEHNKE[1]).

The case in which $\lambda = 0$ is exceptional and more difficult. It would seem that $\sigma = \sigma_p = 1-p$ is still a barrier in all cases *except that of a quadratic θ*, but this we have not been able to establish rigorously.

In this last case, finally, our method reveals the existence of HECKE's poles, though it does not render a complete account of them so readily as that of HECKE himself. This is only natural, as HECKE's method is so much more special and so much deeper than ours.

In view of the length of the memoir, we confine ourselves here to the statement of these results, reserving a fuller discussion for publication elsewhere.

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[1]) H. BEHNKE, Über die Verteilung von Irrationalitäten mod. 1. (Diese Abhandlungen, Bd. I, vorliegendes Heft.)

COMMENTS

The paper falls into two separate halves, one (§ 2) analytical in its approach and the other (§ 3) elementary. Both sections are continuations and developments of corresponding work in 1922, 6.

For references to other related work, see Koksma, pp. 103–6.

Some problems of Diophantine approximation: The analytic character of the sum of a Dirichlet's series considered by Hecke.

By G. H. HARDY in Oxford and J. E. LITTLEWOOD in Cambridge.

1. It has been shown by HECKE[1]) that the function $J(s) = J(s, \theta)$, defined when $\sigma > 1$[2]) by the series

$$(1.1) \qquad J(s, \theta) = \sum \alpha_n n^{-s},$$

where (1.11) $$\alpha_n = \alpha_n(\theta) = n\theta - [n\theta] - \frac{1}{2},$$

and θ is a real quadratic irrational, is a meromorphic function whose only singularities are simple poles. He establishes, by means of the theory of the new "Zeta functions" which he has introduced into analysis, a formula for $J(s)$ which effects its continuation all over the plane and exhibits explicitly the nature of every singularity.

In this note we show how it is possible to attain the same end by an entirely different method. The formula at which we ultimately arrive differs fundamentally from HECKE's in structure, and its relations to his are of considerable formal interest. It is remarkable, moreover, that we are able to establish our formula, for the half-plane $\sigma > 2$, by elementary methods[3]); a transcendental argument is required only to prove that it converges all over the plane.

Our method may be applied to any quadratic θ; but we wish to exhibit its principle without unnecessary formal complications, and we therefore limit ourselves to a particular case, as did HECKE in the memoir to which we have referred. We suppose that

$$(1.2) \qquad \theta = \frac{1}{a+}\frac{1}{a+}\frac{1}{a+}\cdots = \sqrt{1+\frac{a^2}{4}} - \frac{a}{2},$$

where a is an integer.

[1]) E. HECKE, Über analytische Funktionen und die Verteilung von Zahlen mod. eins, *Hamburg. Math. Abh.*, 1 (1921), 54—76.

[2]) $s = \sigma + it$.

[3]) That is to say, without employing the ideas of the theory of analytic functions.

1923, 3 (with J. E. Littlewood) *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, 3, 57–68.

2. If $x \geqq 0,\ y = \theta x$, and $f(0) = g(0) = 0$, we have

$$\sum_{m \leqq x} f([m\theta])\,(g(m) - g(m-1)) + \sum_{n \leqq y} g\left(\left[\frac{n}{\theta}\right]\right)(f(n) - f(n-1)) \tag{2.1}$$
$$= f([y])\,g([x])\ ^{4)}.$$

In (2.1) we write

$f(u) = 1 - e^{-u\xi}, \qquad g(u) = 1 - e^{-u\xi_1}$, where ξ is real and

$$\xi_1 = c - \theta\xi, \tag{2.2}$$

c being positive. Making x tend to infinity, we obtain

$$\frac{e^{-\xi}}{1-e^{-\xi}} \sum_1^\infty e^{-[m\theta]\xi - m\xi_1} + \frac{e^{-\xi_1}}{1-e^{-\xi_1}} \sum_1^\infty e^{-\left[\frac{n}{\theta}\right]\xi_1 - n\xi} \tag{2.3}$$
$$= \frac{e^{-\xi-\xi_1}}{(1-e^{-\xi})(1-e^{-\xi_1})}.$$

We suppose now that

$$\theta = \frac{1}{a_1 + \theta_1}, \qquad \theta_1 = \frac{1}{a_2 + \theta_2}, \ \ldots\ldots,^{5)}$$

and write
$$c_1 = \frac{c}{\theta},$$

$$\beta_n = \alpha_n\left(\frac{1}{\theta}\right) = \alpha_n(\theta_1) = n\theta_1 - [n\theta_1] - \frac{1}{2} = \frac{n}{\theta} - \left[\frac{n}{\theta}\right] - \frac{1}{2}, \tag{2.4}$$

$$F(c, \theta, \xi) = \sum_1^\infty e^{-mc + \alpha_m \xi}, \qquad F(c_1, \theta_1, \xi_1) = \sum_1^\infty e^{-nc_1 + \beta_n \xi_1}; \tag{2.5}$$

and we obtain from (2.3), after some simple reductions

$$F(c, \theta, \xi) = \frac{e^{-\frac{1}{2}\xi - \frac{1}{2}\xi_1}}{2\,sh\frac{1}{2}\xi_1} - \frac{sh\frac{1}{2}\xi}{sh\frac{1}{2}\xi_1} F(c_1, \theta_1, \xi_1). \tag{2.6}$$

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[4]) This is Lemma 7 of our memoir in vol. I of this journal (212—249).

[5]) It is only at a later stage that we introduce the hypothesis (1.2).

Using (2.6) repeatedly, we find

$$F(c,\theta,\xi)=\frac{1}{2}sh\frac{1}{2}\xi\sum_{\nu=0}^{p}(-1)^{\nu}\frac{e^{-\frac{1}{2}\xi_\nu-\frac{1}{2}\xi_{\nu+1}}}{sh\frac{1}{2}\xi_\nu\, sh\frac{1}{2}\xi_{\nu+1}} \tag{2.7}$$

$$+(-1)^{p+1}\frac{sh\frac{1}{2}\xi}{sh\frac{1}{2}\xi_{p+1}}F(c_{p+1},\theta_{p+1},\xi_{p+1}),$$

the sequences (c_p) and (ξ_p) being defined by

$$c_0=c,\quad c_p=\frac{c_{p-1}}{\theta_{p-1}},\quad \xi_0=\xi,\quad \xi_p=c_{p-1}-\theta_{p-1}\xi_{p-1}. \tag{2.71}$$

Suppose that $0<\xi<c$. Then $0<\xi_1<c$. Also $\xi_2=c_1-\theta_1\xi_1<c_1$

and
$$\xi_2=c_1-\theta_1\xi_1>c_1(1-\theta\theta_1)>\frac{1}{2}c_1;$$

so that ξ_2 lies between $\frac{1}{2}c_1$ and c_1. It follows similarly that ξ_p lies between $\frac{1}{2}c_{p-1}$ and c_{p-1}. Hence $\xi_p\to\infty$, and the series in (2.7) converges when continued to infinity. Also $F(c_{p+1},\theta_{p+1},\xi_{p+1})=O(e^{-c_{p+1}+\frac{1}{2}\xi_{p+1}})$ tends to zero when $p\to\infty$. Thus

$$F(c,\theta,\xi)=\frac{1}{2}sh\frac{1}{2}\xi\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{e^{-\frac{1}{2}\xi_\nu-\frac{1}{2}\xi_{\nu+1}}}{sh\frac{1}{2}\xi_\nu\, sh\frac{1}{2}\xi_{\nu+1}}. \tag{2.8}$$

3. Subtracting

$$F(c,\theta,0)=\frac{e^{-c}}{1-e^{-c}}=\frac{e^{-\frac{1}{2}c}}{2sh\frac{1}{2}c}$$

from both sides of (2.8), dividing by ξ, and making $\xi\to 0$,[6]) we obtain

$$\Phi(c,\theta)=\sum_{1}^{\infty}\alpha_m e^{-mc}=w(c,\theta)+\chi(c,\theta), \tag{3.1}$$

[6]) There is no difficulty in justifying this process.

where (3.21) $$w(c, \theta) = \frac{\theta e^{-c}}{(1-e^{-c})^2} - \frac{1}{2}\frac{e^{-c}}{1-e^{-c}},$$

(3.22) $$\chi(c, \theta) = \sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{e^{-\eta_\nu - \eta_{\nu+1}}}{(1-e^{-\eta_\nu})(1-e^{-\eta_{\nu+1}})},$$

and the η's are the values of the ξ's when $\xi_0 = 0$. We write

(3.3) $\eta_\nu = c\beta_\nu$, and β_ν is then defined by

(3.4) $$\beta_1 = 1, \qquad \beta_\nu = \frac{1}{\theta\theta_1 \cdots \theta_{\nu-2}} - \theta_{\nu-1}\beta_{\nu-1}.$$

Since $$\frac{1}{\theta\theta_1 \cdots \theta_{\nu-2}} = q_{\nu-1} + \theta_{\nu-1} q_{\nu-2},$$

where $\frac{p_\nu}{q_\nu}$ is the ν^{th} convergent to θ,[7] the second equation (3.4) is

$$\beta_\nu - q_{\nu-1} = \theta_{\nu-1}(q_{\nu-2} - \beta_{\nu-1});$$

and $\beta_2 = \frac{1}{\theta} - \theta_1 = a_1 = q_1$. Hence

(3.5) $$\beta_\nu = q_{\nu-1}.$$

4. We multiply (3.1) by $\frac{c^{s-1}}{\Gamma(s)}$ (supposing in the first instance that σ is sufficiently large) and integrate from $c = 0$ to $c = \infty$. We find

(4.1) $$J(s, \theta) = \sum_1^{\infty}\frac{\alpha_m}{m^s} = \theta\zeta(s-1) - \frac{1}{2}\zeta(s) + X(s, \theta),$$

where (4.2) $$X(s, \theta) = \sum_{\nu=1}^{\infty}(-1)^{\nu}\chi_\nu(s, \theta)$$

[7]) By Lemma 2 of our memoir already referred to.

and (4.3) $$\chi_\nu(s,\theta) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\eta_\nu - \eta_{\nu+1}}}{(1-e^{-\eta_\nu})(1-e^{-\eta_{\nu+1}})} c^{s-1}\,dc$$

$$= \sum_{h,k=1}^\infty \frac{1}{\Gamma(s)} \int_0^\infty e^{-(hq_{\nu-1}+kq_\nu)c} c^{s-1}\,dc$$

$$= \sum_{h,k=1}^\infty \frac{1}{(hq_{\nu-1}+kq_\nu)^s} = \zeta_2(s, q_{\nu-1}+q_\nu, q_{\nu-1}, q_\nu),$$

$\zeta_2(s, a, w, w')$ being the double Zeta-function of BARNES[8]). The integral

$$\int_0^\infty \sum_0^\infty \left| \frac{e^{-\eta_\nu - \eta_{\nu+1}}}{(1-e^{-\eta_\nu})(1-e^{-\eta_{\nu+1}})} c^{s-1} e^{-c} \right| dc$$

is convergent if $\sigma > 2$, so that our formal process is valid. Thus we find

(4.4) $$J(s,\theta) = \theta\,\zeta(s-1) - \frac{1}{2}\zeta(s) + \sum_{\nu=1}^\infty (-1)^\nu \zeta_2(s, q_{\nu-1}+q_\nu, q_{\nu-1}, q_\nu).$$

This expression of $J(s,\theta)$ as a series of double Zeta-functions is valid for all θ. It is now that we proceed to specialise.

5. Let us suppose in particular that

(5.1) $$\theta = \frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots = \sqrt{1+\frac{a^2}{4}} - \frac{a}{2} = e^{-\gamma},$$

so that $\gamma > 0$ and $\theta_\nu = \theta$ for every ν. An elementary calculation shows that (5.2) $$q_{\nu-1} = \frac{e^{\nu\gamma} + (-1)^{\nu-1} e^{-\nu\gamma}}{e^\gamma + e^{-\gamma}},$$

(5.3) $$hq_{\nu-1} + kq_\nu = \frac{Pe^{\nu\gamma} + (-1)^{\nu-1} Q e^{-\nu\gamma}}{2\,ch\,\gamma},$$

[8]) See p. 216 of our former memoir.

where (5.31) $\qquad P = h + ke^{\gamma}, \qquad Q = h - ke^{-\gamma}.$

It follows that

$$\zeta_2(s, q_{\nu-1} + q_\nu, q_{\nu-1}, q_\nu) = (2\,ch\,\gamma)^s \sum_{h,k} (Pe^{\nu\gamma} + (-1)^{\nu-1} Qe^{-\nu\gamma})^{-s}$$

$$= (2\,ch\,\gamma)^s \sum_{h,k} \sum_{l=0}^{\infty} (-1)^{\nu l} \begin{bmatrix} s \\ l \end{bmatrix} e^{-(s+2l)\nu\gamma} \frac{Q^l}{P^{s+l}},$$

where $$\begin{bmatrix} s \\ l \end{bmatrix} = \frac{s(s+1)\cdots\cdot(s+l-1)}{l!};$$

and so (5.4) $\qquad J(s, \theta) = \theta\zeta(s-1) - \frac{1}{2}\zeta(s) + \Omega(s, \theta),$

where (5.5) $\Omega(s, \theta) = (2\,ch\,\gamma)^s \sum_{\nu=1}^{\infty} \sum_{h,k}' \sum_{l=0}^{\infty} (-1)^{(l+1)\nu} \begin{bmatrix} s \\ l \end{bmatrix} e^{-(s+2l)\nu\gamma} \frac{Q^l}{P^{s+l}}.$

The quadruple series is absolutely convergent if $\sigma > 2$. We have then in fact

$$\left| \begin{bmatrix} s \\ l \end{bmatrix} \right| = \left| \frac{\Gamma(s+l)}{\Gamma(s)\Gamma(l+1)} \right| < Al^{\sigma-1},$$

$$|P^{-s-l}| < A(h+k)^{-\sigma-l}, \qquad |Q^l| < A(h+k)^l,$$

where the A's are constants: and the series may be compared with

$$\sum_{h,k} (h+k)^{-\sigma} \sum_{\nu,l} l^{\sigma-1} e^{-(\sigma+2l)\nu\gamma} = \sum_{h,k} (h+k)^{-\sigma} \sum_{l} \frac{l^{\sigma-1} e^{-(\sigma+2l)\gamma}}{1 - e^{-(\sigma+2l)\gamma}}.$$

We may therefore rearrange the series as we please. Effecting the summation with respect to ν, we obtain

(5.6) $$\Omega(s, \theta) = -(2\,ch\,\gamma)^s \sum_{l=0}^{\infty} (-1)^l \begin{bmatrix} s \\ l \end{bmatrix} \frac{e^{-(s+2l)\gamma}}{1 + (-1)^l e^{-(s+2l)\gamma}} Z_l(s, \theta),$$

where (5.61) $$Z_l(s, \theta) = \sum_{h,k} \frac{Q^l}{P^{s+l}}.$$

These equations are valid for $\sigma > 2$. It will be observed that, up to this point, our argument is entirely "elementary".

6. We shall now show that the equation (5.6) gives the analytical continuation of $\Omega(s, \theta)$, and so of $J(s, \theta)$, all over the plane. We begin by proving the following lemma.

Lemma. *The function $Z_l(s, \theta)$ is regular all over the plane, except possibly for simple poles at the points*

$$s = -l+1, -l+2, \ldots, 0, 1, 2. \tag{6.1}$$

If D is any bounded domain in the plane of s, from which these points are excluded, then

$$|Z_l(s, \theta)| < e^{\varepsilon l}, \tag{6.2}$$

where ε is any positive number, for all values of s in D, and all sufficiently large values of l.

We may obviously suppose that $\sigma \leqq 3$ at all points of D. We have

$$Z = Z_l(s, \theta) = \sum \frac{Q^l}{P^{s+l}} = w^{s+l} \sum \frac{(hw' - kw)^l}{(hw + kw')^{s+l}}, \tag{6.3}$$

if (6.31) $\qquad w = \theta, \qquad w' = 1, \qquad \sigma > 2.$

We write

$$Z = e^{\delta s^2} \frac{(s-2)(s-1)\cdots(s+l-1)}{(s-4)(s-5)\cdots(s-l-5)} w^{-s} Z, \tag{6.4}$$

where $\delta > 0$; and we prove

(i) that Z is regular in a strip

$$-A_1 l \leqq \sigma \leqq 3; \tag{6.5}$$

(ii) that $|Z| \to 0$ when $|t| \to \infty$, for every fixed value of l, uniformly throughout the strip;

(iii) that (6.6) $\qquad |Z| < A_2 \quad (\sigma = 3);$

(iv) that

$$|Z| < e^{A_3 \delta l^2} \quad (\sigma = -A_1 l) \tag{6.7}$$

for all sufficiently large values of l.

In these propositions the A's are appropriately chosen constants (independent of δ, s, and l).

7. To prove (i) we observe that

$$(7.1)\quad s(s+1)\cdots(s+l-1)\,w^{-s-l}\,Z$$

$$= s(s+1)\cdots(s+l-1)\sum\frac{(hw'-kw)^l}{(hw+kw')^{s+l}}$$

$$= D\sum\frac{1}{(hw+kw')^s} = D\zeta_2(s,\, w+w',\, w,\, w') = D\zeta_2(s),$$

if $\sigma > 2$, D being a differential operator

$$(7.2)\qquad D = \sum_{r=0}^{l} H_r\left(\frac{\partial}{\partial w}\right)^r\left(\frac{\partial}{\partial w'}\right)^{l-r} = \sum_{r=0}^{l} H_r\, D_r.$$

As $\zeta_2(s)$ is regular save for simple poles at $s=1$ and $s=2$, the same is true of $D\zeta_2(s)$, and so Z is regular in the strip (6.5).

The truth of (ii) is almost obvious; for $\zeta_2(s)$, and so Z, is of finite order in the strip, while $e^{\delta s^2}$ tends to zero like $e^{-\delta t^2}$.

To prove (iii) we observe that, when $\sigma = 3$,

$$\left|\frac{(s-2)(s-1)\cdots(s+l-1)}{(s-4)(s-5)\cdots(s-l-5)}\right| = 1, \qquad |e^{\delta s^2}| \leqq e^{9\delta},$$

$$|w^{-s}Z| \leqq w^l\sum\frac{|hw'-kw|^l}{(hw+kw')^{3+l}}$$

$$= \sum\left|\frac{w(hw'-kw)}{hw+kw'}\right|^l\frac{1}{(hw+kw')^3} \leqq \sum\frac{1}{(hw+kw')^3}.^{9)}$$

---

[9]) The function

$$\frac{w(w'-wx)}{w+w'x} = \frac{\theta(1-\theta x)}{\theta+x}$$

decreases steadily from 1 to $-\theta^2$ as x increases from 0 to ∞.

It remains to prove (iv). It is plain that, if we write

$$(7.3)\quad Z = e^{\delta s^2} \frac{(s-2)(s-1)}{(s-4)(s-5)\cdots(s-l-5)} w^l D\zeta_2(s) = \eta(s) D\zeta_2(s),$$

then (7.4)
$$|\eta(s)| < e^{\delta\sigma^2-\delta t^2} w^l < e^{A_4\delta l^2-\delta t^2},$$

when $\sigma = -A_1 l$ and l is sufficiently large.

Again, the number of terms in D is $l+1$, and the numerical value of a coefficient H_r is less than A_5^l. Hence, if

$$(7.5)\qquad \psi = \operatorname*{Max}_{0 \leqq r \leqq l} |D_r \zeta_2(s)|,$$

we have (7.6) $|Z| < (l+1) A_5^l e^{A_4\delta l^2-\delta t^2} \psi < e^{A_6\delta l^2-\delta t^2} \psi,$

if l is sufficiently large.

In order to obtain an upper bound for ψ, we use the equation [10]

$$(7.7)\quad \zeta_2(s) = (2\pi)^{s-1} \Gamma(1-s) \left\{ w^{-s} \sum_{m=1}^{\infty} \frac{\sin\left\{\frac{1}{2}(1-s)\pi - \frac{mw'\pi}{w}\right\}}{m^{1-s} \sin \frac{mw'\pi}{w}} \right\}^*.$$

The two series involved in (7.7) are absolutely convergent, when θ is quadratic, for $\sigma < 0$. We have to examine the effect on the series of the operator D_r. Expanding the numerators in (7.7), we obtain four series, two of the type

$$(7.8)\qquad (2\pi)^{s-1} \Gamma(1-s) \sin\frac{1}{2}(1-s)\pi \sum m^{s-1} \cot \frac{mw'\pi}{w},$$

and two of a simpler type (without the cotangent). It will be clear from our analysis that it is sufficient to consider the series (7.8).

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[10]) See p. 218 of our former memoir for this formula, and for an explanation of the notation.

The external factor is independent of w and w', and it is easily verified that, when $s = -A_1 l + it$,

(7.9) $$\left|(2\pi)^{s-1}\Gamma(1-s)\sin\frac{1}{2}(1-s)\pi\right| < (A_7 l^2 + t^2)^{A_6 l}.$$ [11]

8. Again, it is easily verified that

(8.1) $$D_r \cot\frac{mw'\pi}{w} = m^l \sum_j C_j w^{a_j} w'^{b_j} \left(\cos\frac{mw'\pi}{w}\right)^{c_j} \left(\sin\frac{mw'\pi}{w}\right)^{d_j},$$

where the number of terms is less than A_9^l, the coefficients C_j are all less than $l^{A_{10} l}$, a_j and b_j are numerically less than $A_{11} l$, $c_j \geqq 0$, and $d_j \geqq -l-1$. The effect of the operator D_r on the function (7.8) is therefore to produce a series which possesses a majorant of the form

(8.2) $$(A_7 l^2 + t^2)^{A_6 l}\, l^{A_{12} l} \sum m^{\sigma+l-1} \left|\operatorname{cosec}\frac{mw'\pi}{w}\right|^{-l-1}.$$

But $$\left|\operatorname{cosec}\frac{mw'\pi}{w}\right| > \frac{A_{13}}{m};$$

and the series in (8.2) is therefore convergent, and less than A_{14}^l, if $\sigma = -A_1 l$ and l is sufficiently large. It follows that

(8.3) $$\psi < (A_7 l^2 + t^2)^{A_6 l}\, l^{A_{15} l},$$

and, by (7.6), (8.4) $$|Z| < e^{A_4 \delta l^2 - \delta t^2}(A_7 l^2 + t^2)^{A_6 l}\, l^{A_{15} l},$$

if l is sufficiently large.

Taking the maximum of the right hand side of (8.4) for variation of t, we find

$$A_7 l^2 + t^2 = \frac{A_8 l}{\delta},$$

---

[11]) We have in fact

$$|\Gamma(\alpha + i\beta)| < (\alpha^2 + \beta^2)^{\frac{1}{2}(\alpha+1)} \sqrt{\frac{\beta}{\operatorname{sh}\beta\pi}} \quad (\alpha \geqq 3).$$

so that

$$|Z| < e^{A_6 \delta l^p} \left(\frac{A_8 l}{\delta}\right)^{A_9 l} l^{A_{13} l} < e^{A_6 \delta l^p} l^{A_{15} l} < e^{A_3 \delta l^p}, \tag{8.5}$$

if l is sufficiently large. This is (iv) of § 6.

From (i), (ii), (iii) and (iv) of § 6 it follows[12]) that

$$|Z| < A_2^{\frac{\sigma + A_1 l}{3 + A_1 l}} \left(e^{A_3 \delta l^p}\right)^{\frac{3-\sigma}{3+A_1 l}}$$

for $-A_1 l \leqq \sigma \leqq 3$. If now D is the domain of § 6, and

$$-A_{16} \leqq \sigma \leqq 3$$

throughout D, we have in D

$$|Z| < A_{17} \left(e^{A_3 \delta l^p}\right)^{\frac{3+A_{16}}{3+A l}} < e^{A_{18} \delta l},$$

and so

$$|Z| < A_{19}\, e^{A_{18} \delta l} < e^{A_{20} \delta l} < e^{\varepsilon l},$$

if δ is sufficiently small and l sufficiently large. This completes the proof of the lemma.

9. The points $0, -1, -2, \ldots$ were excluded from D in the lemma, as they are possible singularities of some of the functions Z_l. A bounded domain D' can contain at most a finite number of these points. We may suppose it to contain some or all of the points $0, -1, \ldots, -p$, but neither of the points $1, 2$. The function

$$s(s+1)(s+2)\cdots(s+p)\, Z_l(s, \theta)$$

is plainly regular in D', and satisfies an inequality analogous to that satisfied by Z_l in D.

The series (5.6) is uniformly convergent in D', and gives the analytic continuation of $\Omega(s, \theta)$, and so of $J(s, \theta)$, all over the plane.

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[12]) If $f(s)$ is regular and bounded in the strip $\alpha \leqq \sigma \leqq \beta$, and $L(\sigma)$ is the upper bound of its modulus for $s = \sigma + it$, then $\log L(\sigma)$ is a convex function of σ, so that

$$(L(\sigma))^{\beta-\alpha} \leqq (L(\alpha))^{\beta-\sigma} (L(\beta))^{\sigma-\alpha}.$$

The theorem is a variant of Hadamard's "three circle theorem". See G. Doetsch, „Über die obere Grenze des absoluten Betrages einer analytischen Funktion auf Geraden", *Math. Zeitschrift*, 9 (1920), 237—240.

It follows that the function $J(s, \theta)$ is meromorphic, that its poles are simple, and that they lie at some or all of the points given by

$$1+(-1)^l e^{-(s+2l)}$$

(9.1) or
$$s=-2l+\frac{r\pi i}{\log\frac{1}{\theta}},$$

where $l=0,1,2,\ldots.$ and r runs through all even or all odd values, according as l is odd or even.

Our analysis may be extended to the case of a general quadratic θ, but, in view of the existence of HECKE's alternative method, we shall not attempt to carry out the details of the work. The poles of $J(s, \theta)$ lie, in the general case, at some or all of a series of points on the lines $\sigma=0,-2,-4,\ldots,$ at intervals

$$\frac{2\pi i}{\log\frac{1}{\Theta}},$$

where
$$\Theta=\theta_p\,\theta_{p+1}\cdots\theta_{p+q-1},$$

$(a_p, a_{p+1}, \ldots, a_{p+q-1})$ being the periodic part of the continued fraction for θ. It is easy, when $\theta=\sqrt{D}$, to define Θ in terms of the solutions of the PELLIAN equation $\lambda^2-Dy^2=1$ (the unities of the corpus $K(\sqrt{D})$).

We conclude by one remark as to the relation between our formula (5.6) and HECKE's formula[13]). HECKE's formula involves series of the type

$$\sum_{n=-\infty}^{\infty}\frac{\zeta\left(1-\frac{s}{2};\lambda^n v\right)}{\sin\left(\frac{1}{4}s+\frac{1}{2}(1-a)+\frac{ni}{2\log\eta}\right)\pi\,\sin\left(\frac{1}{4}s+\frac{1}{2}(1-a)-\frac{ni}{2\log\eta}\right)\pi}.$$

It is not necessary to explain the meaning of all the symbols. The essential difference between his formula and ours is this, that his collects into one term all poles on a line parallel to the real axis, and ours all on a line parallel to the imaginary axis. We have not been able to find any simple method of passing from one formula to the other.

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13) HECKE, l. c. 63.

COMMENTS

There was a previous brief communication to the London Mathematical Society at its meeting on 9 February 1922 (see *Proc.* 21 (1923), xv–xvi).

The essential difference between the method of this paper and that of Hecke is that Hecke's work is based on the properties of the zeta-functions corresponding to his 'Grössencharactere' for a real quadratic field, whereas Hardy and Littlewood use the periodicity of the continued fraction for a quadratic irrational. The relationship between the two resulting expressions for the series (1.1), each of which exhibits the analytic character of the function, is described at the end of the paper. It seems to be still the case that no simple method has been given of passing from one expression to the other.

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XXVII. SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: THE ANALYTIC PROPERTIES OF CERTAIN DIRICHLET'S SERIES ASSOCIATED WITH THE DISTRIBUTION OF NUMBERS TO MODULUS UNITY

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XXVII. *Some problems of Diophantine approximation: The analytic properties of certain Dirichlet's series associated with the distribution of numbers to modulus unity.*

By G. H. Hardy, Trinity College, and J. E. Littlewood, Trinity College.

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1·1. The series in question are

$$F_1(s) = \Sigma \frac{\alpha_n}{n^s}, \quad F_2(s) = \Sigma \frac{\alpha_n^2 - \frac{1}{12}}{n^s}, \quad F_3(s) = \Sigma \frac{\alpha_n^3 - \frac{1}{4}\alpha_n}{n^s}, \quad \ldots, \quad \ldots\ldots\ldots(1·11)$$

where

$$s = \sigma + it, \quad \alpha_n = \alpha_n(\theta) = \{n\theta\} = n\theta - [n\theta] - \tfrac{1}{2}, \quad \ldots\ldots\ldots(1·111)$$

θ is irrational, $[x]$ is the integral part of x, and the summation (as always unless the contrary is stated) extends over positive integral values of n. The general formula for the kth function is

$$F_k(s) = F_k(s, \theta) = \Sigma \frac{\phi_k(n\theta)}{n^s}, \quad \ldots\ldots\ldots(1·12)$$

where $\phi_k(x)$ is defined by

$$\phi_{2m}(x) = P_{2m}(x) + (-1)^{m-1} B_m, \quad \phi_{2m+1}(x) = P_{2m+1}(x), \quad (0 < x < 1) \ldots(1·131)$$

$$\phi(x+1) = \phi(x) \quad \ldots\ldots\ldots(1·132)$$

and the P's are Bernoulli's polynomials*.

The properties of these functions, which are very remarkable, are intimately bound up with the problem of the distribution of the numbers $n\theta$ to modulus 1†.

1·2. The properties of the function $F_1(s)$ have already been investigated by Hecke‡ when θ is a quadratic surd. Hecke supposes in particular that $\theta = \sqrt{D}$, where D is free from squared factors and congruent to 2 or 3 to modulus 4. He shows that in this case $F_1(s)$ is meromorphic, and that its only possible singularities are simple poles at the points

$$s = -2q + \frac{2r\pi i}{\log \eta}, \quad \ldots\ldots\ldots(1·21)$$

where

$$q = 0, 1, 2, \ldots; \; r = \ldots, -1, 0, 1, 2, \ldots, \quad \ldots\ldots\ldots(1·211)$$

and η is a particular unity of the corpus $K(\sqrt{D})$. His method rests upon the theory of the new 'Zeta-functions' which he has recently introduced into analysis, and there can be no doubt that it is the best for the particular problem with which he is concerned.

It is none the less of interest to discuss the function for general values of θ, and by methods as elementary as possible. When we do this, we find ourselves compelled to treat $F_1(s)$ as the

* We follow the notation of Lindelöf (*Le calcul des résidus et ses applications à la théorie des fonctions*, 32 *et seq.*). The definition of the functions for integral values of x is immaterial.

† In regard to this problem see the following memoirs:

G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine approximation': (1) *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge, 1912, 1, 223—229; (2) 'The fractional part of $n^k\theta$', *Acta Math.*, 37 (1914), 155—190; (3) 'The lattice points of a right-angled triangle', *Proc. London Math. Soc.* (2), 20 (1921), 15—36; (4) 'The lattice points of a right-angled triangle (second memoir)', *Hamburg Math. Abh.*, 1 (1922), 212—249.

H. Weyl, 'Über die Gleichverteilung von Zahlen mod. Eins', *Math. Ann.*, 77 (1916), 313—352.

E. Hecke, 'Über analytische Funktionen und die Verteilung von Zahlen mod. Eins', *Hamburg Math. Abh.*, 1 (1921), 54—76.

A. Ostrowski; (1) 'Bemerkungen zur Theorie der Diophantischen Approximationen', *ibid.*, 77—98; (2) 'Zu meiner Note: Bemerkungen u.s.w.', *ibid.*, 250—251.

H. Behnke, 'Über die Verteilung von Irrationalitäten mod. 1', *ibid.*, 252—267.

‡ *l.c. supra.*

1923, 4 (with J. E. Littlewood) *Transactions of the Cambridge Philosophical Society*, 22, 519–33.

first of the sequence of functions $F_k(s)$. We also find ourselves led to the following classification of irrationals θ.

We suppose, as we may without loss of generality, that $0<\theta<1$, and we write

$$\theta=\frac{1}{a_1+\theta_1},\quad \theta_1=\frac{1}{a_2+\theta_2},\quad \ldots, \qquad (1\cdot22)$$

where $a_1, a_2, \ldots$ are the partial quotients in the expression of θ as a simple continued fraction. We say that θ is *of class* λ if λ is the least number such that

$$(\theta\theta_1\ldots\theta_{n-1})^{\lambda+\epsilon}/\theta_n\to 0 \qquad (1\cdot23)$$

for every positive ϵ, or, what is the same thing, such that

$$n^{1+\lambda+\epsilon}|\sin n\theta\pi|\to\infty$$

for every positive ϵ. If no such number exists, we say that θ is *of infinite class*. A quadratic surd is of class 0, and every algebraic number is of finite class.

Our principal results may be summarised as follows. In the first place, $F_k(s)$ *is regular for*

$$\sigma>\sigma_k=1-\frac{k}{1+\lambda}; \qquad (1\cdot24)$$

in particular, $F_1(s)$ is regular for $\sigma>\lambda/(1+\lambda)$. This we prove for $k>1$ in § 2, and for $k=1$ in § 3.

There are alternative proofs of this theorem. When $k=1$, it may be derived from Theorem 2 of our memoir (4), or from the sharper Theorem 5, due originally to Ostrowski; but the analysis of § 3 is necessary in any case for our further investigations. When $k>1$, it has been proved by Behnke*, by means of the formulae of linear transformation of the Theta-functions. The proof given here is a good deal simpler.

If $\lambda>0$, the result just stated is final; for then $\sigma=\sigma_k$ *is a singular line for the function*. We prove this in § 3. We have no doubt that the line is still singular when $\lambda=0$, *except when θ is quadratic*, so that the case considered by Hecke is completely exceptional; but this we are unable to prove.

In § 4 we consider the question of the convergence or summability of the series (1·11), and show that the regions of convergence or summability are always as extensive as is consistent with the analytic properties of the functions and the order of magnitude of the coefficients. Some theorems concerning convergence have been found already by Behnke†. These are included in ours, which assert the most that can be true.

2·1. THEOREM 1. *If $k>1$, and θ is of class λ, then $F_k(s)$ is regular for*

$$\sigma>\sigma_k=1-\frac{k}{1+\lambda}.$$

We have‡

$$\phi_{2m}(x)=\frac{(-1)^{m+1}2(2m)!}{(2\pi)^{2m}}\sum\frac{\cos 2\nu\pi x}{\nu^{2m}}\quad (m\geq 1), \qquad (2\cdot111)$$

$$\phi_{2m+1}(x)=\frac{(-1)^{m+1}2(2m+1)!}{(2\pi)^{2m+1}}\sum\frac{\sin 2\nu\pi x}{\nu^{2m+1}}\quad (m\geq 0). \qquad (2\cdot112)$$

It is therefore sufficient to show that the functions

$$g_k(s)=\sum\frac{\psi_k(n\theta)}{n^s},\quad h_k(s)=\sum\frac{\psi_k(-n\theta)}{n^s}, \qquad (2\cdot12)$$

where

$$\psi_k(x)=\sum\frac{e(\nu x)}{\nu^k}\quad (k>1)\S, \qquad (2\cdot121)$$

are regular for $\sigma>\sigma_k$. We shall discuss only $g_k(s)$, observing that our argument remains valid with a formal change throughout of $n\theta$ into $-n\theta$.

* Behnke, *l.c.*, 265—266. † Behnke, *l.c.*, 266. ‡ Lindelöf, *l.c.*, 34. § We write $e(x)$ for $e^{2\pi ix}$, following Weyl.

Suppose first that $\sigma > 1$. Then

$$g_k(s) = \sum_1^\infty \frac{1}{n^s} \sum_1^\infty \frac{e(\nu n\theta)}{\nu^k} = \sum_1^\infty \frac{\chi(\nu)}{\nu^k}, \quad \text{.......(2·13)}$$

where

$$\chi(\nu) = \chi(s, \theta, \nu) = \sum_1^\infty \frac{e(\nu n\theta)}{n^s}. \quad \text{.......(2·131)}$$

This function is an integral function of s, and its continuation all over the plane is given by

$$\chi(\nu) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{e^{-x+2\nu\theta\pi i}}{1 - e^{-x+2\nu\theta\pi i}} (-x)^{s-1}\, dx, \quad \text{.......(2·14)}$$

where C is a loop enclosing the positive real axis in the clockwise direction, and passing inside all the poles

$$x = x_m = 2\pi i(m + \nu\theta) \qquad (m = \dots -1, 0, 1, \dots)$$

of the integrand. We write

$$\mathrm{X}(\nu) = \mathrm{X}(s, \theta, \nu) = \frac{\chi(s, \theta, \nu)}{\Gamma(1-s)}, \quad G_k(s) = \frac{g_k(s)}{\Gamma(1-s)}. \quad \text{.......(2·15)}$$

2·2. There is one and only one of the numbers x_m whose modulus is less than π. We define a number $\delta = \delta(\nu)$ as follows. If $\mathbf{x}_m$ is the x_m of least modulus, and $|\mathbf{x}_m| \geqq \frac{1}{2}\pi$, we take $\delta = \frac{1}{4}\pi$. If $|\mathbf{x}_m| < \frac{1}{2}\pi$, we take $\delta = \frac{3}{4}\pi$. We denote by C_0 the contour formed by the semicircle $|x| = \delta$, $|\arg(-x)| \leqq \frac{1}{2}\pi$ and the two lines $\mathbf{R}(x) \geqq 0$, $|\mathbf{I}(x)| = \delta$. The distance of any point of C_0 from the nearest pole is greater than $\frac{1}{4}\pi$. Hence, if we write

$$\mathrm{X}_0(\nu) = \mathrm{X}_0(s, \theta, \nu) = \frac{1}{2\pi i} \int_{C_0} \frac{e^{-x+2\nu\theta\pi i}}{1 - e^{-x+2\nu\theta\pi i}} (-x)^{s-1}\, dx, \quad \text{.......(2·21)}$$

we have

$$\mathrm{X}_0(\nu) = O\left(\int_{C_0} |e^{-x}|\,|(-x)^{s-1}|\,|dx|\right) = O(1), \quad \text{.......(2·22)}$$

uniformly throughout any bounded domain T in the plane of s.

Now

$$\mathrm{X}(\nu) = \mathrm{X}_0(\nu) \quad (|\mathbf{x}_m| \geqq \tfrac{1}{2}\pi), \quad \text{.......(2·231)}$$

and

$$\mathrm{X}(\nu) = \mathrm{X}_0(\nu) + (-\mathbf{x}_m)^{s-1} \quad (|\mathbf{x}_m| < \tfrac{1}{2}\pi). \quad \text{.......(2·232)}$$

The series

$$\sum \frac{\mathrm{X}_0(\nu)}{\nu^k}$$

is, by (2·22), uniformly convergent throughout T, and its sum is an analytic function regular throughout T. It follows, from (2·13), (2·15), (2·231), and (2·232), that $G_k(s)$ is regular in any bounded domain throughout which the series

$$S = \sum \frac{|(-\mathbf{x}_m)^{s-1}|}{\nu^k} \quad (|\mathbf{x}_m| < \tfrac{1}{2}\pi) \quad \text{.......(2·24)}$$

is uniformly convergent. This is certainly so if the series

$$\sum_1^\infty \frac{1}{\nu^k |\overline{\nu\theta}|^{1-\sigma}},$$

where $\overline{\nu\theta}$ is the difference between $\nu\theta$ and the integer nearest to $\nu\theta$, is uniformly convergent; and this series is, by Lemma 3 of our paper (4), uniformly convergent in any half-plane

$$\sigma \geqq 1 - \frac{k}{1+\lambda} + \epsilon > 1 - \frac{k}{1+\lambda}.$$

In other words, $G_k(s)$ is regular for the values of s specified in the theorem.

It follows from (2·15) that $g_k(s)$ is also regular, except perhaps at the poles $s = 1, 2, 3, \dots$ of $\Gamma(1-s)$. Of these, $s = 2, 3, \dots$ are plainly not poles of $g_k(s)$. When $s = 1$, $\mathrm{X}(\nu)$, and therefore $G_k(s)$, vanishes. Thus $g_k(s)$ is regular also for $s = 1$, which completes the proof of the theorem.

3·1. The method of § 2 fails when $k=1$, and more intricate analysis is necessary.

Lemma A. If θ is irrational and positive, $x \geqq 0$, $y=\theta x$, and $f(0)=g(0)=0$, then

$$\sum_{m\leqq x} f([m\theta])(g(m)-g(m-1))+\sum_{n\leqq y} g\left(\left[\frac{n}{\theta}\right]\right)(f(n)-f(n-1))=f([y])\,g([x]). \qquad \text{(3·11)}$$

This is Lemma 7 of our paper (4).

Lemma B. If $c>0$, ξ is real, and

$$c_1=c/\theta,\quad \xi_1=c-\theta\xi,\quad \beta_n=\{n/\theta\}, \qquad \text{(3·12)}$$

then

$$\frac{e^{-\frac{1}{2}\xi}}{1-e^{-\xi}}\sum_1^\infty e^{-mc}(e^{\alpha_m\xi}-1)+\frac{e^{-\frac{1}{2}\xi_1}}{1-e^{-\xi_1}}\sum_1^\infty e^{-nc_1}(e^{\beta_n\xi_1}-1)=W, \qquad \text{(3·13)}$$

where

$$W=W(c,\theta,\xi)=\frac{e^{-\xi-\xi_1}}{(1-e^{-\xi})(1-e^{-\xi_1})}-\frac{e^{-\frac{1}{2}\xi}}{1-e^{-\xi}}\frac{e^{-c}}{1-e^{-c}}-\frac{e^{-\frac{1}{2}\xi_1}}{1-e^{-\xi_1}}\frac{e^{-c_1}}{1-e^{-c_1}}. \qquad \text{(3·131)}$$

In (3·11) take $\qquad f(u)=1-e^{-u\xi},\quad g(u)=1-e^{-u\xi_1},$

where $c=\theta\xi+\xi_1>0$, and make $x\to\infty$. We obtain

$$\frac{e^{-\xi}}{1-e^{-\xi}}\sum_1^\infty e^{-[m\theta]\xi-m\xi_1}+\frac{e^{-\xi_1}}{1-e^{-\xi_1}}\sum_1^\infty e^{-[n/\theta]\xi_1-n\xi}=\frac{e^{-\xi-\xi_1}}{(1-e^{-\xi})(1-e^{-\xi_1})}.$$

Substituting for $[m\theta]$ and $[n/\theta]$ in terms of α_m and β_n, and making some simple reductions, we obtain (3·13).

Taking the limit of (3·13) as $\xi\to 0$, we obtain

Lemma C. If $c>0$ and $c_1=c/\theta$, then

$$\sum_1^\infty \alpha_m e^{-mc}+\frac{e^{-\frac{1}{2}c}}{1-e^{-c}}\sum_1^\infty (e^{\beta_n c}-1)e^{-nc_1}=w, \qquad \text{(3·14)}$$

where

$$w=w(c,\theta)=\frac{\theta e^{-c}}{(1-e^{-c})^2}-\frac{1}{2}\frac{e^{-c}}{1-e^{-c}}-\frac{e^{-\frac{1}{2}c}}{1-e^{-c}}\frac{e^{-c_1}}{1-e^{-c_1}}. \qquad \text{(3·141)}$$

Lemma D. We have

$$\sum_1^\infty \alpha_m e^{-mc}+\sum_1^\infty \beta_n e^{-nc_1}+\tfrac{1}{2}c\sum_1^\infty (\beta_n^2-\tfrac{1}{12})e^{-nc_1}=v+O(ce^{-c}) \qquad \text{(3·15)}$$

for all positive values of c, where

$$v=v(c,\theta)=\frac{\theta e^{-c}}{(1-e^{-c})^2}-\frac{1}{2}\frac{e^{-c}}{1-e^{-c}}-\frac{1}{c}\frac{e^{-c_1}}{1-e^{-c_1}}. \qquad \text{(3·151)}$$

The left-hand side of (3·15) is $O(e^{-c})=O(ce^{-c})$ if $c\geqq 1$. We may therefore suppose $c<1$. In (3·14) we may write

$$\frac{e^{-\frac{1}{2}c}}{1-e^{-c}}=\frac{1}{c}-\frac{c}{24}+O(c^3), \qquad \text{(3·16)}$$

$$e^{\beta_n c}-1=\beta_n c+\tfrac{1}{2}\beta_n^2c^2+O(c^3). \qquad \text{(3·17)}$$

Since $c_1>c$ and $|\beta_n|<1$, we have

$$c^p\sum\beta_n^q e^{-nc_1}=O(c^{p-1}e^{-c})$$

for all positive integral values of p and q. Hence the left-hand side of (3·14) takes the form

$$\sum_1^\infty \alpha_m e^{-mc}+\sum_1^\infty \beta_n e^{-nc_1}+\tfrac{1}{2}c\sum_1^\infty \beta_n^2 e^{-nc_1}+O(ce^{-c}). \qquad \text{(3·18)}$$

Also, by (3·16),

$$\frac{e^{-\frac{1}{2}c}}{1-e^{-c}}\frac{e^{-c_1}}{1-e^{-c_1}}=\left(\frac{1}{c}-\frac{c}{24}\right)\frac{e^{-c_1}}{1-e^{-c_1}}+O(ce^{-c})$$
$$=\frac{1}{c}\frac{e^{-c_1}}{1-e^{-c_1}}-\frac{c}{24}\sum_1^\infty e^{-nc_1}+O(ce^{-c}). \qquad (3\cdot19)$$

Hence (3·14) takes the form (3·15).

3·2. *Lemma E. If σ is sufficiently large,*

$$\frac{1}{\Gamma(s)}\int_0^\infty c^{s-1}v(c,\theta)\,dc=\theta\zeta(s-1)-\tfrac{1}{2}\zeta(s)-\frac{\theta^{s-1}\zeta(s-1)}{s-1}. \qquad (3\cdot21)$$

This function is an integral function of s.

The equation (3·21) follows at once from (3·151) by direct integration. It may be verified at once that the right-hand side is regular at its only possible singularities, viz. $s=1$ and $s=2$.

3·3. In what follows we denote by $\mathbf{D}(\alpha)$ a finite domain in the plane of s, all of whose points satisfy $\sigma\geqq\alpha+\delta>\alpha$; and by $R(s,\alpha)$ a function regular and bounded in $\mathbf{D}(\alpha)$. It is to be understood that the upper bound of such a function depends upon the form of $\mathbf{D}$, and in particular upon δ, *but not upon* θ, and that the O's which we use are also uniform with respect to θ.

Lemma F. If $\psi(c,\theta)=O(c^q e^{-c})$, where $q\geqq0$, then

$$\chi(s,\theta)=\frac{1}{\Gamma(s)}\int_0^\infty\psi(c,\theta)\,c^{s-1}dc=R(s,-q). \qquad (3\cdot31)$$

For the integral is uniformly convergent in $\mathbf{D}(-q)$.

Lemma G. The function

$$F_1(s,\theta)+\theta^s F_1(s,\theta_1)+\tfrac{1}{2}s\theta^{s+1}F_2(s+1,\theta_1) \qquad (3\cdot32)$$

is regular for $\sigma>-1$.

Supposing first σ sufficiently large, multiply (3·15) by $c^{s-1}/\Gamma(s)$, and integrate from $c=0$ to $c=\infty$. The result then follows immediately from Lemmas E and F. We obtain in fact

$$F_1(s,\theta)+\theta^s F_1(s,\theta_1)+\tfrac{1}{2}s\theta^{s+1}F_2(s+1,\theta_1)=\theta\zeta(s-1)-\tfrac{1}{2}\zeta(s)-\frac{\theta^{s-1}\zeta(s-1)}{s-1}+R(s,-1), \qquad (3\cdot33)$$

and

$$Z_1(s)=Z_1(s,\theta)=\theta\zeta(s-1)-\tfrac{1}{2}\zeta(s)-\frac{\theta^{s-1}\zeta(s-1)}{s-1} \qquad (3\cdot34)$$

is an integral function.

As a corollary, we have

THEOREM 2. *The function $F_1(s,\theta)+\theta^s F_1(s,\theta_1)$ is regular for $\sigma>0$.*

For $F_2(s+1,\theta_1)$ is plainly a function $R(s,0)$.

3·4. We have, from (3·33),

$$F_1(s,\theta)+\theta^s F_1(s,\theta_1)=Z_1(s,\theta)+R(s,0). \qquad (3\cdot41)$$

Similarly

$$F_1(s,\theta_1)+\theta_1^s F_1(s,\theta_2)=Z_1(s,\theta_1)+R(s,0), \qquad (3\cdot411)$$

and so on generally. From the first n such equations we deduce

$$F_1(s,\theta)+(-1)^{n-1}(\theta\theta_1\ldots\theta_{n-1})^s F_1(s,\theta_n)=\Phi_n+\Psi_n, \qquad (3\cdot42)$$

where $$\Phi_n = \sum_{\nu=0}^{n-1} (-1)^\nu (\theta\theta_1 \dots \theta_{\nu-1})^s Z_1(s, \theta_\nu), \quad \text{.......................(3·421)}$$

$$\Psi_n = \sum_{\nu=0}^{n-1} (-1)^\nu (\theta\theta_1 \dots \theta_{\nu-1})^s R(s, 0)^*. \quad \text{.......................(3·422)}$$

We suppose for the moment that $\sigma > 2$.

Then $$|F_1(s, \theta_n)| < A$$
where A is independent of n and s, and the second term on the left-hand side of (3·42) tends to zero. Similarly the functions Φ_n and Ψ_n tend to

$$\Phi(s) = \sum_{\nu=0}^{\infty} (-1)^\nu (\theta\theta_1 \dots \theta_{\nu-1})^s Z_1(s, \theta_\nu), \quad \text{.......................(3·431)}$$

$$\Psi(s) = \sum_{\nu=0}^{\infty} (-1)^\nu (\theta\theta_1 \dots \theta_{\nu-1})^s R(s, 0), \quad \text{.......................(3·432)}$$

respectively; and $$F_1(s, \theta) = \Phi(s) + \Psi(s), \quad \text{...................................(3·44)}$$
for $\sigma > 2$. This relation between analytic functions holds throughout any region in which each of them is regular. The function $\Psi(s)$ is plainly regular for $\sigma > 0$, since $\theta_\nu\theta_{\nu+1} < \frac{1}{2}$. We thus obtain

THEOREM 3. *The function* $$F_1(s, \theta) - \Phi(s) \quad \text{.......................................(3·45)}$$
is regular for $\sigma > 0$.

The study of the singularities of $F_1(s, \theta)$, for $\sigma > 0$, is thus reduced to that of the singularities of $\Phi(s)$ in the same region.

3·5. THEOREM 4. *If θ is of class λ, then each of the functions*
$$F_1(s, \theta), \quad \Phi(s)$$
is regular for $$\sigma > \sigma_1 = 1 - \frac{1}{1+\lambda} = \frac{\lambda}{1+\lambda}.$$
If $\lambda > 0$, then the line $\sigma = \sigma_1$ is a singular line for each function.

We observe first that $$Z_1(s) = O(\theta^{\sigma-1}) + O(1), \quad \text{................................(3·51)}$$
uniformly in θ, on any closed curve C which does not pass through either of the points $s = 1$ or $s = 2$. The series for $\Phi(s)$ is thus the sum of two series, of the types
$$\Sigma\, O(\theta\theta_1 \dots \theta_{\nu-1})^\sigma, \quad \Sigma\, O(\theta\theta_1 \dots \theta_{\nu-1})^\sigma \theta_\nu^{\sigma-1}$$
respectively. The first series is uniformly convergent on C if C lies in any half-plane $\sigma \geqq \delta > 0$. The second is convergent if
$$\sigma + (\sigma - 1)\lambda > 0,$$
i.e. if $\sigma > \sigma_1$, and is uniformly convergent on C if C lies in any half-plane $\sigma \geqq \sigma_1 + \delta > \sigma_1$. It follows that $\Phi(s)$, and therefore $F_1(s, \theta)$, is regular inside any curve C subject to these conditions, and therefore for $\sigma > \sigma_1$.

It remains to show that, when $\lambda > 0$, $\sigma = \sigma_1$ is a singular line, and it is plainly enough, after what precedes, to show that the line is singular for

$$\Phi_1(s) = \frac{\zeta(s-1)}{s-1} \Sigma (-1)^\nu (\theta\theta_1 \dots \theta_{\nu-1})^s \theta_\nu^{s-1} = \frac{\zeta(s-1)}{s-1} \mathrm{X}(s), \quad \text{...........(3·52)}$$

or for $\mathrm{X}(s)$. We choose $\delta > 0$, and divide the ν's into two classes ν', ν'', writing $\nu = \nu'$ if
$$\theta_\nu < (\theta\theta_1 \dots \theta_{\nu-1})^{\lambda-\delta} \quad \text{.....................................(3·53)}$$

* $R(s, 0)$ is of course a different function in different terms of this series.

and $\nu = \nu''$ in the contrary case. In virtue of the definition of λ, there are, for every δ, an infinity of ν''s.

We write
$$X(s) = \sum_{\nu} = \sum_{\nu'} + \sum_{\nu''} = X'(s) + X''(s). \qquad (3{\cdot}54)$$

The series for X'' is absolutely convergent if $\sigma + (\sigma - 1)(\lambda - \delta) > 0$ or
$$\sigma > \frac{\lambda - \delta}{1 + \lambda - \delta},$$
and the number on the right-hand side is less than σ_1. Hence X'' is regular across the line $\sigma = \sigma_1$. It is therefore sufficient to prove the line singular for X'.

Suppose that the values of ν' are $\nu_1, \nu_2, \ldots, \nu_k, \ldots$, and write
$$e^{-\lambda_k} = \theta\theta_1 \ldots \theta_{\nu_k}. \qquad (3{\cdot}55)$$

Then the series for $X'(s)$, viz.
$$\Sigma \frac{(-1)^{\nu_k}}{\theta_{\nu_k}} (\theta\theta_1 \ldots \theta_{\nu_k})^s = \Sigma \frac{(-1)^{\nu_k}}{\theta_{\nu_k}} e^{-\lambda_k s}$$
is a Dirichlet's series of the type $\Sigma a_k e^{-\lambda_k s}$, and
$$\lambda_{k+1} - \lambda_k = \log \frac{1}{\theta_{\nu_k+1}\theta_{\nu_k+2} \ldots \theta_{\nu_{k+1}}} \geqslant \log \frac{1}{\theta_{\nu_{k+1}}} > (\lambda - \delta) \log \frac{1}{\theta\theta_1 \ldots \theta_{\nu_{k+1}-1}} \to \infty$$
when $k \to \infty$. It follows, by a theorem of Wennberg*, that the line $\sigma = \sigma_1$ is singular for X', which completes the proof of the theorem.

We have supposed $0 < \lambda < \infty$. When $\lambda = \infty$ the result is still valid, $\sigma = 1$ being a singular line; and only trivial modifications are needed in the proof. The case $\lambda = 0$ is much more difficult. It appears to be true that $\sigma = 0$ is then a singular line, except in the special case in which θ is a quadratic surd; but we are unable to prove this rigorously. The exceptional case is that studied by Hecke.

3·6. Suppose in particular that θ is a quadratic surd. The continued fraction for θ is then periodic, and we have
$$\theta_{r+km} = \theta_r \quad (r \geqq \rho,\ k = 1, 2, \ldots),$$
if ρ is the number of non-repeated θ's and m the length of the period.

In this case $F_2(s, \theta)$ is, by Theorem 1, regular for $\sigma > -1$. It follows from Lemma G that each of the functions
$$F_1(s, \theta) + (-1)^{\rho-1} (\theta\theta_1 \ldots \theta_{\rho-1})^s F_1(s, \theta_\rho),$$
$$F_1(s, \theta_\rho) + (-1)^{m-1} (\theta_\rho\theta_{\rho+1} \ldots \theta_{\rho+m-1})^s F_1(s, \theta_{\rho+m})$$
is regular for $\sigma > -1$. But the last function is
$$(1 + (-1)^{m-1} \Theta^s) F_1(s, \theta_\rho).$$
It follows that $F_1(s, \theta_\rho)$, and therefore $F_1(s, \theta)$, is regular for $\sigma > -1$, except possibly where
$$1 + (-1)^{m-1} \Theta^s = 0,$$
at which points it may have simple poles. These points are the points
$$s = \frac{k\pi i}{\log \Theta}$$
where k is an arbitrary odd or arbitrary even integer, according as m is odd or even.

* Wennberg, 'Zur Theorie der Dirichlet'schen Reihen', *Inaugural dissertation*, Upsala, 1920, 3—7. It has been shown by Carlson and Landau that the result is true under the more general conditions
$$\lambda_{n+1} - \lambda_n > \Delta, \quad \lambda_n/n \to \infty.$$
See F. Carlson and E. Landau, 'Neuer Beweis und Verallgemeinerungen des Fabryschen Lückensatzes', *Göttinger Nachrichten*, 1921, 184—188.

3·7. There appears to be no doubt of the truth of the following propositions:

(ak) *$F_k(s)$ is regular for $\sigma > \sigma_k$;*

(bk) *$\sigma = \sigma_k$ is a singular line for $F_k(s)$ whenever $\lambda > 0$;*

(ck) *$\sigma = \sigma_k$ is singular even when $\lambda = 0$, except when θ is quadratic;*

(dk) *$F_k(s)$ is meromorphic when θ is quadratic; its poles are all simple; and they are situated at some or all of a doubly infinite system of points distributed at equal distances along the lines*

$$\sigma = 1 - k - 2p \quad (p = 0, 1, \ldots);$$

(ek) *$F_k(s, \theta) + (-1)^{k-1}\theta^{s+k-1}F_k(s, \theta_1)$ is regular for $\sigma > \sigma_{k+1} - 1$; and $\sigma = \sigma_{k+1} - 1$ is a singular line for the function when $\lambda > 0$:*

and a complete theory of the functions would contain proofs of these propositions in full generality.

Of these propositions we have proved (ak), in § 2 when $k > 1$ and in § 3·5 when $k = 1$.

We are unable to prove (ck) in any case. The case in which θ is quadratic is doubtless best treated by the deeper methods of Hecke. We have however shown, in § 3·6, that our method will accomplish *something* in the direction indicated by (dk).

There remain the propositions (bk) and (ek), of which, at present, we have proved ($b\,1$) only. We proceed now to the general proof. The particular case contains most of the leading ideas, and we have condensed the general argument wherever the ground is familiar. In what follows the A's, O's, and $R(s, \alpha)$'s depend on k in addition to the regions $\mathbf{D}$; they are either independent of θ, as in § 3·3, or at any rate, when we have to consider a sequence of irrationals $\theta, \theta_1, \theta_2, \ldots$, of the n in θ_n.

Lemma H. If $k > 1$ we have, throughout $\mathbf{D}(\sigma_k)$,

$$|F_k(s, \theta)| < A\Sigma \nu^{-k}|\overline{\nu\theta}|^{\sigma-1-\frac{1}{2}\delta} + A = AT + A \leqq A\Sigma\nu^{-k}|\overline{\nu\theta}|^{\sigma_k-1+\frac{1}{2}\delta} + A.$$

This is a straightforward deduction from the results of §§ 2·1, 2·2. By (2·231) and (2·232),

$$|G_k(s)| \leqq \Sigma |\mathrm{X}(\nu)|\nu^{-k} \leqq \Sigma(|\mathrm{X}_0(\nu)| + A|\mathbf{x}_m|^{\sigma-1})\nu^{-k}.$$

Also $|\mathrm{X}_0(\nu)| < A$, by (2·22); and, since $A|\overline{\nu\theta}| < |\mathbf{x}_m| < A$, we have

$$|\mathbf{x}_m|^{\sigma-1} < A + A|\overline{\nu\theta}|^{\sigma-1}.$$

Hence
$$|G_k(s)| < A\Sigma\nu^{-k}|\overline{\nu\theta}|^{\sigma-1} + A < AT + A. \qquad (3·71)$$

Let $\mathbf{D}'$ be the domain obtained by removing from $\mathbf{D}$ circles $C_1, C_2, \ldots$ of radius $\frac{1}{4}\delta$ surrounding such poles $1, 2, \ldots$ of $\Gamma(1-s)$ as fall in $\mathbf{D}$. Then

$$|g_k(s)| = |\Gamma(1-s)|\,|G_k(s)| < A\Sigma\nu^{-k}|\overline{\nu\theta}|^{\sigma-1} + A < AT + A \qquad (3·72)$$

in $\mathbf{D}'$. On C_1, $1 - \frac{1}{4}\delta \leqq \sigma \leqq 1 + \frac{1}{4}\delta$, and it is easily deduced that

$$|g_k(s)| < A\Sigma\nu^{-k}|\overline{\nu\theta}|^{-\frac{1}{4}\delta} + A < AT + A \qquad (3·73)$$

on C_1. The middle term here is independent of σ, and $g_k(s)$ is regular for $s = 1$, so that the inequalities (3·73) are valid also inside C_1. Similarly it may be shown that $|g_k(s)| < AT + A$ throughout $C_2, C_3, \ldots$, and so, by (3·72), throughout $\mathbf{D}$. A similar argument may be applied to $h_k(s)$, and the lemma follows, since $F_k(s)$ is a linear combination of the two functions.

Lemma K. Throughout $\mathbf{D}(\sigma_{k+1})$

$$|F_{k+r}(s + r, \theta_n)| < A + A(\theta\theta_1 \ldots \theta_{n-1})^{(\sigma-\frac{1}{2}\delta)(\lambda+\delta)} \quad (1 \leqq r \leqq k)^*.$$

* The important case is $\sigma < 0$. When $\sigma \geqq \delta$ the second term may be absorbed in the first; the proof will be clearer if $\sigma < 0$ is thought of as the standard case.

The left-hand side is less than

$$A\Sigma\nu^{-k-r}\,|\,\overline{\nu\theta_n}\,|^{\sigma+r-1-\frac{1}{2}\delta}+A<A\Sigma\nu^{-k-1}\,|\,\overline{\nu\theta_n}\,|^{\sigma-\frac{1}{2}\delta}+A,$$

by Lemma H. Let $\epsilon=\frac{1}{2}\delta/(k+1)<\delta$, and $h=\lambda+1+\epsilon$. Then

$$\frac{1}{\theta_{n+l}}<\frac{A}{(\theta\theta_1\ldots\theta_{n+l-1})^{h-1}}<\frac{At_n^{-1}}{(\theta_n\ldots\theta_{n+l-1})^{h-1}},\quad\text{......(3·74)}$$

where $$t_n=(\theta\theta_1\ldots\theta_{n-1})^{h-1}=(\theta\ldots\theta_{n-1})^{\lambda+\epsilon}.\quad\text{......(3·75)}$$

If now P_m/Q_m is the mth convergent of the continued fraction for θ_n, (3·74) implies that $Q_m<At_n^{-1}Q^h_{m-1}$, by Lemma 2 of our paper (4), and therefore that

$$|\,\overline{\nu\theta_n}\,|>At_n\nu^{-h}.$$

Hence

$$|F_{k+r}(s+r,\theta_n)|<\sum_\nu\nu^{-k-1}\{A+(At_n\nu^{-h})^{\sigma-\frac{1}{2}\delta}\}<A+At_n^{\sigma-\frac{1}{2}\delta}\sum_\nu\nu^{-k-1-h(\sigma-\frac{1}{2}\delta)}\leqq A+At_n^{\sigma-\frac{1}{2}\delta}\sum_\nu\nu^{-k-1-h(\sigma_{k+1}+\frac{1}{2}\delta)}.$$

The index of ν is

$$-k-1-(1+\lambda+\epsilon)\left(1-\frac{k+1}{1+\lambda}+\tfrac{1}{2}\delta\right)=-1-\epsilon-\left(\tfrac{1}{2}\delta-\frac{(k+1)\epsilon}{1+\lambda}\right)-\lambda-\tfrac{1}{2}\delta\lambda-\delta\epsilon<-1-\epsilon,$$

so that the series last written is convergent. Hence

$$|\,F_{k+r}(s+r,\theta_n)\,|<A+A\,(\theta\ldots\theta_{n-1})^{(\sigma-\frac{1}{2}\delta)(\lambda+\epsilon)}<A+A\,(\theta\ldots\theta_{n-1})^{(\sigma-\frac{1}{2}\delta)(\lambda+\delta)},$$

the result of the lemma.

3·8. We return now to the identity (3·13), and we equate the coefficients of $\xi^{k-1}/k!$ in the Laurent expansions of the two sides. If we define $\phi_0(x)$ to be unity, then

$$\frac{e^{xz}}{e^z-1}=\sum_{r=0}^{\infty}\frac{\phi_r(x)}{r!}z^{r-1}.$$

Hence the coefficient in the first term on the left is

$$\Sigma\{\phi_k(\alpha_m+\tfrac{1}{2})-\phi_k(\tfrac{1}{2})\}\,e^{-mc}=\Sigma\phi_k(m\theta)\,e^{-mc}-\phi_k(\tfrac{1}{2})\,\Sigma e^{-mc}.\quad\text{......(3·81)}$$

The coefficient of $\xi^{k-1}/k!$ in the second term is

$$k(-\theta)^{k-1}\Sigma\,u_ne^{-nc_1},\quad\text{......(3·82)}$$

where $$u_n=u_n(c)=\left(\frac{d}{dc}\right)^{k-1}\left(\frac{e^{-\frac{1}{2}c}}{1-e^{-c}}(e^{\beta_nc}-1)\right)=\left(\frac{d}{dc}\right)^{k-1}\left(\frac{e^{\omega c}-e^{\frac{1}{2}c}}{e^c-1}\right),\quad\text{......(3·821)}$$

$$\omega=\omega_n=\beta_n+\tfrac{1}{2}=(n\theta_1).\quad\text{......(3·822)}$$

Now $$u_n(c)=\sum_{r=0}^{k+1}\frac{u_n^{(r)}(0)}{r!}c^r+\frac{u_n^{(k+2)}(\vartheta c)}{(k+2)!}c^{k+2}\quad(0<\vartheta<1)$$

$$=\sum_{r=0}^{k+1}\left[\left(\frac{d}{dc}\right)^{k+r-1}\left(\frac{e^{\omega c}-e^{\frac{1}{2}c}}{e^c-1}\right)\right]_0\frac{c^r}{r!}+\left[\left(\frac{d}{dc}\right)^{2k+1}\left(\frac{e^{\omega c}-e^{\frac{1}{2}c}}{e^c-1}\right)\right]_{\vartheta c}\frac{c^{k+2}}{(k+2)!}$$

$$=\sum_{r=0}^{k+1}\frac{1}{k+r}\{\phi_{k+r}(\omega)-\phi_{k+r}(\tfrac{1}{2})\}\frac{c^r}{r!}+\Phi(\vartheta c)\frac{c^{k+2}}{(k+2)!},\quad\text{......(3·83)}$$

say; and it is easily verified that

$$|\,\Phi(x)\,|<A\quad(x>0).\quad\text{......(3·84)}$$

Summing up from (3·81), (3·82), (3·83), and (3·84), we find that the coefficient of $\xi^{k-1}/k!$ in the left-hand side of (3·13) is

$$\Sigma\phi_k(m\theta)\,e^{-mc}+(-\theta)^{k-1}\Sigma\,\phi_k(n\theta_1)\,e^{-nc_1}+(-\theta)^{k-1}\sum_{r=1}^{k+1}\frac{k}{k+r}\frac{c^r}{r!}\Sigma\,\phi_{k+r}(n\theta_1)\,e^{-nc_1}$$

$$-\phi_k(\tfrac{1}{2})\,\Sigma e^{-mc}-(-\theta)^{k-1}\sum_{r=0}^{k+1}\frac{k}{k+r}\phi_{k+r}(\tfrac{1}{2})\frac{c^r}{r!}\Sigma\,e^{-nc_1}+O(c^{k+2})\,\Sigma e^{-nc_1}.\quad\text{......(3·85)}$$

We consider next the coefficient of $\xi^{k-1}/k!$ in W, the right-hand side of (3·13). Now W is regular at $\xi=0$, and, expanding formally, we have

$$W=\left\{\sum_{r=0}^{\infty}\frac{\phi_r(0)}{r!}\xi^{r-1}\right\}\left\{\sum_{r=0}^{\infty}\frac{(-\theta\xi)^r}{r!}\left(\frac{d}{dc}\right)^r\left(\frac{e^{-c}}{1-e^{-c}}\right)\right\}$$
$$-\frac{e^{-c}}{1-e^{-c}}\left\{\sum_{r=0}^{\infty}\frac{\phi_r(\frac12)}{r!}\xi^{r-1}\right\}-\frac{e^{-c_1}}{1-e^{-c_1}}\left\{\sum_{r=0}^{\infty}\frac{(-\theta\xi)^r}{r!}\left(\frac{d}{dc}\right)^r\left(\frac{e^{-\frac12 c}}{1-e^{-c}}\right)\right\}.$$

Collecting the coefficient of $\xi^{k-1}/k!$, and equating it to (3·85), we obtain

$$\Sigma\phi_k(m\theta)e^{-mc}+(-\theta)^{k-1}\Sigma\phi_k(n\theta_1)e^{-nc_1}+\sum_{r=1}^{k+1}(-\theta)^{k-1}\frac{k}{k+r}\frac{c^r}{r!}\Sigma\phi_{k+r}(n\theta_1)e^{-nc_1}$$
$$=O(c^{k+2})\frac{1}{e^{c_1}-1}+V_k(c), \quad \text{......(3·861)}$$

where

$$V_k=\sum_{r=0}^{k}\binom{k}{r}\phi_r(0)(-\theta)^{k-r}\left(\frac{d}{dc}\right)^{k-r}\left(\frac{1}{e^c-1}\right)-\frac{k(-\theta)^{k-1}}{e^{c_1}-1}\left\{\left(\frac{d}{dc}\right)^{k-1}\left(\frac{e^{\frac12 c}}{e^c-1}\right)-\sum_{r=0}^{k+1}\frac{\phi_{k+r}(\frac12)}{k+r}\frac{c^r}{r!}\right\}.$$
$$\text{.........(3·862)}$$

In (3·862) we associate a term $-\dfrac{1}{c}$ with $\dfrac{1}{e^c-1}$ and $\dfrac{e^{\frac12 c}}{e^c-1}$, perform some trivial rearrangements and reductions*, and obtain the alternative expressions

$$V_k=\sum_{r=0}^{k}\binom{k}{r}\phi_r(0)(-\theta)^{k-r}\left(\frac{d}{dc}\right)^{k-r}\left(\frac{1}{e^c-1}-\frac1c\right)-\frac{k!}{\theta}\left(\frac{c}{\theta}\right)^{-k}\left\{\frac{1}{e^{c_1}-1}-\sum_{r=0}^{k}\frac{\phi_r(0)}{r!}\left(\frac{c}{\theta}\right)^{r-1}\right\}$$
$$+\frac{(-1)^k k\theta^{k-1}}{e^{c_1}-1}\left\{\left(\frac{d}{dc}\right)^{k-1}\left(\frac{e^{\frac12 c}}{e^c-1}-\frac1c\right)-\sum_{r=0}^{k}\frac{\phi_{k+r}(\frac12)}{k+r}\frac{c^r}{r!}\right\} \quad \text{.........(3·863)}$$

or

$$V_k=V_{k,1}+V_{k,2}+O(c^{k+2})\frac{1}{e^{c_1}-1}, \quad \text{.........................(3·864)}$$

where $V_{k,1}$ and $V_{k,2}$ are the first and second terms on the right-hand side of (3·863)†.

We now multiply (3·861) by $c^{s-1}/\Gamma(s)$, integrate from $c=0$ to $c=\infty$, and obtain

$$F_k(s,\theta)+(-1)^{k-1}\theta^{k-1+s}F_k(s,\theta_1)+(-1)^{k-1}\sum_{r=1}^{k+1}\theta^{k-1+s+r}\frac{k}{k+r}\frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r+1)}F_{k+r}(s+r,\theta_1)$$
$$=R(s,-k-1)+\frac{1}{\Gamma(s)}\int_0^\infty\{V_{k,1}(c)+V_{k,2}(c)\}c^{s-1}dc$$
$$=R(s,-k-1)+Z_k(s,\theta), \quad \text{...(3·871)}$$

say. By (3·864) and (3·862),

$$V_{k,1}+V_{k,2}=V_k+O(c^{k+2}e^{-c})=O(c^{k+2}e^{-c}) \quad (c>1).$$

Further, the formulae which define $V_{k,1}$ and $V_{k,2}$ show that these functions tend to limits as $c\to 0$. It follows that

$$\int_0^\infty(V_{k,1}+V_{k,2})c^{s-1}dc$$

exists, and defines a function of s regular for $\sigma>0$.

Next, we observe that

$$\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{d}{dc}\right)^{k-r}\left(\frac{1}{e^c-1}-\frac1c\right)c^{s-1}dc=\eta_r(s),$$

and
$$\frac{1}{\Gamma(s)}\int_0^\infty V_{k,2}(c)c^{s-1}dc=-\frac{k!\,\theta^{s-1}}{\Gamma(s)}\int_0^\infty\left\{\frac{1}{e^{c_1}-1}-\sum_{r=0}^{k}\frac{\phi_r(0)}{r!}c_1^{\,r-1}\right\}c_1^{\,-k+s-1}dc_1=\theta^{s-1}z(s)$$

* Observing in particular that the term for which $r=k+1$ vanishes, since $\phi_{2k+1}(\frac12)=0$.

† The expression in curly brackets in the third term is $O(c^{k+2})$.

exist and define functions regular for $0 < \sigma < 1$. Hence

$$Z_k(s, \theta) = \sum_{r=0}^{k} A_r \theta^{k-r} \eta_r(s) + \theta^{s-1} z(s), \qquad (3\cdot88)$$

this equation giving, moreover, the analytic continuation of Z_k wherever η_r and z are regular. Now η_r and z do not contain θ; they belong to a well-known class of integrals; and it may be shown that they are regular everywhere, except possibly for simple poles at certain *positive* integral values of s*. This being so, and if we suppose positive integral values of s to be excluded from $\mathbf{D}(-k-1)$ by circles of radius δ, we have

$$\sum_{r=0}^{k} A_r \theta^{k-r} \eta_r(s) = R(s, -k-1);$$

for only positive powers of θ occur on the left-hand side. Hence, from (3·871),

$$F_k(s, \theta) + (-1)^{k-1} \theta^{k-1+s} F_k(s, \theta_1) = R(s, -k-1) + \theta^{s-1} z(s) + Q(s, \theta_1), \qquad (3\cdot89)$$

where $$Q(s, \theta_1) = (-1)^k \sum_{r=1}^{k+1} \theta^{k-1+s+r} \frac{k}{k+r} \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r+1)} F_{k+r}(s+r, \theta_1). \qquad (3\cdot891)$$

3·9. We are now in a position to prove that $\sigma = \sigma_k$ is a barrier for $F_k(s, \theta)$ when $\lambda > 0$. The case $\lambda = \infty$ is comparatively trivial, and we suppose that $0 < \lambda < \infty$. Then

$$\sigma_k > 1 - k, \quad \sigma_k > \sigma_{k+1}.$$

We write θ_{n-1} for θ in (3·89), multiply by $(-1)^{kn}(\theta\theta_1 \ldots \theta_{n-2})^{k-1+s}$, and sum as in §3·4. We then apply our former argument, which shows on the one hand that

$$\Sigma (-1)^{kn} (\theta \ldots \theta_{n-2})^{k-1+s} R(s, -k-1)$$

is regular for $\sigma > 1 - k$ (except possibly for certain positive integral values of s), and therefore regular across $\sigma = \sigma_k$; and on the other that

$$\Sigma (-1)^{kn} (\theta \ldots \theta_{n-2})^{k-1+s} \theta_{n-1}^{s-1} z(s)$$

has $\sigma = \sigma_k$ for a barrier. To complete the proof of (bk) it is sufficient to show that

$$\Sigma (-1)^{kn} (\theta \ldots \theta_{n-2})^{k-1+s} Q(s, \theta_n)$$

is regular across $\sigma = \sigma_k$; and this is true provided that, for some $\sigma' < \sigma_k$, the series

$$\Sigma (\theta \ldots \theta_{n-2})^{k-1+\sigma} \,|\, Q(s, \theta_n) \,| \qquad (3\cdot91)$$

is uniformly convergent in the part of $D(\sigma_{k+1})$ for which $\sigma \geqq \sigma'$. Now every term in (3·891) contains θ to the power $k - 1 + \sigma$ at least; hence, by Lemma K,

$$|\, Q(s, \theta_n) | < A \theta_{n-1}^{k-1+\sigma} \{1 + (\theta \ldots \theta_{n-1})^{(\sigma - \frac{1}{2}\delta)(\lambda+\delta)}\},$$

and the general term in (3·91) is less than

$$A(\theta \ldots \theta_{n-1})^{k-1+\sigma'} + A(\theta \ldots \theta_{n-1})^{k-1+\sigma' + (\sigma' - \frac{1}{2}\delta)(\lambda+\delta)}.$$

The first index is positive if $\sigma_k - \sigma'$ is small enough, since $\sigma_k > 1 - k$; and the second is
$k - 1 + \sigma_k + \sigma_k(\lambda + \delta) - (\sigma_k - \sigma')(1 + \lambda + \delta) - \frac{1}{2}\delta(\lambda + \delta) = \lambda + \delta\sigma_k - (\sigma_k - \sigma')(1 + \lambda + \delta) - \frac{1}{2}\delta(\lambda + \delta)$,
and is positive if δ and $\sigma_k - \sigma'$ are small enough. This establishes the uniform convergence of the series (3·91), and so finally the general result (bk).

There remains (ek); and for the proof of this the material we have already is sufficient. We observe that

$$\sigma_{k+1} - 1 \geqq -k - 1, \quad \sigma_{k+r} \leqq \sigma_{k+1} \qquad (r \geqq 1).$$

* See for example A. Hurwitz, 'Ueber die Anwendung eines functionentheoretischen Principes auf gewisse bestimmte Integrale', *Math. Annalen*, 53 (1900), 220—224.

From these facts, and from (3·871) and (3·88), it follows that the only possible singularities of

$$G_k(s) = F_k(s, \theta) + (-1)^{k-1}\theta^{s+k} F_k(s, \theta_1),$$

in $\sigma > \sigma_{k+1} - 1$, are the singularities of Z_k in this region. These can occur only for positive integral values of s. On the other hand (3·871) shows that (when $\lambda < \infty$)* Z_k is regular in $\sigma \geqq \sigma'$, where $\sigma' < 1$†. Hence Z_k and G_k are regular in $\sigma > \sigma_{k+1} - 1$.

On the other hand, if $\lambda > 0$,

$$\sigma_{k+1} - 1 > -k,$$

and σ_{k+r} is a strictly decreasing function of r. It follows from (3·871) that G_k and

$$(-1)^k \theta^{s+k} \frac{k}{k+1} s F_{k+1}(s+1, \theta_1)$$

are equi-singular in the region $\sigma > \text{Max}(-k-1, \sigma_{k+2} - 1)$. Since $F_{k+1}(s+1, \theta_1)$ has a barrier $\sigma = \sigma_{k+1} - 1$ in this region, this line is also a barrier for G_k.

We have therefore proved

THEOREM 5. *If $\lambda > 0$, the line $\sigma = \sigma_k$ is a singular line of $F_k(s, \theta)$.*

THEOREM 6. *If $\lambda > 0$ and $\theta = 1/(a_1 + \theta_1)$, then the function*

$$F_k(s, \theta) + (-1)^{k-1}\theta^{s+k-1} F_k(s, \theta_1)$$

is regular for $\sigma > \sigma_{k+1} - 1$; and the line $\sigma = \sigma_{k+1} - 1$ is a singular line of the function.

4·1. We conclude with a brief discussion of the problem of the convergence or summability of the series $\Sigma\phi_k(n\theta) n^{-s}$ in the region of existence of the corresponding function $F_k(s, \theta)$. It will be seen that our conclusions may be roughly expressed by saying that whatever *could* be true *is* true. A Dirichlet's series cannot be summable outside its half-plane of regularity, and it cannot be summable (C, r) unless its nth term is of the form $o(n^r)$: we shall show that our series is summable (with least possible order) except when these restrictions apply.

THEOREM 7. *The series $\Sigma\phi_k(n\theta) n^{-s}$ is convergent if $\sigma > \sigma_k$, $\sigma > 0$; and summable $(C, -\sigma + \delta)$, for every positive δ, if $\sigma > \sigma_k$, $\sigma < 0$.*

The case $\lambda = \infty$ is trivial, since $\sigma_k = 1$, and we suppose that $\lambda < \infty$. We may confine ourselves also to the case $k > 1$; for the result for $k = 1$ is an immediate deduction from the formula

$$\sum_{n<x} a_n = O\left(x^{1 - \frac{1}{1+\lambda} + \epsilon}\right) = O\left(x^{\sigma_1 + \epsilon}\right)‡. \qquad (4·11)$$

We shall in fact prove rather more than we have stated, when $k > 1$, viz. that the series is summable $(C, -\sigma' + \delta)$, where $\sigma' = \text{Min}(\sigma, 1)$.

When $k > 1$, $\phi_k(n\theta)$ is of the form $A(\psi_k(n\theta) \pm \psi_k(-n\theta))$, where $\psi_k(n\theta)$ is the function (2·121). It is therefore sufficient for our purpose to show that the series

$$\Sigma\psi_k(n\theta) n^{-s}, \quad \Sigma\psi_k(-n\theta) n^{-s},$$

are summable $(C, -\sigma' + \delta)$ for $\sigma > \sigma_k$. Further, it is enough to prove this for real values of s, and hence also, since $\psi_k(n\theta)$ and $\psi_k(-n\theta)$ are then conjugate imaginaries, enough to prove it for the first series. This we do by a series of lemmas.

* The case $\lambda = \infty$ is a trivial deduction from (3·89).

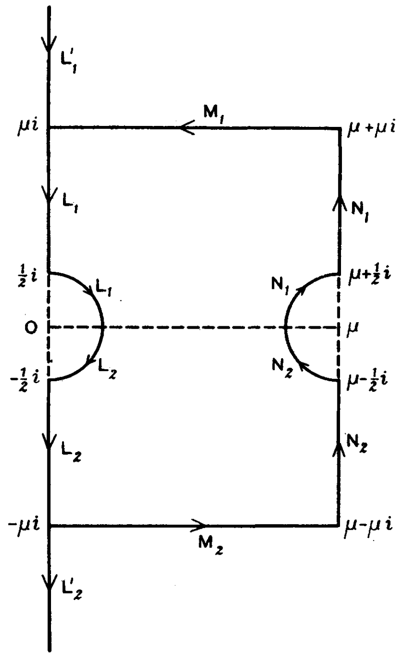
† Thus Z_k is an integral function of s when $\lambda < \infty$. This conclusion may be extended also to the case $\lambda = \infty$. For our argument shows that, in any case, Z_k can have as a singularity at most a simple pole at $s = 1$, the residue being a rational function $r(\theta)$ of θ. Since $r(\theta)$ vanishes for every θ for which $0 < \lambda < \infty$, it must be identically zero.

‡ See our paper (4), Theorem 2.

4·2. Let μ be a (large) positive integer, and let $0 < \phi < 1$, $-1 < \alpha < \beta \leqq \alpha + 1$;

$$S(\mu, \phi) = \sum_{n<\mu} n^{\alpha} e^{2n\pi i\phi} \left(1 - \frac{n}{\mu}\right)^{\beta},$$

$$S(\phi) = \lim_{r \to 1-0} \Sigma n^{\alpha} e^{2n\pi i\phi} r^{n}.$$



Let Λ_1 be the contour $L_1 + L_1'$ of the figure (in which the sense of description is indicated by an arrow), Λ_2 be $L_2 + L_2'$, and Λ be $\Lambda_1 + \Lambda_2$. Finally, let C be the indented rectangle

$$L_1 + L_2 + M_2 + N_2 + N_1 + M_1.$$

In what follows A's denote positive constants depending only on k, α, and β, and the O's have a corresponding meaning.

Lemma L: $|S(\phi)| < A\, \phi^{-(\alpha+1)}$.

This is a particular case of a known result. In fact the function $f(z)$ defined, for $|z| < 1$, by the series $\Sigma n^{\alpha} z^{n}$, has $z = 1$ for its sole singularity, and

$$|f(z)| < A\, |1 - z|^{-(\alpha+1)},$$

so that
$$|S(\phi)| = |f(e^{2\pi i\phi})| < A\, |1 - e^{2\pi i\phi}|^{-(\alpha+1)} < A\, \phi^{-(\alpha+1)}.$$

Lemma M: $S(\phi) = \displaystyle\int_{\Lambda_1+\Lambda_2} z^{\alpha} \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1}\, dz.$

This is a particular case of a very general formula in the theory of residues*.

* See Lindelöf, *l.c.*, ch. v, § 53.

4·3. *Lemma N. If* $0 < |\phi| < \frac{1}{2}$, *then* $|S(\mu, \phi) - S(\phi)| < A|\phi|^{-(\beta+1)}\mu^{\alpha-\beta}$.

We may suppose that $0 < \phi < \frac{1}{2}$, and we begin by showing that

$$S(\mu, \phi) = \int_{L_1+L_2} z^\alpha \left(1 - \frac{z}{\mu}\right)^\beta \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1} dz + O(T), \quad \text{(4·31)}$$

$$S(\phi) = \int_{L_1+L_2} z^\alpha \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1} dz + O(T), \quad \text{(4·32)}$$

where $T = \phi^{-(\beta+1)}\mu^{\alpha-\beta}$. In the first place, by the theorem of residues, we have

$$S(\mu, \phi) = \int_C z^\alpha \left(1 - \frac{z}{\mu}\right)^\beta \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1} dz. \quad \text{(4·33)}$$

Now
$$\left|\frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1}\right| < \begin{cases} Ae^{-2\pi\phi y} & (y > A) \\ Ae^{2\pi(1-\phi)y} & (y < -A), \end{cases}$$

and so
$$\left|\frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1}\right| < Ae^{-A\phi|y|} \quad (|y| > A).$$

Hence the contributions of M_1 and M_2 to the right-hand side of (4·33) are of the form

$$O(\mu^{\alpha+1}e^{-A\mu\phi}) = O(T)(\mu\phi)^{\beta+1}e^{-A\mu\phi} = O(T),$$

since $\beta + 1 > 0$, so that $(\mu\phi)^{\beta+1}e^{-A\mu\phi} < A$. Similarly the straight portions of N_1, N_2 contribute

$$O\left(\int_{\frac{1}{2}}^{\mu} |\mu + iy|^\alpha \left(\frac{y}{\mu}\right)^\beta e^{-Ay\phi} dy\right) = O(\mu^{\alpha-\beta})\int_{\frac{1}{2}}^{\mu} y^\beta e^{-Ay\phi} dy = O(\mu^{\alpha-\beta}\phi^{-(\beta+1)}) = O(T).$$

Lastly, the curved portions of N_1 and N_2 contribute $O(\mu^{\alpha-\beta}) = O(T)$. Thus the contour C, less $L_1 + L_2$, contributes $O(T)$, and (4·31) is proved.

For $S(\phi)$ we have

$$S(\phi) - \int_{L_1+L_2} z^\alpha \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1} dz = \int_{L_1'+L_2'} z^\alpha \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1} dz = O\left(\int_\mu^\infty y^\alpha e^{-Ay\phi} dy\right)$$

$$= O\left(e^{-A\mu\phi}\int_0^\infty y^\alpha e^{-Ay\phi} dy\right) = O\left(e^{-A\mu\phi}\phi^{-\alpha-1}\right) = O\left(e^{-A\mu\phi}(\mu\phi)^{\beta-\alpha}T\right) = O(T).$$

This is (4·32).

From (4·31) and (4·32) we deduce

$$S(\mu, \phi) - S(\phi) = \int_{L_1+L_2} z^\alpha \left\{\left(1 - \frac{z}{\mu}\right)^\beta - 1\right\} \frac{e^{2z\pi i\phi}}{e^{2z\pi i} - 1} dz + O(T).$$

Now
$$\left|\left(1 - \frac{z}{\mu}\right)^\beta - 1\right| < A\left|\frac{z}{\mu}\right|$$

on $L_1 + L_2$. Hence the straight portions of $L_1 + L_2$ contribute

$$O\left(\int_{\frac{1}{2}}^{\mu} y^\alpha . \frac{y}{\mu} . e^{-Ay\phi} dy\right) = O\left(\int_{\frac{1}{2}}^{\mu} y^\alpha . \left(\frac{y}{\mu}\right)^{\beta-\alpha} . e^{-Ay\phi} dy\right) = O(\mu^{\alpha-\beta})\int_0^\infty y^\beta e^{-Ay\phi} dy = O(T),$$

since $\beta - \alpha \leq 1$ and $y/\mu \leq 1$. Finally the curved portions contribute

$$O\left(\frac{1}{\mu}\right) = O(\mu^{\alpha-\beta}) = O(T).$$

This completes the proof of Lemma N.

4·4. We can now deduce that $\Sigma\psi_k(n\theta)n^{-\sigma}$ is summable $(C, -\sigma' + \delta)$ for $\sigma > \sigma_k$. We may suppose that $\sigma < 1$, since a convergent series, whose general term is $O(1/n)$, is summable $(C, -1+\delta)$*. Also $\sigma_k < 1$, since $\lambda < \infty$. Hence we may suppose that $\sigma_k < \sigma = \sigma' < 1$. This being

* G. H. Hardy and J. E. Littlewood, "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1912), 411—478 (462, Theorem 37).

so, we take $\alpha = -\sigma' = -\sigma > -1$, $\beta = -\sigma' + \delta = \alpha + \delta > \alpha$. We shall further suppose, as we may, that $\delta < \sigma - \sigma_k$ and $\delta < 1$, so that $\beta - \alpha < 1$.

Now

$$\sum_{n=1}^{\mu-1} \psi_k(n\theta)\, n^{-\sigma}\left(1-\frac{n}{\mu}\right)^{\beta} = \sum_{\nu=1}^{\infty} \nu^{-k} \sum_{n=1}^{\mu-1} e(\nu n\theta)\, n^{\alpha}\left(1-\frac{n}{\mu}\right)^{\beta}$$

$$= \sum_{\nu=1}^{\infty} \nu^{-k} S((\nu\theta)) + \sum_{\nu=1}^{\infty} \nu^{-k}\{S(\mu, (\nu\theta)) - S((\nu\theta))\}, \quad \text{.........(4·41)}$$

provided that *one* of the series on the right converges. But

$$|S(\mu, (\nu\theta)) - S((\nu\theta))| = |S(\mu, \overline{\nu\theta}) - S(\overline{\nu\theta})| < A\,|\overline{\nu\theta}|^{-(\beta+1)}\mu^{\alpha-\beta},$$

by Lemma N. Also, since $-\beta = \sigma - \delta > \sigma_k$, the series

$$\Sigma\, \nu^{-k}|\overline{\nu\theta}|^{-\beta-1}$$

is convergent. Hence

$$\sum_{\nu=1}^{\infty} \nu^{-k}|S(\mu, \overline{\nu\theta}) - S(\overline{\nu\theta})|$$

is convergent, and its sum tends to zero (like $\mu^{\alpha-\beta}$) as $\mu \to \infty$. It now follows from (4·41) that

$$\Sigma\, \nu^{-k} S((\nu\theta)) \quad \text{...(4·42)}$$

converges, and that the series $\Sigma\, \psi_k(n\theta)\, n^{-\sigma}$

is summable (C, β), *i.e.* $(C, -\sigma' + \delta)$, the sum being given by (4·42). This completes the proof of Theorem 7.

COMMENTS

This paper is concerned with the convergence and analytic character of $\Sigma\{n\theta\}n^{-s}$, considered in 1923, 3 when θ is a quadratic irrational, but now treated for general irrational θ. The problem stated in the penultimate paragraph of § 1.2 is one of the unsolved problems mentioned in the Introduction.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: AN ADDITIONAL NOTE ON THE TRIGONOMETRICAL SERIES ASSOCIATED WITH THE ELLIPTIC THETA-FUNCTIONS.

By

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1. In this note we give an alternative and more instructive proof of the fundamental theorem on which our earlier researches in this field[1] were based. The theorem may be stated as follows.

Theorem A. *Suppose that*

$$0<x<1, \quad 0\leqq\theta\leqq 1, \quad \omega>1 \tag{1.1}$$

and

$$s(\omega)=s(\omega, x, \theta)=\sum_{0\leqq n^2\leqq\omega} e^{-n^2\pi i x}\cos 2n\pi\theta. \tag{1.2}$$

Then

$$s(\omega, x, \theta)-\frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}e^{\frac{\pi i\theta^2}{x}}s\left(x^2\omega, -\frac{1}{x}, \frac{\theta}{x}\right)=O\left(\frac{1}{\sqrt{x}}\right) \tag{1.3}$$

uniformly in ω and θ.[2]

[1] G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine Approximation', *Acta mathematica,* 37 (1914), 193—238, and *Proc. Cambridge Phil. Soc.,* 21 (1923), 1—5.

[2] That is to say, the absolute value of the left hand side is less than $Ax^{-\frac{1}{2}}$, where A is an absolute constant.

Our earlier proof, which followed the classical lines of the calculus of residues, as exposed in Lindelöf's book[1], was fairly straightforward, but very long. Two other proofs have been given recently by VAN DER CORPUT.[2] The proof which we give here proceeds on lines different from any of these, and seems to us in some ways the most natural. It has also the advantage of being applicable, in principle at any rate, to the sums associated with any power of a a theta-function, such as the sum

$$\sum_{0\leqq n\leqq \omega} r(n) e^{-n\pi i x},$$

where $r(n)$ is the number of representations of n as a sum of two squares.

The proof which we give here owes very much of its comparative simplicity to the criticism of Mr. A. E. INGHAM, to whom we submitted our original version. In particular Mr. Ingham pointed out to us the usefulness of the elementary identity (5. 2), and we have rewritten the whole of §§ 5—7 in accordance with his suggestions.

2. We begin by showing that we may assume certain supplementary hypotheses without prejudice to the generality of the theorem.

In the first place, since we are aiming at a result which holds uniformly in θ, we may suppose that $0<\theta<1$. The result for $\theta=0$ or $\theta=1$ will then follow by continuity.

Next, we may suppose that

$$\lambda=\sqrt{\omega} \tag{2. 1}$$

is excluded from each of the sets of intervals

$$(i_m) \quad \frac{m-\theta}{x}-\frac{\delta}{\sqrt{x}}\leqq\lambda\leqq\frac{m-\theta}{x}+\frac{\delta}{\sqrt{x}}, \tag{2. 21}$$

$$(j_m) \quad \frac{m+\theta}{x}-\frac{\delta}{\sqrt{x}}\leqq\lambda\leqq\frac{m+\theta}{x}+\frac{\delta}{\sqrt{x}}, \tag{2. 22}$$

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[1] E. LINDELÖF, *Le calcul des résidus*, 1905.

[2] J. G. VAN DER CORPUT, 'Über Summen, die mit den elliptischen ϑ-Funktionen zusammenhängen', *Math. Annalen*, 87 (1922), 66—77, and 90 (1923), 1—18. van der Corput proves a good deal more than is asserted by the theorem, and his proofs appear for this reason to be more elaborate than they are. The first proof is based on the theory of Fourier series, while the second follows lines more like those of our original proof, on which it is (when reduced to its simplest terms) a considerable improvement.

where δ is an appropriate positive constant. Here $m = 0, 1, 2, \ldots$, and any negative part of any interval is to be discarded as irrelevant.

To prove this, consider the interval I defined by

$$0 \leqq M < \lambda < M + \left[\frac{1}{\sqrt{x}}\right] = M + \mu.$$

If δ is small enough, this interval will necessarily include a λ external to the i's and j's. For each i is of length $\frac{2\delta}{\sqrt{x}}$ and is followed by a complementary interval k of length

$$\frac{1}{x} - \frac{2\delta}{\sqrt{x}} > \frac{1-2\delta}{\sqrt{x}}.$$

There may be no complete k inside I. In this case there is at most one (complete or incomplete) i, and at most

$$\frac{2\delta}{\sqrt{x}} < 4\delta\mu$$

of I is inside i's. Otherwise I contains at least one complete k, and the number of complete or incomplete i's does not exceed twice the number of complete k's. The ratio of the total length of the i's to that of the k's is accordingly less than

$$\frac{4\delta}{1-2\delta}$$

which is less than 8δ if $\delta < \frac{1}{4}$; and then the length of the i's is not greater than $8\delta\mu$. A similar argument applies to the j's, and the part of I, inside one interval or another of the two systems, can in no case exceed $16\delta\mu$. Our conclusion follows if $\delta < \frac{1}{16}$.[1]

Suppose now that Theorem A has been established for λ's excluded from the i's and j's. Any given value of λ lies in an I, and there is therefore a λ' subject to our restrictive conditions and differing from λ by less than μ. It is plain that, if we change λ into λ', the alteration of the left hand side of (1.3) is

[1] It would naturally be easy to improve on this number if it were necessary.

$$O\left(\frac{1}{\sqrt{x}}\right)+O\left(\frac{1}{\sqrt{x}}\right)O(1)=O\left(\frac{1}{\sqrt{x}}\right),$$

so that the theorem, if true for λ', is true for λ.

3. We write

$$f(s)=f(s,\theta)=1+2\sum_{1}^{\infty}e^{-n^2\pi s}\cos 2n\pi\theta, \tag{3.1}$$

where $s=\sigma+it$, $\sigma>0$. It is well-known[1] that

$$f(s)=\frac{1}{\sqrt{s}}\sum_{-\infty}^{\infty}e^{-\frac{\pi}{s}(n-\theta)^2} \tag{3.2}$$

We write also

$$\varepsilon_n=1 \quad (n=0), \qquad \varepsilon_n=2 \quad (n>0).$$

Then, if c is positive, we have[2]

$$\pi\sum_{0\leqq n\leqq\lambda}\varepsilon_n(\omega-n^2)e^{-n^2\pi ix}\cos 2n\pi\theta=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\omega\pi s}}{s^2}f(s+ix)ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\omega\pi(s-ix)}}{(s-ix)^2\sqrt{s}}\sum_{-\infty}^{\infty}e^{-\frac{\pi(n-\theta)^2}{s}}ds,$$

on writing $s-ix$ for s and using (3.2).

We may invert the order of integration and summation; for, when t is large,

$$\left|\frac{1}{(s-ix)^2\sqrt{s}}\right|=O\left(|t|^{-\frac{5}{2}}\right),$$

$$\sum\left|e^{-\frac{\pi(n-\theta)^2}{s}}\right|=\sum exp\left(-\frac{\pi\sigma(n-\theta)^2}{\sigma^2+t^2}\right)$$

$$=O(\sqrt{\sigma^2+t^2})=O(|t|),$$

and

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[1] See for example E. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, 277.

[2] See G. H. HARDY and M. RIESZ, *The general theory of Dirichlet's series*, 50 (Theorem 39). We require the result of the theorem for absolutely convergent series only.

$$\int^{\infty} t^{-\frac{5}{2}}.t.dt$$

is convergent. We have thus

(3. 3)
$$\pi \sum_{0 \leqq n \leqq \lambda} \varepsilon_n (\omega - n^2) e^{-n^2 \pi i x} \cos 2 n \pi \theta = \sum_{-\infty}^{\infty} I_n,$$

where

(3. 4)
$$I_n = \frac{1}{2 \pi i} \int_{c-i\infty}^{c+i\infty} e^{\Phi(s)} \frac{ds}{(s-ix)^2 \sqrt{s}},$$

(3. 41)
$$\Phi(s) = \omega \pi (s - ix) - \frac{\pi (n-\theta)^2}{s}.$$

If we suppose now that $\lambda = \sqrt{\omega}$ is non-integral, as plainly we may do without loss of generality, and differentiate (3. 3) formally with respect to ω, we obtain

(3. 5)
$$-1 + 2 s(\omega) = \sum_{-\infty}^{\infty} J_n,$$

where

(3. 51)
$$J_n = \frac{d I_n}{d \omega} = \frac{1}{2 \pi i} \int_{c-i\infty}^{c+i\infty} e^{\Phi(s)} \frac{ds}{(s-ix) \sqrt{s}}.$$

This process is certainly legitimate if ΣJ_n is uniformly convergent in a neighbourhood of the particular value of ω considered. That this is so will appear incidentally in the sequel.

4. Our main idea is to approximate to J_n, in (3. 5), by the saddle-point method or 'method of steepest descents'.

The saddle-points of $e^{\Phi(s)}$ are given by

$$\Phi'(s) = 0$$

or

$$s = \pm i \frac{|n-\theta|}{\lambda} = \pm i N,$$

---

[1] See G. N. Watson, *Theory of Bessel Functions*, 235, for a general account of the method.

say. The curve of 'zero level', given by $\Re\,\Phi(s)=0$, has the equation

$$\sigma(\sigma^2+t^2-N^2)=0,$$

and consists of the imaginary axis and the circle described on the line joining the saddle-points as diameter. It divides the plane into four regions, the 'low' regions, for which $\Re\,\Phi(s)<0$, being the right hand inside and the left hand outside regions.

We define the path $C=C_1+C_2$ by the lines C_1 and C_2 from the point $s=\frac{1}{2}N$ to infinity through the upper and lower saddle-points respectively. The whole path lies in low ground, except at the saddle-points, and it cannot pass through $s=ix$, since (owing to the restrictions of § 2)

$$x \neq \frac{|n-\theta|}{\lambda}$$

for any value of n. We can deform the path of integration in (3. 51) into C, if we introduce the appropriate correction when this deformation involves crossing a pole. This is so if and only if $x>N$, *i. e.* if $|n-\theta|<\lambda x$, and the correction required is accordingly

$$\frac{1}{\sqrt{ix}}\sum_{|n-\theta|<\lambda x} e^{\frac{\pi i(n-\theta)^2}{x}},$$

which differs from

$$2\frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}\,e^{\frac{\pi i\theta^2}{x}}\,s\left(x^2\omega,\,-\frac{1}{x},\,\frac{\theta}{x}\right)$$

by $O\left(\frac{1}{\sqrt{x}}\right)$. We thus obtain

$$s(\omega,x,\theta)-\frac{e^{-\frac{1}{4}\pi i}}{\sqrt{x}}\,e^{\frac{\pi i\theta^2}{x}}\,s\left(x^2\omega,\,-\frac{1}{x},\,\frac{\theta}{x}\right)=\sum_{-\infty}^{\infty}\bar{J}_n+O\left(\frac{1}{\sqrt{x}}\right),$$

where $\bar{J}_n$ differs from J_n in being taken along C; and the proof of Theorem A is reduced to a proof that

$$\Sigma\bar{J}_n=O\left(\frac{1}{\sqrt{x}}\right). \tag{4. 1}$$

5. It is convenient to write

$$\xi = |n - \theta| = N\lambda > 0,$$

$$X = \frac{x}{N} = \frac{\lambda x}{\xi} > 0, \quad Y = \lambda \xi = \omega N > 0,$$

and to transform $\bar{J}_n$ by writing Ns for s. We thus obtain

$$\bar{J}_n = e^{-\omega \pi i x} \sqrt{\frac{\lambda}{\xi}} \frac{1}{2\pi i} \int e^{\pi Y \left(s - \frac{1}{s}\right)} \frac{ds}{(s - iX)\sqrt{s}}, \tag{5.1}$$

where the path of integration $\Gamma = \Gamma_1 + \Gamma_2$ is $C = C_1 + C_2$ reduced in the ratio $1 : N$, so that it passes through the points $-i, \frac{1}{2}, i$.

Since

$$\frac{1}{s - iX} = \frac{i(s-i)}{2s(1+X)} - \frac{i(s+i)}{2s(1-X)} + \frac{i(s^2+1)X}{s(s-iX)(1-X^2)}, \tag{5.2}$$

we may write (5.1) in the form

$$\bar{J}_n = e^{-\omega \pi i x} \sqrt{\frac{\lambda}{\xi}} (K_{n,1} - K_{n,2} + L_n); \tag{5.3}$$

where

$$K_{n,1} = \frac{1}{4\pi(1+X)} \int e^{\pi Y \left(s - \frac{1}{s}\right)} \frac{s-i}{s^{\frac{3}{2}}} ds, \tag{5.31}$$

$$K_{n,2} = \frac{1}{4\pi(1-X)} \int e^{\pi Y \left(s - \frac{1}{s}\right)} \frac{s+i}{s^{\frac{3}{2}}} ds, \tag{5.32}$$

$$L_n = \frac{X}{2\pi(1-X^2)} \int e^{\pi Y \left(s - \frac{1}{s}\right)} \frac{s^2+1}{(s-iX)s^{\frac{3}{2}}} ds. \tag{5.33}$$

6. The integrals $K_{n,1}$ and $K_{n,2}$ are linear combinations of Bessel functions of orders $\frac{1}{2}$ and $-\frac{1}{2}$, and may accordingly be evaluated as elementary functions.[1] We have in fact

[1] See WATSON, *loc. cit.*, 175 *et seq.*

$$K_{n,1} = \frac{i}{2\pi(1+X)\sqrt{Y}} e^{-2\pi i Y} = \frac{i}{2\pi}\sqrt{\frac{\xi}{\lambda}} \frac{e^{-2\pi i \lambda \xi}}{\lambda x + \xi},$$

$$K_{n,2} = \frac{i}{2\pi(1-X)\sqrt{Y}} e^{2\pi i Y} = -\frac{i}{2\pi}\sqrt{\frac{\xi}{\lambda}} \frac{e^{2\pi i \lambda \xi}}{\lambda x - \xi}.$$

Thus the contribution to ΣJ_n of these integrals is

$$\frac{i}{2\pi} e^{-\omega\pi i x}\left(\sum_{-\infty}^{\infty} \frac{e^{-2\pi i \lambda |n-\theta|}}{\lambda x + |n-\theta|} + \sum_{-\infty}^{\infty} \frac{e^{2\pi i \lambda |n-\theta|}}{\lambda x - |n-\theta|}\right)$$

$$= \frac{i}{2\pi} e^{-\omega\pi i x}\left(\sum_{-\infty}^{\infty} \frac{e^{-2\pi i \lambda (n-\theta)}}{\lambda x + n - \theta} + \sum_{-\infty}^{\infty} \frac{e^{2\pi i \lambda (n-\theta)}}{\lambda x - n + \theta}\right),$$

when we combine the positive half of each series with the negative half of the other.

Now[1]

$$\sum_{-\infty}^{\infty} \frac{e^{-2\pi i \lambda (n-\theta)}}{\lambda x + n - \theta} = \pi e^{2\pi i \lambda \theta} \frac{e^{2\pi i \Lambda (\lambda x - \theta)}}{\sin(\lambda x - \theta)\pi},$$

where

$$\Lambda = \lambda - [\lambda] - \frac{1}{2},$$

and this is

$$O\left(\frac{1}{|\sin(\lambda x - \theta)\pi|}\right) = O\left(\frac{1}{\sqrt{x}}\right),$$

in virtue of the restrictions of § 2. The second series may be treated in the same way, so that the total contribution of $K_{n,1}$ and $K_{n,2}$ is $O\left(\frac{1}{\sqrt{x}}\right)$.

7. It remains only to discuss the contribution of L_n. It appears at once from a figure that, on either Γ_1 or Γ_2,

$$(7.1) \qquad |s - iX| > A|1 - X|, \quad \left|s^{\frac{3}{2}}\right| > A, \quad |s^2 + 1| < A|\sigma|$$

if $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$, and

[1] See, for example, T. J. I'A. Bromwich, *Infinite series*.

(7. 2) $$|s-iX|>A|1-X|,\quad \left|s^{\frac{3}{2}}\right|>A|\sigma|^{\frac{3}{2}},\quad |s^2+1|<A\sigma^2$$

if $\sigma<-\frac{1}{2}$, the A's being absolute constants. Also

$$\Re\left(s-\frac{1}{s}\right)=\sigma\left(1-\frac{1}{\sigma^2+t^2}\right)=\sigma\left(1-\frac{1}{\sigma^2+(1-2\sigma)^2}\right)$$

is negative except when $\sigma=0$, and

(7. 3) $$\Re\left(s-\frac{1}{s}\right)<-A\sigma^2\quad\left(-\frac{1}{2}\leqq\sigma\leqq\frac{1}{2}\right),\quad \Re\left(s-\frac{1}{s}\right)<A\sigma\quad\left(\sigma<-\frac{1}{2}\right).$$

It follows from (7. 1) — (7. 3) that

$$|L_n|<\frac{AX}{(1-X)^2(1+X)}\int_{-\frac{1}{2}}^{\frac{1}{2}}|\sigma|e^{-AY\sigma^2}d\sigma+\frac{AX}{(1-X)^2(1+X)}\int_{-\infty}^{\frac{1}{2}}\sqrt{|\sigma|}\,e^{AY\sigma}d\sigma$$

$$<\frac{AX}{(1-X)^2(1+X)}\left(\frac{1}{Y}+\frac{1}{Y^{\frac{3}{2}}}\right)<\frac{A\sqrt{X}}{(1-X)^2\,Y}+\frac{AX}{|1-X|^3\,Y^{\frac{3}{2}}}$$

$$<A\sqrt{\frac{\xi}{\lambda}}\frac{\sqrt{x}}{(\lambda x-\xi)^2}+A\sqrt{\frac{\xi}{\lambda}}\frac{x}{\lambda|\lambda x-\xi|^3},$$

Thus the contribution of L_n is

$$O\left(\sqrt{x}\sum_{-\infty}^{\infty}\frac{1}{(|n-\theta|-\lambda x)^2}\right)+O\left(x\sum_{-\infty}^{\infty}\frac{1}{||n-\theta|-\lambda x|^3}\right).$$

In these series there are at most four terms in which the denominator is less than unity, and the contribution of these terms is

$$O\left(\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)^2\right)+O\left(x\left(\frac{1}{\sqrt{x}}\right)^3\right)=O\left(\frac{1}{\sqrt{x}}\right)$$

in virtue of the restrictions on λ. The contribution of the remaining terms is obviously $O(1)$. Thus the total contribution of L_n is $O\left(\frac{1}{\sqrt{x}}\right)$.

This completes the proof of the theorem. It is only necessary to add one word in justification of the assumption made provisionally in § 3, that the series ΣJ_n is uniformly convergent in a neighbourhood of any particular value of ω under consideration. The series has in fact been decomposed into a number of parts, all of which have been proved uniformly convergent by direct estimation of their terms, except those which were summed in § 6. The uniform convergence of these last series is classical.

COMMENTS

The proof of the approximate functional equation of the theta-function given here is much simpler than the original proof given in 1914, 3. But it is not entirely self-contained, in that the functional equation for $\theta(x)$ is assumed. See the references to Mordell and to Wilton in the comments on 1914, 3.

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: A SERIES OF COSECANTS.

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1. We consider here some properties of the series

$$(1\cdot1) \qquad \text{(S)} \sum \frac{1}{n \sin n\theta\pi}, \ \text{(T)} \sum \frac{1}{\sin n\theta\pi},$$

where θ is irrational. The analytical and arithmetical peculiarities of these series resemble those of the series

$$\sum e^{n^2\theta\pi i}, \ \sum (n\theta-[n\theta]-\tfrac{1}{2}),^{1}$$

which have been discussed very thoroughly both by ourselves and by others.[2] The analysis connected with the series (1·1) is however in some ways particularly simple, and, although many writers, ourselves included, have considered similar series from time to time, there are some obvious questions concerning them which have remained unanswered.

We do not aim primarily at generality. The case in which we shall be interested particularly is that in which

$$\theta=a+\frac{1}{2a+}\,\frac{1}{2a+\dots}=\sqrt{(a^2+1)},$$

where a is an odd integer. It will however sometimes be obvious that our arguments apply to wider classes of irrationals. In particular a good many of our results hold for the class Θ of irrationals whose continued fractions have bounded co-efficients.

[1] $[x]$ is the integral part of x.

[2] Especially Hecke, Ostrowski, and Behnke. See the list of papers at the end.

1930, 3 (with J. E. Littlewood) *Bulletin of the Calcutta Mathematical Society*, 20, 251–66.

We denote the general term of S by s_n, and the sum of its first n terms by S_n, and use a similar notation in T and other letters. It is plain that there is no universal upper bound for S_n, valid for all irrational θ; $|S_n|$ will be, for appropriate θ and n, larger than any assigned function $\phi(n)$. It is also plain that S cannot converge for any θ, since $|\sin n\theta\pi| < A/n$[1] for any θ and an infinity of n. The first question which suggests itself is whether S_n is *bounded* for any θ, and, if so, whether S is then summable by Cesàro's or other means. Our first object is to answer this question by proving Theorem I below.

Theorem I. *There are quadratic θ for which* S *has the following properties:*

(i) S *oscillates finitely*;

(ii) S *is not summable by any Cesàro mean*;

(iii) S *is summable by Riesz's logarithmic means of any positive order.*

In particular all this is true when

$$\theta = \sqrt{(a^2+1)}$$

where a is an odd integer, and the (*Rieszian*) *sum of the series is then*

$$-\frac{\pi}{12}\sqrt{(a^2+1)}.$$

The theorem shews that the behaviour of the series in these respects is like that of

$$1-1+0+1+0+0+0-1+0+\ldots$$

(where the ranks of the non-zero terms are 1, 2, 4, 8,...). The most difficult part of the proof is the proof of (i), which we defer to § 3. In § 2 we prove the rest of the theorem, taking (i) for granted where it is necessary.

Proof of Theorem I, (ii) *and* (iii).

2·1. It is convenient to work in terms of the series.

$$(2\cdot11) \qquad (\mathrm{U})\ \Sigma\ \frac{(-1)^n}{n \sin n\theta\pi},\ (\mathrm{V})\ \Sigma\ \frac{(-1)^n}{\sin n\theta\pi}$$

[1] Here and later $A = A(\theta)$ denotes a positive number depending only on θ, whose precise value is immaterial.

(which differ only trivially from S and T, becoming S and T when θ is replaced by $\theta+1$). We shall also have to consider the series

$$(2{\cdot}12) \qquad (\mathrm{W}) \ \Sigma \frac{1}{\sin^2 n\theta\pi}.$$

Lemma 1. *If ω and ω' are positive, and $\theta=\omega/\omega'$ belongs to Θ, or again if θ is algebraic, then*

$$(2{\cdot}13) \qquad f(s)=f(s,\theta)=\sum \frac{(-1)^n}{n^s \sin n\theta\pi}$$

is absolutely convergent when the real part σ of $s=\sigma+it$ is sufficiently large; and

$$(2{\cdot}14) \qquad \omega^{s-1} f(s,\theta)+\omega'^{\,s-1} f\left(s,\frac{1}{\theta}\right)$$

$$=\frac{(2\pi)^s}{\Gamma(s)\sin\frac{1}{2}s\pi}\,\zeta_2\left(1-s,\frac{\omega+\omega'}{2},\ \omega,\ \omega'\right),$$

where ζ_2 is the double zeta-function of Barnes.[1]

For the proof, see Hardy and Littlewood, 3 (Lemma β, p. 29). There we consider the algebraic case. If θ belongs to Θ (when of course it is not generally algebraic) the result holds for $\sigma>1$. See Hardy and Littlewood, 4 (Lemma 3, p. 216).

Suppose now in particular that

$$(2{\cdot}15) \qquad \theta=\frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\cdots=\sqrt{(a^2+1)}-a,$$

where a is odd. We take

$$\omega^2=\sqrt{(a^2+1)}-a, \qquad \omega'^2=\sqrt{(a^2+1)}+a.$$

Then $\quad f(s,\theta)=f(s,1/\theta)$ and

$$(2{\cdot}16) \qquad f(s)=\frac{(2\pi)^s\zeta_2(1-s,\ \frac{1}{2}\omega+\frac{1}{2}\omega',\ \omega,\ \omega')}{2\Gamma(s)\sin\frac{1}{2}s\pi\cosh\{(s-1)\log\omega'\}}$$

[1] Barnes, 7.

The zeta-function is regular all over the plane, except for simple poles at $s=0$ and $s=-1$, and vanishes when $s=2, 4, 6, \ldots\ldots\ldots\ldots$[1] Hence $f(s)$ is meromorphic, with poles at $s=0$, $s=-1$, and

$$s=1+(l+\tfrac{1}{2})\frac{\pi i}{\log \omega},$$

where l is an integer, positive or negative. If now we write

$$g(s)=g(s, \theta)=f(s+1, \theta+1)=\sum \frac{1}{n^{s+1} \sin n\theta\pi}, \tag{2·17}$$

we obtain

Lemma 2. *If $\theta=\sqrt{(a^2+1)}$, where a is an odd integer, then $g(s)$ is a meromorphic function of s, regular except for poles at $s=-1$, $s=-2$, and*

$$s=\frac{(2l+1)\pi i}{\log\left(\sqrt{(a^2+1)}-a\right)} \tag{2·18}$$

We may remark in passing that the main result of the lemma, that $g(s)$ is meromorphic, holds for all quadratic θ. The proof is in principle the same as that which we have given in the special case, but it is naturally more elaborate and (except when, as here, all the partial quotients of the continued fraction for θ are even) it is necessary to treat the two functions

$$\sum \frac{(-1)^n}{n^s \sin n\theta\pi}, \quad \sum \frac{1}{n^s \sin n\theta\pi}$$

simultaneously.[2]

2·2. The Rieszian mean of $\sum u_n$, of logarithmic type and order k, is[3]

$$U^k(w)=w^{-k} \sum_{\log n \leq w} u_n\, (w-\log n)^k \tag{2·21}$$

[1] See Barnes, pp. 338, 340. That the zeta-function which occurs here vanishes for $s=2, 4, \ldots\ldots$ follows from the regularity of $f(s)$ at those points, and may be verified directly from Barnes' contour integral.

[2] Compare Cooper's discussion of the function $\sum e^{n^2\theta\pi i}\, n^{-s}$ (Cooper, 10).

[3] Hardy and Riesz, 11, p. 21.

When $k=0$ and e^w is an integer, $U^k(w)$ reduces to U_n. If we suppose that, as here, the series $g(s)=\sum u_n n^{-s}$ is absolutely convergent for $\sigma>0$, we have [1]

$$(2{\cdot}22) \qquad U^k(w)=\frac{\Gamma(1+k)}{2\pi i\ w^k}\int_{c-i\infty}^{c+i\infty}\frac{g(s)}{s^{1+k}}e^{ws}\,ds.$$

The zeta-function in (2·16), and so $g(s)$, is of finite order (in the sense of Lindelöf and Bohr)[2] in any strip $\sigma_1\leq\sigma\leq\sigma_2$.[3] It follows that, if k is a sufficiently large integer, we may deform the path of integration into the line $(-b-i\infty,\ -b+i\infty)$, where $0<b<1$, if we introduce the appropriate corrections for the residues. We thus obtain.

$$(2{\cdot}23) \qquad U^k(w)=\frac{\Gamma(1+k)}{2\pi i\ w^k}\int_{-b-i\infty}^{-b+i\infty}\frac{g(s)}{s^{1+k}}e^{ws}\,ds+\frac{\Gamma(1+k)}{w^k}(R^*+\sum R_l),$$

where R^* is the residue of the integrand at the origin and R_l a typical residue at a pole (2·18).

The residue R^* is the coefficient of s^k in $g(s)\,e^{ws}$, which is a polynomial of degree k in w whose leading term is

$$\frac{w^k}{k!}\,g(0).$$

The series $\sum R_l$ is

$$(2{\cdot}24) \qquad \sum\left\{\left(l+\tfrac{1}{2}\right)\frac{\pi i}{\log\omega}\right\}^{-1-k}\exp\left\{\left(l+\tfrac{1}{2}\right)\frac{w\pi i}{\log\omega}\right\}G_l,$$

where G_l is the residue of $g(s)$. Since $g(s)$ is of finite order, this series converges, absolutely, and uniformly in w, when k is sufficiently large, and is a periodic function of w, with period $4\log\omega$. Finally the integral in (2·23) is, for the same reason, the product of w^{-k} by an absolutely and uniformly convergent integral. It follows that

$$U^k(w)=g(0)+O(w^{-1})+O(w^{-k})=g(0)+o(1).$$

so that U is summable, to sum $g(0)$, by Riesz's means of sufficiently high order. So far our argument demands no unproved assumption. If we now assume the

[1] Hardy and Riesz, 11, p. 50.
[2] Hardy and Riesz, 11, p. 14.
[3] Hardy and Littlewood, 3, p. 31.

truth of clause (i) of Theorem 1, it follows, by the 'convexity theorem' for Rieszian means,[1] that U is summable, to sum $g(0)$, by means of any positive order.

2·3. We have thus (subject to our provisional assumption) proved clause (iii) of the theorem. To prove clause (ii) we consider Riesz's 'arithmetic' means, known to be equivalent to Cesàro's.[2] The arithmetic mean of U, of order k, is [3]

$$(2\cdot31)\quad U^{(k)}(w)=w^{-k}\sum_{n\leq w} u_n(w-n)^k,$$

and [4]

$$(2\cdot32)\quad U^{(k)}(w)=\frac{\Gamma(1+k)}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\frac{\Gamma(s)}{\Gamma(s+1+k)}\, w^s\, ds,$$

if $w>0$, $c>0$. The characteristic difference between this formula and (2·22) lies in the absence of the factor w^{-k} on the righthand side.

It is plain that we may transform (2·32) as we transformed (2·22). But there will be no factor w^{-k} multiplying the series which corresponds to the series (2·24) and which is, like that series, periodic in w. It will follow that $U^{(k)}(w)$ oscillates finitely for sufficiently large k, when $w\longrightarrow\infty$, and this will prove (ii).

It is important to observe that we have proved incidentally, and without assuming the truth of (i), that the arithmetic means (and therefore the Cesàro means) of sufficiently high order are bounded. As this observation plays an essential part in the proof of (i), we state it formally in a lemma.

Lemma 3. *The Cesàro means of U, of sufficiently high order, are bounded.*

Proof of Theorem I (i).

3·1. We pass to the proof of (i). It now becomes necessary to take account of the properties of the continued fraction for θ.

[1] See Riesz, 15. Particular cases of the theorem had been proved before by ourselves and other writers.

[2] The only complete proof of the equivalence is that given (after Riesz) by Hobson, 13, pp. 90-98.

[3] Hardy and Riesz, 11, p. 23.

[4] Hardy and Riesz, 11, p. 51.

Throughout what follows O and o refer to the limit process $n \longrightarrow \infty$ or $s \longrightarrow \infty$;[1] the constants implied by the O's depend on θ (*i.e.* on a) only. Also $A = A(\theta)$ denotes generally a positive number depending only on θ, $\zeta = \zeta(\theta)$ a number between 0 and 1 depending only on θ.

We suppose that θ is defined by (2·15), and that

$$C_0 = \frac{p_0}{q_0} = \frac{0}{1},\ C_1 = \frac{p_1}{q_1} = \frac{1}{2a},\ \ldots C_s = \frac{p_s}{q_s},\ \ldots$$

are the convergents to θ. Then $p_s = q_{s-1}$, and p_s and q_s are of opposite parity. We denote by $2a'$ the complete quotient corresponding to the partial quotient $2a$, so that

$$2a' = 2a + \theta = \sqrt{(a^2+1)} + a,$$

and write

$$q'_{s+1} = 2a'q_s + q_{s-1}.$$

Then

(3·11) $$q_{s+1} < Aq_s,\ q_{s+1} > (1+A)q_s,\ q'_{s+1} > (1+A)q_{s+1}.$$

It is familiar that

(3·12) $$C_{s+1} - C_s = \frac{(-1)^s}{q_s q_{s+1}},\ \theta - C_s = \frac{(-1)^s}{q_s q'_{s+1}}$$

If ν is an integer less than q_s, we have [2]

(3·13) $$|\nu\theta - i| \geqq |q_{s-1}\theta - p_{s-1}| = \frac{1}{q_s'}$$

for all integers i. Also

(3·14) $$\operatorname{cosec} \nu\theta\pi = O(\nu)$$

for all ν, and

(3·15) $$|\operatorname{cosec} \nu\theta\pi| > A\nu$$

for an infinity of ν (*e.g.* $\nu = q_s$).

3·2. We shall make repeated use of the following lemma, the form of which was suggested to us by Mr. E. C. Titchmarsh.

[1] s is a positive integer; the complex s of §2 does not appear again.
[2] Perron, 14, p. 52.

Lemma 4. *Suppose that $s>1$, that $\nu<q_{s+1}$, and that ν is not a multiple of q_s. Then there is a positive $\zeta(\theta)$ less than 1 such that*

$$1-\zeta<\left|\frac{\sin \nu\theta\pi}{\sin \nu C_s\pi}\right|<1+\zeta \tag{3·21}$$

for all ν for which either $\nu\theta$ or νC_s differs from an integer by less then $\frac{1}{4}$. And for all ν

$$\operatorname{cosec} \nu\theta\pi=O(|\operatorname{cosec} \nu C_s\pi|). \tag{3·22}$$

Write $\nu=rq_s+\mu$, where $r=[\nu/q_s]$, so that $0<\mu<q_s$ if $r<2a$ and $0<\mu<q_{s-1}$ if $r=2a$; and write

$$\nu C_s=\xi_\mu+f_\mu,\ \nu\theta=\xi'_\mu+f'_\mu,$$

where ξ_μ and ξ'_μ are integers and $|f_\mu|$ and $|f'_\mu|$ do not exceed $\frac{1}{2}$.[1] Then

$$f_\mu-f'_\mu=\xi'_\mu-\xi_\mu+\frac{(-1)^{s+1}\nu}{q_s q'_{s+1}}.$$

The lefthand side is *ex hypothesi* numerically less than $\frac{3}{4}$ and the last term on the righthand side is numerically less than $1/q_s<\frac{1}{4}$. It follows that $\xi_\mu'=\xi_\mu$ and that

$$|f_\mu-f_\mu'|<\frac{q_{s+1}}{q_s q'_{s+1}}<\frac{1}{(1+A)q_s}.$$

But $|f_\mu|\geq 1/q_s$. Hence f_μ and $_\mu$ have the same sign and

$$1-\zeta<f_\mu'/f_\mu<1+\zeta,$$

a result plainly equivalent to (3·21). As regards (3·22), this follows from (3·21) if either f_μ or f'_μ is less than $\frac{1}{4}$, and is trivial if neither is less than $\frac{1}{4}$.

[1] $|f_\mu|$ may be $\frac{1}{2}$, in which case there is ambiguity; we may agree then to take f_μ positive.

3·3. **Lemma 5.** $\sum\limits_{1}^{q_s-1} \dfrac{1}{\sin^2 \nu C_s \pi} = O(q_s^2)$

When ν varies from 1 to q_s-1, f_μ assumes, each once, the values

$$\pm\lambda/q_s \ (\lambda=1, 2, \ldots\ldots, \lambda \leq \tfrac{1}{2} q_s)$$

the value $\frac{1}{2}q_s$ occurring, if at all, with one sign only. Hence

$$\Sigma \operatorname{cosec}^2 \nu C_s\pi = O\left\{q_s^2 \left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)\right\} = O(q_s^2),$$

which proves the lemma.

Lemma 6 [1]: $$W_n(\theta) = \sum_1^n \frac{1}{\sin^2 \nu\theta\pi} = O(n^2).$$

Since q_{s+1}/q_s is bounded, it is enough to prove this when $n=q_s$. The term $n=q_s$ contributes $O(q_s^2)$. The remainder, by Lemmas 4 and 5, contribute

$$O(\Sigma \operatorname{cosec}^2 \nu C_s\pi) = O(q_s^2)$$

3·4. **Lemma 7.** *If p and q are coprime integers of opposite parity, and*

$$\text{(3·41)} \qquad B(p, q) = \sum_{\nu=1}^{q-1} (-1)^\nu \operatorname{cosec} \frac{\nu p\pi}{q},$$

then

$$\text{(3·42)} \quad \frac{1}{q}B(p, q) + \frac{1}{p} B(q, p) = \frac{2}{\pi}\int_{-\infty}^{\infty} \left\{ \frac{x^{p+q-1}}{(x^{2p}-1)(x^{2q}-1)} - \frac{1}{4pq}\frac{1}{(x-1)^2} - \frac{1}{4pq} \frac{1}{(x+1)^2} \right\} dx.$$

It is easily verified that the integrand is bounded for $x=1$ and $x=-1$. It follows from Cauchy's Theorem that the value of the integral is $4i$ times the sum of the residues of the integrand at poles above the real axis. Calculating these residues, we obtain (3·42).

Lemma 8. *If $C_s = p_s/q_s$ is a convergent to θ, then*

$$\text{(3·43)} \qquad B(q_s) = B(p_s, q_s) = O(q_s).$$

[1] This lemma is not actually used, and we include it because it is interesting in itself.

We denote the integral in (3·42) by $J(p, q)$. Making some obvious elementary transformations we obtain

$$J(p, q)=\frac{1}{2q}\int_0^\infty\left\{\frac{1}{\sinh w \sinh \phi w}-\frac{1}{4pq}\left[\frac{1}{\cosh^2(w/2q)}+\frac{1}{\sinh^2(w/2q)}\right]\right\}dw,$$

$$=\frac{1}{2q}\int_0^\infty\left\{\left[\frac{1}{\sinh w \sinh \phi w}-\frac{1}{\phi w^2}\right]-\frac{1}{4pq}\left[\frac{1}{\cosh^2(w/2q)}+\frac{1}{\sinh^2(w/2q)}-\left(\frac{2q}{w}\right)^2\right]\right\}dw,$$

where $\phi=p/q$. The integral here is

$$\int_0^\infty\left(\frac{1}{\sinh w \sinh \phi w}-\frac{1}{\phi w^2}\right)dw-\frac{1}{2\phi q}\int_0^\infty\frac{dw}{\cosh^2 w}$$

$$+\frac{1}{2\phi q}\int_0^\infty\left(1-\frac{w^2}{\sinh^2 w}\right)\frac{dw}{w^2}=\int^\infty\left(\frac{1}{\sinh w \sinh \theta w}-\frac{1}{\theta w^2}\right)dw+o(1)=O(1)$$

when $q\longrightarrow\infty$, $p/q\longrightarrow\theta$. It therefore follows from (3·42) that

$$(3{\cdot}44)\quad \frac{1}{q_s}B(p_s, q_s)+\frac{1}{p_s}B(q_s, p_s)=O\left(\frac{1}{q_s}\right)$$

when $s\longrightarrow\infty$.

Now

$$B(q_s, p_s)=B(2aq_{s-1}+q_{s-2}, q_{s-1})=B(q_{s-2}, q_{s-1})=B(p_{s-1}, q_{s-1}),$$

and so, from (3·44),

$$B(p_s, q_s)=O(1)+O\left(\frac{q_s}{q_{s-1}}\right)B(p_{s-1}, q_{s-1})$$

$$=O(1)+O\left(\frac{q_s}{q_{s-1}}\right)+O\left(\frac{q_s}{q_{s-2}}\right)+\ldots\ldots,$$

on repeating the argument. Since q_s increases more rapidly than a geometrical progression, this is $O(q_s)$.

We may observe that the results of Lemmas 6 and 8 are in fact true for any θ of Θ and in particular any quadratic θ. Only slight changes are required in the proof of Lemma 6, or in that of Lemma 8 when all the partial quotients of θ are even. But in the general case the proof of Lemma 8 becomes more complicated because (as in the extension of Lemma 2, referred to at the end of § 2·1) it is then necessary to consider simultaneously sums of two slightly different types.

3·5. **Lemma 9.** $V_{q_s}(\theta)=\sum_1^{q_s}\frac{(-1)^\nu}{\sin\nu\theta\pi}=O(q_s)$.

The contribution of the term $\nu=q_s$ is $O(q_s)$. If $0<\nu<q_s$, we have

$$\sin\nu\theta\pi-\sin\nu C_s\pi=O\left(\left|\sin\tfrac{1}{2}\nu\pi(\theta-C_s)\right|\right)=O\left(\frac{1}{q_s}\right),$$

so that

$$V_{q_s}(\theta)-B(p_s,q_s)=O(q_s)+O\left\{\frac{1}{q_s}\sum_1^{q_s-1}\frac{1}{|\sin\nu\theta\pi\sin\nu C_s\pi|}\right\}$$

$$=O(q_s)+O\left(\frac{1}{q_s}\sum_1^{q_s-1}\frac{1}{\sin^2\nu C_s\pi}\right)=O(q_s),$$

by Lemmas 4 and 5. The result now follows from Lemma 8.

Lemma 10. $$V_n(\theta)=\sum_1^n\frac{(-1)^\nu}{\sin\nu\theta\pi}=O(n).$$

This is the principal lemma. It has been proved (Lemma 9) when $n=q_s$. If $q_s<n<q_{s+1}$, we can write

$$n=bq_s+n_1,$$

where $1\leq b\leq 2a$ and $0\leq n_1<q_s$ if $b<2a$, $0\leq n_1<q_{s-1}$ if $b=2a$ (and also $n_1>0$ if $b=1$). It is enough to prove

$$V_n=O(q_s)\pm V_{n_1}, \tag{3·51}$$

since then, repeating the argument, we obtain

$$V_n=O(q_s)+O(q_{s-1})+\dots=O(q_s)=O(n).$$

We write

$$(3\cdot 52)\qquad V_n=\sum_{r=0}^{b-1}\sum_{rq_s+1}^{(r+1)q_s}+\sum_{bq_s+1}^{bq_s+n_1}=\sum_{r=0}^{b-1}V_{n,r}+V'_{n_1},$$

say. We first consider $V_{n,r}$. We can omit the term $\nu=(r+1)q_s$ with error $O(q_s)$: in the remaining terms $\nu=rq_s+\mu$, where $0<\mu<q_s$. For such ν we have

$$\operatorname{cosec}\nu\theta\pi-\operatorname{cosec}\nu C_s\pi=O\left(\left|\sin\frac{\nu\pi}{2q_sq'_{s+1}}.\operatorname{cosec}\nu\theta\pi.\operatorname{cosec}\nu C_s\pi\right|\right)$$

$$=O\left(\frac{1}{q_s}\operatorname{cosec}^2\nu C_s\pi\right),$$

by Lemma 4. Hence

$$(3\cdot 53)\quad V_{n,r}=O(q_s)+O\left(\frac{1}{q_s}\sum_{\mu=1}^{q_s-1}\frac{1}{\sin^2\mu C_s\pi}\right)$$

$$+(-1)^{r(q_s+p_s)}\sum_{\mu=1}^{q_s-1}\frac{-(1)^{\mu}}{\sin\mu C_s\pi}=O(q_s),$$

by Lemmas 5 and 8. It follows from (3·52) and (3·53) that

$$(3\cdot 54)\qquad V_n=O(q_s)+V'_{n_1}.$$

In the last term on the right we have $\nu=bq_s+\mu$, where $0<\mu\leqq n_1$, and

$$\nu\theta=(bq_s+\mu)\theta=bp_s+\mu C_s+O\left(\frac{1}{q_s}\right)=bp_s+\mu\theta+O\left(\frac{1}{q_s}\right),$$

$$\sin\nu\theta\pi-(-1)^{bp_s}\sin\mu\theta\pi=O\left(\frac{1}{q_s}\right),$$

$$\operatorname{cosec}\nu\theta\pi-(-1)^{bp_s}\operatorname{cosec}\mu\theta\pi=O\left(\frac{1}{q_s}\,|\operatorname{cosec}\nu\theta\pi|\;|\operatorname{cosec}\mu\theta\pi|\right)$$

$$=O\left(\frac{1}{q_s}\operatorname{cosec}^2\mu C_s\pi\right),$$

by Lemma 4. Hence

$$(3\cdot55)\quad V'_{n_1}=O\left(\frac{1}{q_\nu}\sum_{\mu=1}^{q_\nu-1}\frac{1}{\sin^2\mu C_\nu\pi}\right)+(-1)^{b(p_\nu+q_\nu)}\sum_{\mu=1}^{n_1}\frac{(-1)^\mu}{\sin\mu\theta\pi}$$

$$=O(q_\nu)\pm V_{n_1},$$

by Lemma 5. From (3·54) and (3·55) we deduce (3·51) and so the lemma.

3·6. **Lemma 11.** *If some Cesàro mean of* U *is bounded, and*

$$u_1+2u_2+\ldots+nu_n=O(n),$$

then U_n *is bounded.*

We write for convenience $u_0=0$. Then

$$U_n=\frac{U_0+U_1+\ldots+U_n}{n+1}+\frac{u_1+2u_2+\ldots nu_n}{n+1}.$$

Hence the difference between the Cesàro means of U, of orders 0 and 1, is bounded. If follows that the difference between the means of orders k and $k+1$ is bounded, and this proves the Lemma.

3·7. We can now complete the proof of the theorem. By Lemma 3, the Cesàro means of U, of sufficiently high order, are bounded. By Lemma 10, the second hypothesis of Lemma 11 is satisfied. It follows from Lemma 11 that U oscillates finitely.

Further Results.

4·1. We add a few remarks about the behaviour of our series for general quadratic θ or general θ of class Θ, without attempting to justify all that we say in detail.

The arguments of §§ 3·2—3·5 are extensible, without new difficulties of principle, to any θ of Θ; the results

$$(4\cdot11)\qquad V_n=O(n),\qquad W_n=O(n^2)$$

of Lemmas 10 and 6 hold for all such θ. When the quotients of θ are all even, but little elaboration is needed; in the general case there is the complication alluded to in § 3·4. The results (4·11) are obviously 'best possible' for any θ.

The situation in regard to U is a little more complex. It is easy to prove that

$$U_n = O(\log n) \tag{4·12}$$

for all θ of Θ. For some θ, as we have seen, more is true (U_n being bounded), but (4·12) is the most that is true even for all *quadratic* θ. For quadratic θ, in fact, there are only two possibilities, *viz.* $U_n = O(1)$ and

$$U_n = A \log n + O(1) \qquad (A \neq 0). \tag{4·13}$$

It is interesting to give an example of the second case.

Recurring to the analysis of §2·1, let

$$\theta = \frac{1}{2a+}\frac{1}{2b+}\frac{1}{2a+}\frac{1}{2b+}\cdots$$

where $a \neq b$. Then

$$\theta = \frac{1}{2a+\theta_1},\ \theta_1 = \frac{1}{2b+\theta},\ \ldots$$

where

$$\theta = \sqrt{\left(\frac{b}{a}\right)}\{\sqrt{(ab+1)} - \sqrt{(ab)}\},\ \theta_1 = \sqrt{\left(\frac{a}{b}\right)}\{\sqrt{(ab+1)} - \sqrt{(ab)}\}.$$

If

$$h(s) = \sum \frac{(-1)^n}{n^{s+1} \sin n\theta\pi},$$

we obtain, by analysis similar to that of §2,

$$h(s) = \frac{(2\pi)^{s+1}}{\Gamma(1+s)\cos\frac{1}{2}s\pi}\,\frac{1}{1-(\theta\theta_1)^s}\left\{\zeta_2\left(-s, \frac{1+\theta}{2}, 1, \theta\right)\right.$$
$$\left. -\theta^s \zeta_2\left(-s, \frac{1+\theta_1}{2}, 1, \theta_1\right)\right\}.$$

The arithmetic mean of U of order k is given by (2·32), with $h(s)$ in place of $g(s)$. The difference is that there is now a double pole at the origin. Otherwise we may argue as in § 2·3, and we find that $U^{(k)}(w)$ is, for sufficiently large k, of the form

$$A \log w + O(1),$$

where

$$A = -\frac{(b-a)\pi}{12 \log\{\sqrt{(ab+1)} - \sqrt{(ab)}\}}$$

It follows (substantially, as in the proof of Lemma 11) that U_n itself is of the same form. If in particular $a=2$, $b=1$, we find

$$\theta=\tfrac{1}{2}\sqrt{6}-1,\quad A=\frac{\pi}{12\log(\sqrt{3}-\sqrt{2})},$$

$$\sum_1^n \frac{1}{\nu\sin\frac{1}{2}\nu\pi\sqrt{6}}=\frac{\pi\log n}{12\log(\sqrt{3}-\sqrt{2})}+O(1).$$

We add in conclusion that

$$\sum_1^n \frac{1}{|\sin\nu\theta\pi|}=O(n\log n),\quad \sum_1^n \frac{1}{\nu|\sin\nu\theta\pi|}=O((\log n)^2)$$

for all θ of Θ, and that these results also are the best possible of their kind.[1]

Bibliographical Note.

Our own relevant writings have been published in various journals under the general title 'Some Problems of Diophantine Approximation.' We may refer in particular to

1. Some Problems of Diophantine Approximation: *Proc. Fifth International Congress* (1912), 223-229 (a preliminary sketch).

2. The Trigonometrical Series associated with the Elliptic Theta-functions: *Acta. Math.*, 37 (1914), 193-238 [an additional note in *Proc. Camb. Phil. Soc.*, 21 (1923), 1-5].

3. The Lattice Points of a Right-angled Triangle: *Proc. London Math. Soc.* (2), 20 (1921), 15-36.

4. The Lattice Points of a Right-angled Triangle (second memoir): *Hamburg Math. Abhandlungen*, 1 (1922), 212-249.

5. The Analytic Character of the Sum of a Dirichlet's Series considered by Hecke, *ibid*, 3 (1923), 57-68.

6. The Analytic Properties of Certain Dirichlet's Series associated with the Distribution of Numbers to Modulus Unity: *Trans. Camb. Phil. Soc.*, 22 (1923), 519-533.

[1] See Hardy and Littlewood, 4, p. 216.

The other books and memoirs referred to in the text are—

7. E. W. Barnes: A Memoir on the Double Gamma-function: *Phil. Trans. Royal Soc.* (A), 196 (1901), 265-387.

8. H. Behnke: Über die Verteilung von Irrationalitäten mod. 1: *Hamburg Math. Abhandlungen*, 1 (1922), 252-267.

9. H. Behnke: Zur Theorie der diophantischen Approximationen: *ibid*, 3 (1924), 261-318 and 4 (1926), 33-46.

10. R. Cooper: The Behaviour of Certain Series associated with limiting Cases of Elliptic Theta-functions: *Proc. London Math. Soc.*, 27 (1928), 410-426.

11. G. H. Hardy and M. Riesz: The General Theory of Dirichlet's Series: *Camb. Math. Tracts*, 18 (1915).

12. E. Hecke: Über analytische Functionen und die Verteilung von Zahlen mod. Eins: *Hamburg Math. Abhandlungen*, 1 (1922), 54-76.

13. E. W. Hobson: The Theory of Functions of a Real Variable, Vol. 2, second edition, 1926.

14. O. Perron: Die Lehre von den Kettenbrüchen, 1913.

15. M. Riesz: Sur un théorème de la moyenne et ses applications: *Acta Univ. Hungaricae*, 1 (1923), 3-15

Further references to the older literature connected with series of the type considered here will be found in a paper by Hardy, 'On Certain Series of Discontinuous Functions connected with the Modular Functions,' *Quarterly Journal*, 36 (1905), 93-123.

CORRECTIONS

p. 254, footnote (2). Read: $e^{n^2\theta\pi i}n^{-s}$.
p. 258, line 4 from below. Read: Hence f_μ and f'_μ have . . .

COMMENTS

As mentioned in the Introduction, it would be desirable to have a simpler and more direct proof of the results of this paper.

NOTES ON THE THEORY OF SERIES (XXIV): A CURIOUS POWER-SERIES

BY G. H. HARDY AND J. E. LITTLEWOOD

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1. This note originates from a question put to us by Mr W. R. Dean concerning series of the type

$$(1\cdot 1) \qquad \Sigma a_n \frac{x^n}{\sin\theta\pi \sin 2\theta\pi \ldots \sin n\theta\pi},$$

where θ is irrational*. Series of the type

$$(1\cdot 2) \qquad \Sigma a_n \frac{x^n}{\sin n\theta\pi}$$

are familiar: it is well known, for example, that

$$(1\cdot 3) \qquad \Sigma \frac{x^n}{\sin n\theta\pi}$$

may have any radius of convergence from 0 to 1 inclusive, according to the arithmetic nature of θ. It is natural to ask how this radius is connected with that of

$$(1\cdot 4) \qquad \Sigma \frac{x^n}{\sin\theta\pi \sin 2\theta\pi \ldots \sin n\theta\pi},$$

and, more generally, how those of (1·1) and (1·2) are connected.

We show here that the radius of convergence of (1·1) is usually half that of (1·2). We prove this by two methods each of which has points of interest. The first, which is rather oddly indirect, depends on an algebraical identity, the second on arguments of a type more usual in the theory of Diophantine approximation.

Actually we shall use not (1·3) but

$$(1\cdot 5) \qquad \Sigma \frac{x^n}{n \sin n\theta\pi},$$

which has obviously the same radius of convergence.

* See *Proc. Cambridge Phil. Soc.* 42 (1946), 24. The actual series encountered by Dean was

$$\Sigma \frac{\cos\theta\pi \cos 2\theta\pi \ldots \cos n\theta\pi}{\sin\theta\pi \sin 2\theta\pi \ldots \sin (n+1)\theta\pi} \frac{x^n}{n!}.$$

2. We begin by proving

THEOREM 1. *If the radius of convergence of* (1·3) *is* ρ, *where* $0 \leqq \rho \leqq 1$, *then that of* (1·4) *is* $\frac{1}{2}\rho$.

We show first that if $|x| < 1$, $|q| < 1$, and

$$(2\cdot1)\quad f(x,q) = \frac{x}{1-q} + \frac{x^2}{2(1-q^2)} + \dots, \qquad (2\cdot2)\quad F(x,q) = 1 + \frac{x}{1-q} + \frac{x^2}{(1-q)(1-q^2)} + \dots,$$

then

$$(2\cdot3)\qquad F(x,q) = e^{f(x,q)}.$$

In fact

$$(2\cdot4)\quad f(x,q) = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)} = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{m=1}^{\infty} q^{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(xq^m)^n}{n} = \sum_{m=1}^{\infty} \log \frac{1}{1-xq^m},$$

and so

$$(2\cdot5)\qquad e^{f(x,q)} = \frac{1}{(1-x)(1-qx)(1-q^2x)\dots} = G(x,q),$$

say. Also $G(x,qx) = (1-x)\,G(x,q)$ and, if

$$G(x,q) = 1 + c_1x + c_2x^2 + \dots,$$

then
$$1 + c_1qx + c_2q^2x^2 + \dots = (1-x)(1 + c_1x + c_2x^2 + \dots).$$

Equating coefficients, we find that $c_n(1-q^n) = c_{n-1}$ and so $G(x,q) = F(x,q)$*.

If (1·5) diverges for all x, then (1·4) plainly does the same. We may therefore suppose $\rho > 0$; and we shall show that (2·3) is then still true when $|x| < \rho$ and

$$q = q_0 = e^{2\pi i\theta},$$

so that $|q| = 1$.

The series
$$f(x,q_0) = \Sigma \frac{x^n}{n(1-q_0^n)} = \tfrac{1}{2}i\,\Sigma \frac{(xe^{-\pi i\theta})^n}{n \sin n\pi\theta}$$

has also radius ρ. If
$$q = \zeta q_0 = \zeta e^{2\pi i\theta} \quad (0 \leqq \zeta \leqq 1)$$

then
$$\left|\frac{1-q_0^n}{1-q^n}\right| \leqq A,$$

where A is an absolute constant†. Since $f(x,q_0)$ is uniformly convergent for

$$|x| \leqq \rho - \delta < \rho,$$

it follows that $f(x,q)$ is uniformly convergent for $|x| \leqq \rho - \delta$, $0 \leqq \zeta \leqq 1$; and that

$$f(x,q) \to f(x,q_0)$$

when $\zeta \to 1$, uniformly for $|x| \leqq \rho - \delta$. Hence also

$$F(x,q) = e^{f(x,q)} \to e^{f(x,q_0)}$$

with the same uniformity.

* The identity (2·3) is naturally not a new one, though we cannot refer to a quite explicit statement of it. The expression of $F(x,q)$ as an infinite product goes back to Euler (see for example Macmahon, *Combinatory analysis*, vol. 2, 2), and the transformation in (2·4) is one of a type familiar in the theory of elliptic functions.

† For
$$\left|\frac{1-e^{2n\pi i\theta}}{1-\zeta^n e^{2n\pi i\theta}}\right|^2 = \frac{4\sin^2 n\pi\theta}{(1-\zeta^n)^2 + 4\zeta^n \sin^2 n\pi\theta},$$

which does not exceed 16 if $\zeta^n \leqq \frac{1}{2}$ or 2 if $\zeta^n \geqq \frac{1}{2}$. Actually the best value of A is 2.

It now follows from a familiar theorem of Weierstrass* that, if

$$F(x,q) = \Sigma C_n(q)\,x^n, \quad e^{f(x,q_0)} = \Sigma C_n(q_0)\,x^n,$$

then $C_n(q) \to C_n(q_0)$ when $\zeta \to 1$; so that

$$C_n(q_0) = \lim C_n(q) = \lim \frac{1}{(1-q)\ldots(1-q^n)} = \frac{1}{(1-q_0)\ldots(1-q_0^n)}.$$

Thus the functions in (2·3) remain regular, and the identity valid, for $|x| < \rho$, when q is replaced by q_0. In particular, the radius of convergence of $F(x,q_0)$ is at least ρ, and therefore that of (1·4) is at least $\frac{1}{2}\rho$.

3. We have still to prove that the radius of convergence of $F(x,q_0)$ does not exceed ρ. From this point on we drop the suffix in q_0 and suppose that $q = e^{2\pi i\theta}$.

If $F(x,q)$ has a radius σ greater than ρ, and $\rho < \tau < \sigma$, then $F(x,q)$ is regular, and has at most a finite number of zeros, in $|x| \leqq \tau$; and hence

$$f(x,q) = \sum_{l=1}^{k} p_l \log(a_l - x) + \phi(x), \tag{3·1}$$

where $|a_l| \leqq \tau$, the p_l are positive integers, and $\phi(x)$ is regular for $|x| \leqq \tau$. If $a_1, a_2, \ldots, a_s$ are the a_l with smallest modulus, say ϖ, and

$$a_l = \varpi e^{-i\alpha_l} \quad (1 \leqq l \leqq s,\ 0 \leqq \alpha_l < 2\pi,\ \alpha_l \neq \alpha_{l'} \text{ if } l \neq l'),$$

then
$$f(x,q) = \sum_{l=1}^{s} p_l \log(\varpi e^{-i\alpha_l} - x) + \psi(x),$$

where $\psi(x)$ is regular for $|x| \leqq \varpi$. It follows on equating coefficients that

$$\frac{\varpi^n}{1-q^n} = -\sum_{l=1}^{s} p_l e^{ni\alpha_l} + O(\delta^n),$$

where $0 < \delta < 1$, for $n > 1$; and so that

$$\frac{\varpi^{2n}}{4\sin^2 n\theta\pi} = \left|\frac{\varpi^n}{1-q^n}\right|^2 = P + 2Q + O(\delta^n), \tag{3·2}$$

where
$$P = \sum_{l=1}^{s} p_l^2 > 0, \quad Q = \sum_{l \neq l'} p_l p_l' \cos n(\alpha_l - \alpha_{l'}).$$

It is now easy to derive a contradiction. It is plain that (3·2) is impossible if $\varpi \geqq 1$, since then the right-hand side is bounded and the left is not. We may therefore suppose $\varpi < 1$. Summing (3·2) from $n = 1$ to $n = N$, and observing that $\alpha_l \neq \alpha_{l'}$, we obtain

$$S_N = \sum_{n=1}^{N} \frac{\varpi^n}{\sin^2 n\theta\pi} = 4PN + O(1). \tag{3·3}$$

Hence, first,
$$\frac{\varpi^n}{\sin^2 n\theta\pi} = 4P + O(1) < H,$$

say, for all n. Secondly, since θ is irrational, the numbers $n\theta$, taken mod 1, are uniformly

* See for example Harkness and Morley, *Introduction to the theory of analytic functions*, 134–6. The theorem is stated there in the form in which the coefficients depend on an integral parameter m which tends to infinity ('Weierstrass's double series theorem').

Alternatively, we may appeal to the representation of $C_n(q)$ as an integral by means of Cauchy's theorem.

distributed in $(0, 1)$*. Hence, if N is large, and η independent of n, the number of n in $(1, N)$ for which $\sin^2 n\theta\pi < \eta$ is less than $N\zeta(\eta)$, where $\zeta(\eta) \to 0$ with η. Thus

$$S_N < HN\zeta(\eta) + \sum_{n=1}^{N} \frac{\varpi^n}{\eta} < HN\zeta(\eta) + \frac{\varpi}{(1-\varpi)\eta},$$
$$\overline{\lim_{N\to\infty}}\, N^{-1}S_N \leqq H\zeta(\eta) < \epsilon$$

for small η, in contradiction to (3·3).

It follows that the radius of convergence of $F(x, q)$ does not exceed ρ, and this completes the proof of Theorem 1. It will be observed that this is a theorem about all irrational θ, whereas those in what follows are theorems about almost all†.

4. The radius of (1·3) is 1 for almost all θ, in fact whenever

$$\theta = a_0 + \frac{1}{a_1 + \dots}, \tag{4·1}$$

p_r/q_r is the rth convergent to θ, and

$$a_{r+1} = O(e^{\epsilon q_r}) \tag{4·2}$$

for every positive ϵ, a condition satisfied by all but very abnormal θ. It follows that the radius of convergence of (1·4) is $\frac{1}{2}$ for all such θ, so that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\operatorname{cosec}\pi\theta \operatorname{cosec} 2\pi\theta \dots \operatorname{cosec} n\pi\theta|} = 2,$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{m=1}^{n} \log|\operatorname{cosec} m\pi\theta| = \log 2, \tag{4·3}$$

for all such θ.

On the other hand it is easy to show that

$$\underline{\lim_{n\to\infty}} \frac{1}{n} \sum_{m=1}^{n} \log|\operatorname{cosec} m\pi\theta| \geqq \log 2 \tag{4·4}$$

for all irrational θ. For, if we define a function $f_\delta(x)$, with period 1, as $\log \operatorname{cosec} \pi x$ in $(\delta, 1-\delta)$ and 0 in the rest of $(0, 1)$, then $f_\delta(x)$ is Riemann integrable, and therefore‡

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f_\delta(m\theta) = \int_0^1 f_\delta(x)\,dx = \int_\delta^{1-\delta} \log \operatorname{cosec} \pi x\,dx.$$

Hence
$$\underline{\lim} \frac{1}{n} \sum_{m=1}^{n} \log|\operatorname{cosec} m\pi\theta| \geqq \lim \frac{1}{n} \sum_{m=1}^{n} f_\delta(m\theta)$$
$$= \int_\delta^{1-\delta} \log \operatorname{cosec} \pi x\,dx > \int_0^1 \log \operatorname{cosec} \pi x\,dx - \epsilon = \log 2 - \epsilon,$$

for any ϵ and a corresponding δ.

Combining our results, we obtain

THEOREM 2. *If θ satisfies* (4·2), *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \log|\operatorname{cosec} m\pi\theta| = \log 2; \tag{4·5}$$

and this is true for almost all θ.

* By a well-known theorem of Bohl, Sierpinski and Weyl: see Weyl, *Math. Annalen*, 77 (1916), 313–52 (314), or Hardy and Wright, *Introduction to the theory of numbers*, 378–81.

† Except Theorem 5, which refers to a special class of θ.

‡ By another theorem of Weyl, substantially equivalent to the theorem of uniform distribution already quoted.

It is also plain that, when (4·5) is true, the result of Theorem 1 may be extended to the more general series (1·1) and (1·2).

THEOREM 3. *The radius of convergence of* (1·1) *is half that of* (1·2) *for almost all* θ.

5. It is natural to ask for a more direct proof of Theorem 2, and for more general results of the type

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} f(m\theta)=\int_0^1 f(x)\,dx, \tag{5·1}$$

where $f(x)$ is a function with infinities at $x=0$ and $x=1$. We suppose throughout what follows that $f(x)$ has period 1, so that

$$f(m\theta)=f(\overline{m\theta}),$$

where $\bar{x}$ is the difference of x from the nearest integer, that it is Riemann integrable in $(\delta, 1-\delta)$ for every positive δ, and that it increases steadily to infinity when $x\to 0$ and $x\to 1$. We prove

THEOREM 4. *If*

$$\int_0^1 f(x)\left(\log^2\frac{1}{x}+\log^2\frac{1}{1-x}\right)dx<\infty \tag{5·2}$$

then (4·1) *is true for almost all* θ.

THEOREM 5. *If the partial quotients* a_r *of* θ *are bounded, then* (5·1) *is true whenever the integral on the right is finite.*

We might replace the logarithmic factor in (5·2) by

$$\psi\left(\frac{1}{x}\right)+\psi\left(\frac{1}{1-x}\right),$$

where ψ is any positive increasing function for which

$$\int^{\infty}\frac{du}{u\psi(u)}<\infty.$$

We use two lemmas, in which $N>0$, $h\geqq 2$, and the results are asserted for almost all θ, viz.

(*a*) *the number of n for which*

$$N\leqq n\leqq N+h,\quad |\overline{n\theta}|<h^{-1},$$

is $O(\log^2 h)$, *uniformly in* N;

(*b*) *the number of m for which*

$$1\leqq m\leqq n,\quad |\overline{n\theta}|<h^{-1},$$

is $O(n\log^2 h/h)$.

It is plain that (*b*) is a corollary of (*a*). To prove (*a*) we observe that, if it is false, there are, for large h, numbers n_1 and n_2 such that

$$N\leqq n_1<n_2\leqq N+h,\quad \mu=n_2-n_1<\frac{h}{\log^2 h},\quad \overline{\mu\theta}=O\left(\frac{1}{h}\right).$$

But then
$$h>\mu\log^2 h\geqq \mu\log^2\mu,\quad \overline{\mu\theta}=O\left(\frac{1}{\mu\log^2\mu}\right),$$

and μ is large with h, since $\overline{\mu\theta}=O(h^{-1})$ is small. It is well known that this is false for almost all θ*.

* See for example, Hardy and Wright, 168.

If now $f_\delta(x)$ is periodic, is $f(x)$ in $(\delta, 1-\delta)$, and is 0 in the rest of $(0, 1)$, then

$$S = \frac{1}{n}\sum_1^n f(m\theta) = \frac{1}{n}\sum_1^n f_\delta(m\theta) + \frac{1}{n}\sum_1^n \{f(m\theta) - f_\delta(m\theta)\} = S_1 + S_2, \tag{5·3}$$

where
$$S_2 = \frac{1}{n}\sum_{|\overline{m\theta}|<\delta} f(m\theta) = \frac{1}{n}\sum_{k=0}^{\infty} \sum_{2^{-k-1}\delta \leqq |\overline{m\theta}| < 2^{-k}\delta} f(m\theta) = \frac{1}{n}\sum_{k=0}^{\infty} T_k,$$

say. When δ is small, $f(m\theta)$ is positive throughout S_2.

The number of m in T_k is, by (b),

$$O\left\{n\frac{\log^2(2^k/\delta)}{2^k/\delta}\right\} = O\left\{n\frac{\log^2(2^{k+1}/\delta)}{2^{k+1}/\delta}\right\},$$

and one or other of

$$f(m\theta) \leqq f(2^{-k-1}\delta), \quad f(m\theta) \leqq f(1-2^{-k-1}\delta),$$

is true for each such m, according to the sign of $\overline{m\theta}$. Hence

$$S_2 \leqq H\left\{\sum_{k=0}^{\infty}\frac{\delta}{2^{k+1}}\log^2\frac{2^{k+1}}{\delta} f\left(\frac{\delta}{2^{k+1}}\right) + \sum_{k=0}^{\infty}\frac{\delta}{2^{k+1}}\log^2\frac{2^{k+1}}{\delta} f\left(1-\frac{\delta}{2^{k+1}}\right)\right\}$$
$$\leqq H\left(\int_0^\delta f(x)\log^2\frac{1}{x}\,dx + \int_{1-\delta}^1 f(x)\log^2\frac{1}{1-x}\,dx\right),$$

for an appropriate H depending only on θ. Thus

$$0 < S_2 < \epsilon \tag{5·4}$$

for $\delta \leqq \delta_0(\epsilon)$ and sufficiently large n.

On the other hand
$$S_1 \to \int_0^1 f_\delta(x)\,dx = \int_\delta^{1-\delta} f(x)\,dx$$

when $n \to \infty$, so that

$$\left|S_1 - \int_0^1 f(x)\,dx\right| < \epsilon \tag{5·5}$$

for $\delta \leqq \delta_1(\epsilon)$ and sufficiently large n; and it follows from (5·3)–(5·5) that

$$\left|S - \int_0^1 f(x)\,dx\right| < 2\epsilon$$

for sufficiently large n.

This proves Theorem 4. The proof of Theorem 5 is similar but simpler, since we can omit the logarithmic factors throughout the argument.

6. We end with a remark about the actual series encountered by Dean. The effect of the factor $(n!)^{-1}$ is naturally to make $\rho = \infty$ for almost all θ (though it may still be 0 for sufficiently eccentric θ). If we omit this factor and consider

$$\sum \frac{\cos\theta\pi \cos 2\theta\pi \ldots \cos n\theta\pi}{\sin\theta\pi \sin 2\theta\pi \ldots \sin(n+1)\theta\pi} x^n, \tag{6·1}$$

then the odd factor in the denominator does not affect the results. The effect of the cosines is to replace $\int \log|\operatorname{cosec}\pi x|\,dx$ by $\int \log|\cot \pi x|\,dx$, $\log 2$ by 0, and $\frac{1}{2}$ by 1, so that (6·1) will have the same radius of convergence as (1·3).

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CORRECTION

p. 89. In the statement of Theorem 4, for (4.1) read (5.1).

COMMENTS

It is surprising that the problem raised should admit of such a simple answer. Apart from that, the main interest of the paper lies in the variety of types of reasoning that are shown to be relevant.

Theorem 4 is perhaps deserving of separate statement, being a useful supplement to a classical result. If $f(x)$ is periodic with period 1, and is bounded and Riemann integrable in $(0, 1)$, it is well known that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(m\theta) = \int_0^1 f(x)\,dx$$

for any irrational θ. Now suppose f is no longer bounded but increases steadily to ∞ as $x \to 0$ from the right or from the left. Then the result still holds *for almost all θ* provided that

$$\int_0^1 f(x)\left\{\psi\left(\frac{1}{x}\right)+\psi\left(\frac{1}{1-x}\right)\right\} dx$$

is finite, where $\psi(u)$ is any positive increasing function for which

$$\int^{\infty} \frac{du}{u\psi(u)}$$

converges.

2. ADDITIVE NUMBER THEORY

(*a*) Combinatory Analysis and Sums of Squares

INTRODUCTION TO PAPERS ON COMBINATORY ANALYSIS AND SUMS OF SQUARES

It is no accident that of all Hardy's papers on additive number theory, those on modular forms come first in chronological order. The general idea is to determine (either exactly or approximately) an arithmetical function $r(n)$ by means of the formula

$$r(n) = \frac{1}{2\pi i}\int f(x)x^{-n-1}\,dx,$$

where the 'generating function'

$$f(x) = \sum_{n=0}^{\infty} r(n)x^n$$

is regular in the interior of the unit circle and the integral is taken along a circle $|x| = r < 1$.

The most interesting case arises when the unit circle is the natural boundary of the function $f(x)$. In many important cases the function $f(e^{\pi i\tau})$ is a modular form in τ in the upper half-plane, so that the behaviour of $f(e^{\pi i\tau})$ as τ approaches a rational number h/k is easily determined from the behaviour of $f(e^{\pi i\tau})$ as τ approaches 0.

The first example of this type which Hardy dealt with, in collaboration with S. Ramanujan, was the partition function p_n, the number of partitions of n into positive integers without respect to the order of the terms, but with repetitions allowed.* The generating function in this case is

$$\prod_{m=1}^{\infty}(1-x^m)^{-1} = 1+\sum_{n=1}^{\infty} p_n x^n,$$

which is closely related to the well-known modular 'discriminant'

$$\Delta(\tau) = e^{\pi i\tau}\prod_{m=1}^{\infty}(1-e^{\pi i m\tau})^{24}.$$

In their first paper on the subject (1917, 4) Hardy and Ramanujan had already obtained the result

$$\log p_n \sim \pi(2n/3)^{\frac{1}{2}}.$$

Their method consisted of the application of a Tauberian theorem to $f(x)$ near $x = 1$ only. Though the method can and did lead to fairly general theorems, it could not produce results as deep as those found in their later work.

In the paper (1918, 5), for which the papers (1916, 10) and (1917, 1) are preliminary announcements,† Hardy and Ramanujan obtained a rapidly convergent asymptotic formula giving p_n with an error term $O(n^{-\frac{1}{4}})$. Twenty years later H. Rademacher (*Proc. London Math. Soc.* (2), 43 (1937), 241–54), by a slight but far-reaching modification of the proof, replaced the asymptotic expansion by a convergent series for p_n.

* J. V. Uspensky also investigated the asymptotic behaviour of p_n by means of the contour integral; see *Bull. de l'Acad. des Sciences de Russie*, (6) 14 (1920), 199–218.

† An abstract of the paper itself was published in *Proc. London Math. Soc.* (2), 16 (1917), xxii.

‡ There was also an announcement in *Proc. London Math. Soc.* (2), 17 (1918) xxii–xxiv.

The next paper (1918, 2) deals with functions $f(x)$ based on modular functions which have poles, but no essential singularities, in the upper half-plane.

The paper (1920, 10) deals with the number $r_s(n)$ of representations of n as a sum of s squares. [(1918, 10) is a preliminary announcement.‡] The relevant function is

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau},$$

which satisfies the functional equation

$$\left(\frac{\tau}{i}\right)^{\frac{1}{2}} \theta(\tau) = \theta\left(-\frac{1}{\tau}\right).$$

Hardy and Littlewood had already studied the asymptotic behaviour of $\theta(\tau)$, particularly in (1914, 3).

By the use of the formula

$$\theta(\tau)^s = \sum_{n=0}^{\infty} r_s(n) e^{\pi i n \tau}$$

and Cauchy's integral formula, Hardy obtained an exact expression for $r_s(n)$ if $5 \leqslant s \leqslant 8$ and an asymptotic expression if $s \geqslant 9$. Hardy was, of course, aware of the fact that his results for $5 \leqslant s \leqslant 8$ were not new, though he stresses quite rightly that his method deals simultaneously with even and odd values of s and presents a unified approach to the problem for all values of s, though the classical cases $s = 4$ and especially $s = 3$ present some special difficulties. On the case $s = 3$, see T. Estermann, *Proc. London Math. Soc.* (3), 9 (1959), 575–94.

The whole matter was put into a more general context by C. L. Siegel [*Annals of Math.* (2) 36 (1935), 527–606]. He proved not only that Hardy's formula holds, *mutatis mutandis*, asymptotically for the representation of a number by any definite quadratic form with rational integral coefficients, but that it holds exactly if a system of forms is considered, which has one representative from each class in a fixed genus. As the number of classes of positive definite quadratic forms of discriminant 1 is one if and only if $s \leqslant 8$, it is now apparent why Hardy's formula holds exactly for $s \leqslant 8$ only.

H. H.

NOTE

The four papers [1917, 1; 1917, 4; 1918, 2; 1918, 5] written in collaboration with Ramanujan are reprinted from *Collected Papers of S. Ramanujan* (Cambridge, 1927).

The reader is referred to Hardy's own comments given there, and to Chapters 3, 8, 9 of his *Ramanujan* (Cambridge, 1940).

ASYMPTOTIC FORMULAE IN COMBINATORY ANALYSIS.

By

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1. The researches of which I propose to give a summary here are the joint work of the distinguished Indian mathematician, Mr. S. RAMANUJAN, and myself. They are the result of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory of analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represent certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$f(x) = 1 + \sum_1^\infty p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}.$$

If we write

$$x = q^2 = e^{2\pi i \tau},$$

where the imaginary part of τ is positive, we see that $f(x)$ is

1916, 10 *Quatrième Congrès des Mathématiciens Scandinaves*, Stockholm, 1916, 45–53.

substantially the reciprocal of the modular function called by TANNERY and MOLK $h(\tau)$; that in fact

$$h(\tau)=q^{\frac{1}{12}}q_0=q^{\frac{1}{12}}\prod_1^\infty(1-q^{2n})=\frac{x^{\frac{1}{24}}}{f(x)}.$$

The theory of partitions has, from the time of EULER onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities — many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formulae, one may fairly say, there are none. So true is this, in fact, that we have been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of $p(n)$.

2. The function $p(n)$ may of course be expressed in the form of an integral

$$p(n)=\frac{1}{2\pi i}\int\frac{f(x)}{x^{n+1}}\,dx \tag{1}$$

by means of CAUCHY's Theorem, the path of integration enclosing the origin and lying entirely inside the unit circle. The idea which dominates this paper is that of obtaining asymptotic formulae for $p(n)$ by a detailed study of the integral (1). This idea is an extremely obvious one: it is the idea which has dominated nine-tenths of modern research in the analytic theory of numbers; and it may seem very strange that it should never have been applied to this particular problem before. Of this there are no doubt two explanations. The first is that the theory of partitions has received its most important developments, since its foundation by EULER, at the hands of a series of mathematicians whose interests have lain primarily in algebra. The second and more fundamental reason is to be found in the extreme complexity of the behaviour of the generating function $f(x)$ near a point of the unit circle.

It is instructive to contrast this problem with the corresponding problems which arise for the arithmetical functions $\pi(n)$, $\vartheta(n)$, $\psi(n)$, $\mu(n)$, $d(n)$, ... which have their genesis in RIEMANN's Zeta-function and the functions allied to it. In the latter problems we are dealing with functions defined by

DIRICHLET's series. The study of such functions presents difficulties far more fundamental than any which confront us in the theory of the modular functions. These difficulties, however, relate to the distribution of the zeros of the functions and their general behaviour at infinity: no difficulties whatever are occasioned by the crude singularities of the functions in the finite part of the plane. The single finite singularity of $\zeta(s)$, for example, the pole at $s=1$, is a singularity of the simplest possible character. It is this pole which gives rise to the *dominant* terms in the asymptotic formulae for the arithmetical functions associated with $\zeta(s)$. To prove such a formula rigorously is often exceedingly difficult; to determine precisely the order of the error which it involves is in many cases a problem which still defies the utmost resources of analysis. But to write down the dominant terms involves, as a rule, no difficulty more formidable than that of deforming a path of integration over a pole of the subject of integration and calculating the corresponding residue.

In the theory of partitions, on the other hand, we are dealing with functions which do not exist at all outside the unit circle. Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a 'dominant term'.

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulae of the theory of the linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behaviour of $f(x)$ near any assigned point of the unit circle.[1] It is to an appropriate use of these formulae that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

3. It is very important, in dealing with such a problem as this, to distinguish clearly the various stages to which we can progress by arguments of a progressively 'deeper' and less

[1] See G. H. HARDY and J. E. LITTLEWOOD, 'Some problems of Diophantine approximation (II: The trigonometrical series associated with the elliptic Theta-functions)', *Acta Mathematica*, vol. 37, 1914, pp. 193—238, for applications of the formulae to different but not unrelated problems.

elementary character. The earlier results are naturally (so far as the particular problem is concerned) superseded by the later. But the more elementary methods are likely to be applicable to other problems in which the more subtle analysis is impracticable.

We have attacked this particular problem by a considerable number of different methods, and cannot profess to have reached any very precise conclusions as to the possibilities of each. A detailed comparison of the results to which they lead would moreover expand this summary to a quite unreasonable length. I shall therefore content myself with referring to a brief note entitled 'Une formule asymptotique pour le nombre des partitions de n', which appeared in the *Comptes Rendus* of 2 Jan. 1917, and to a full account of our researches which is to appear in the *Proceedings of the London Mathematical Society;* [1] and pass on at once to an account of the more powerful methods which give the best results in this particular problem.

4. In order to obtain a good approximation to $p(n)$, we begin by constructing an auxiliary function $F(x)$ which is regular at all points of the unit circle except $x=1$ and has there a singularity of a type as near as possible to that of $f(x)$. We may then hope to obtain a fairly close approximation by applying CAUCHY's Theorem to $f(x)-F(x)$ instead of to $F(x)$. For, although every point of the circle is a singular point of $f(x)$, the point $x=1$ is, to put it roughly, much the *heaviest* singularity. It follows from the formulae of the transformation-theory that $f(x)$ satisfies the equation

$$f(x)=\frac{x^{\frac{1}{24}}}{\sqrt{2\pi}}\sqrt{\log\frac{1}{x}}\,exp\left(\frac{\pi^2}{6\log\frac{1}{x}}\right)f(x'), \tag{2}$$

where

$$x'=exp\left(-\frac{4\pi^2}{\log\frac{1}{x}}\right);$$

so that, when $x\to 1$ by real values, $f(x)$ tends to infinity like an exponential

$$exp\left(\frac{\pi^2}{6(1-x)}\right).$$

[1] Since this was written, the memoir here referred to has been published. See *Proc. London Math. Soc.*, ser. 2, vol. 17, 1918, pp. 75–115.

On the other hand it may be shown that, when $x = re^{\frac{2\pi i p}{q}}$, p and q being co-prime integers, and $r \to 1$, $f(x)$ tends to infinity like an exponential

$$exp\left(\frac{\pi^2}{6q^2(1-r)}\right);$$

and if $x = re^{2\theta\pi i}$, where θ is irrational, $f(x)$ can become infinite at most like an exponential of the type

$$exp\left(o\left(\frac{1}{1-r}\right)\right).$$

5. The function required is

$$F(x) = \frac{1}{\pi\sqrt{2}} \sum_1^\infty \psi(n)\, x^n,$$

where

$$\psi(n) = \frac{d}{dn}\left(\frac{\cosh C\lambda_n - 1}{\lambda_n}\right),$$

$$C = \pi\sqrt{\frac{2}{3}}, \quad \lambda_n = \sqrt{n - \frac{1}{24}}.$$

This function may be transformed into an integral by means of a general formula given by Lindelöf; and it is then easy to prove that the 'principal branch' of $F(x)$ is regular all over the plane except at $x = 1$; and that

$$F(x) - X(x),$$

where

$$X(x) = \frac{x^{\frac{1}{24}}}{\sqrt{2\pi}} \sqrt{\log\frac{1}{x}} \left\{ exp\left(\frac{\pi^2}{6\log\frac{1}{x}}\right) - 1 \right\}, \tag{3}$$

is regular for $x = 1$. If we compare (2) and (3), and observe that $f(x')$ tends to unity with extreme rapidity when x tends to 1 along any regular path which does not touch the circle of convergence, we can see at once the very close similarity between the behaviour of $f(x)$ and $F(x)$ inside the unit circle and in the neighbourhood of $x = 1$.

Applying CAUCHY's Theorem to $f(x)-F(x)$, we obtain

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+O\left(e^{D\sqrt{n}}\right), \tag{4}$$

where D is any number greater than

$$\frac{1}{2}C=\frac{1}{2}\pi\sqrt{\frac{2}{3}}.$$

6. The error term in (4) is of an exponential type, and may be expected ultimately to increase with very great rapidity. It was therefore with considerable surprise that we found what exceedingly good results the formula gives for fairly large values of n. For $n=61, 62, 63$ it gives 1121538.972, 1300121.359, 1505535.606, while the correct values are 1121505, 1300156, 1505499. The errors 33.972, -34.641, 36.606 are relatively very small, and alternate in sign.

The next step is naturally to direct our attention to the singular point of $f(x)$ next in importance after that at $x=1$, viz. that at $x=-1$; and to subtract from $f(x)$ a second auxiliary function, related to this point as $F(x)$ is to $x=1$. No new difficulty of principle is involved, and we find that

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+\frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right)+O\left(e^{D\sqrt{n}}\right), \tag{5}$$

where D is now any number greater than $\frac{1}{3}C$. It now becomes obvious why our earlier approximation gave errors alternately of excess and of defect.

It is obvious that this process may be repeated indefinitely. The singularities next in importance are those at $x=e^{\frac{2}{3}\pi i}$ and $e^{\frac{4}{3}\pi i}$; the next those at $x=i$ and $x=-i$; and so on. The next two terms in the approximate formula are found to be

$$\pi\sqrt{\frac{3}{2}}\cos\left(\frac{2}{3}n\pi-\frac{1}{18}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)$$

and

$$\pi\sqrt{2}\cos\left(\frac{1}{2}n\pi-\frac{1}{8}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{4}C\lambda_n}}{\lambda_n}\right).$$

As we proceed further, the complexity of the calculations increases. The auxiliary function associated with the point $x = e^{\frac{2p\pi i}{q}}$ involves a certain $24q$-th root of unity, connected with the linear transformation which must be used in order to elucidate the behaviour of $f(x)$ near the point; and the explicit expression of this root in terms of p and q, though known, is somewhat complex. But it is plain that, by taking a sufficient number of terms, we can find a formula in which the error is

$$O\left(e^{\frac{C\lambda_n}{\nu}}\right),$$

where ν is a fixed, but arbitrarily large, integer.

7. A final question remains. We have still the recourse of making ν a function of n, that is to say of making the number of terms in our approximate formula itself a function of n. In this way we may reasonably hope, at any rate, to find a formula in which the error is of order less than that of *any* exponential of the type e^{an}; of the order of a power of n, for example, or even bounded.

When, however, we proceeded to test this hypothesis by means of numerical data most kindly provided for us by Major P. A. MACMAHON, we found a correspondence between the real and the approximate values of such astonishing accuracy as to lead us to hope for even more. Taking $n = 100$, we found that the first six terms of our formula gave

$$\begin{array}{r} 190568944.783 \\ +\ 348.872 \\ -\quad 2.598 \\ +\quad .685 \\ +\quad .318 \\ -\quad .064 \\ \hline 190569291.996, \end{array}$$

while

$$p(100) = 190569292;$$

so that the error after six terms is only .004. We then proceeded to calculate $p(200)$, and found

$$\begin{array}{r} 3,972,998,993,185.896 \\ +\ 36,282.978 \\ -87.555 \\ +\ 5.147 \\ +\ 1.424 \\ +\ 0.071 \\ 0.000^{1} \\ +\ 0.043 \\ \hline 3,972,999,029,388.004; \end{array}$$

and Major MacMahon's subsequent calculations showed that $p(200)$ is in fact

$$3,972,999,029,388.$$

These results suggest very forcibly that it is possible to obtain a formula for $p(n)$ which not only exhibits its order of magnitude and structure but may be used to calculate its *exact* value for any value of n. That this in fact is so is shown by the following theorem.

Theorem. *Suppose that*

$$\psi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{\frac{C\lambda_n}{q}}}{\lambda_n} \right),$$

where

$$C = \pi \sqrt{\frac{2}{3}}, \quad \lambda_n = \sqrt{n - \frac{1}{24}},$$

for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula

[1] This term vanishes identically.

$$\omega_{p,q} = \left(\frac{-q}{p}\right) exp\left(-\left[\frac{1}{4}(2-pq-p) + \frac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right]\pi i\right),$$

and when q is odd by the formula

$$\omega_{p,q} = \left(\frac{-p}{q}\right) exp\left(-\left[\frac{1}{4}(q-1) + \frac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right]\pi i\right),$$

where $\left(\frac{a}{b}\right)$ *is the symbol of* LEGENDRE *and* JACOBI, *and p' is any positive integer such that $1 + pp'$ is divisible by q; that*

$$A_q(n) = \sum_{(p)} \omega_{p,q}\, e^{-\frac{2np\pi i}{q}};$$

and that α is any positive constant and ν the integral part of $\alpha\sqrt{n}$.

Then

$$p(n) = \sum_{1}^{\nu} A_q \psi_q + O(n^{-\frac{1}{4}});$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\sum_{1}^{\nu} A_q \psi_q.$$

The proof of this theorem is naturally intricate, but it involves no fundamental idea beyond those which I have explained.

UNE FORMULE ASYMPTOTIQUE POUR LE NOMBRE DES PARTITIONS DE n

(*Comptes Rendus*, 2 Jan. 1917)

1. Les divers problèmes de la théorie de la partition des nombres ont été étudiés surtout par les mathématiciens anglais, Cayley, Sylvester et Macmahon†, qui les ont abordés d'un point de vue purement algébrique. Ces auteurs n'y ont fait aucune application des méthodes de la théorie des fonctions, de sorte qu'on ne trouve pas, dans la théorie en question, de formules asymptotiques, telles qu'on en rencontre, par exemple, dans la théorie des nombres premiers. Il nous semble donc que les résultats que nous allons faire connaître peuvent présenter quelque nouveauté.

2. Nous nous sommes occupés surtout de la fonction $p(n)$, nombre des partitions de n. On a

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots} = \sum_0^\infty p(n)\,x^n \qquad (|x| < 1).$$

Nous avons pensé d'abord à faire usage de quelque théorème de caractère *Taubérien*: on désigne ainsi les théorèmes réciproques du théorème classique d'Abel et de ses généralisations. A cette catégorie appartient l'énoncé suivant:

Soit $g(x) = \Sigma a_n x^n$ une série de puissances à coefficients POSITIFS, *telle qu'on ait*

$$\log g(x) \sim \frac{A}{1-x},$$

quand x tend vers un par des valeurs positives. Alors on a

$$\log s_n = \log(a_0 + a_1 + \dots + a_n) \sim 2\sqrt{(An)},$$

quand n tend vers l'infini‡.

† Voir le grand traité *Combinatory Analysis* de M. P. A. Macmahon (Cambridge, 1915–16).

‡ Nous avons donné des généralisations étendues de ce théorème dans un mémoire qui doit paraître dans un autre recueil.

1917, 1 (with S. Ramanujan) *Comptes Rendus*, 164, 35–38.

En posant $g(x) = (1-x) f(x)$, on a

$$A = \frac{\pi^2}{6};$$

et nous en tirons

$$p(n) = e^{\pi\sqrt{(\frac{2}{3}n)}(1+\epsilon)}, \quad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

où ϵ tend vers zéro avec $1/n$.

3. Pour pousser l'approximation plus loin, il faut recourir au théorème de Cauchy. Des formules

$$p(n) = \frac{1}{2\pi i}\int \frac{f(x)}{x^{n+1}}\,dx,$$

avec un chemin d'intégration convenable intérieur au cercle de rayon un, et

$$f(x) = \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\exp\left(\frac{\pi^2}{6\log(1/x)}\right) f\left\{\exp\left(-\frac{4\pi^2}{\log(1/x)}\right)\right\}\dots\dots(2)$$

(fournie par la théorie de la transformation linéaire des fonctions elliptiques), nous avons tiré, en premier lieu, la formule vraiment asymptotique

$$p(n) \sim P(n) = \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{(\frac{2}{3}n)}}. \quad \dots\dots\dots\dots\dots\dots(3)$$

On a

$p(10) = 42$, $p(20) = 627$, $p(50) = 204\,226$, $p(80) = 15\,796\,476$;
$P(10) = 48$, $P(20) = 692$, $P(50) = 217\,590$, $P(80) = 16\,606\,781$.

Les valeurs correspondantes de $P(n) : p(n)$ sont

$$1{\cdot}145;\quad 1{\cdot}104;\quad 1{\cdot}065;\quad 1{\cdot}051:$$

la valeur approximative est toujours en excès.

4. Mais nous avons abouti plus tard à des résultats beaucoup plus satisfaisants. Nous considérons la fonction

$$F(x) = \frac{1}{\pi\sqrt{2}}\sum_1^\infty \frac{d}{dn}\left\{\frac{\cosh\left[\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}\right]-1}{\sqrt{(n-\frac{1}{24})}}\right\} x^n. \quad \dots\dots\dots(4)$$

En faisant usage des formules sommatoires que démontre M. E. Lindelöf dans son beau livre *Le calcul des résidus*, on trouve aisément que $F(x)$ (on parle, il va sans dire, de la branche principale) a pour seul point singulier le point $x = 1$, et que la fonction

$$F(x) - \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\left[\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\}-1\right]$$

est régulière pour $x = 1$. On est conduit naturellement à appliquer le théorème de Cauchy à la fonction $f(x) - F(x)$, et l'on trouve

$$p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\frac{e^{\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}} + O(e^{k\sqrt{n}}) = Q(n) + O(e^{k\sqrt{n}}), \dots\dots(5)$$

où k désigne un nombre quelconque supérieur à $\pi/\sqrt{6}$. L'approximation, pour des valeurs assez grandes de n, est très bonne : on trouve, en effet,

$$p(61)=1\,121\,505, \quad p(62)=1\,300\,156, \quad p(63)=1\,505\,499;$$
$$Q(61)=1\,121\,539, \quad Q(62)=1\,300\,121, \quad Q(63)=1\,505\,536.$$

La valeur approximative est, pour les valeurs suffisamment grandes de n, alternativement en excès et en défaut.

5. On peut pousser ces calculs beaucoup plus loin. On forme des fonctions, analogues à $F(x)$, qui présentent, pour les valeurs

$$x=-1, \quad e^{\frac{2}{3}\pi i}, \quad e^{-\frac{2}{3}\pi i}, \quad i, \quad -i, \quad e^{\frac{2}{5}\pi i}, \quad \ldots,$$

des singularités d'un type très analogue à celles que présente $f(x)$. On soustrait alors de $f(x)$ une somme d'un nombre fini convenable de ces fonctions. On trouve ainsi, par exemple,

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\frac{e^{\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}}+\frac{(-1)^n}{2\pi}\frac{d}{dn}\frac{e^{\frac{1}{2}\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}}$$
$$+\frac{\sqrt{3}}{\pi\sqrt{2}}\cos\left(\frac{2n\pi}{3}-\frac{\pi}{18}\right)\frac{d}{dn}\frac{e^{\frac{1}{3}\pi\sqrt{\{\frac{2}{3}(n-\frac{1}{24})\}}}}{\sqrt{(n-\frac{1}{24})}}+O(e^{k\sqrt{n}}), \quad \ldots\ldots(6)$$

où k désigne un nombre quelconque plus grand que $\frac{1}{4}\pi\sqrt{\frac{2}{3}}$.

ASYMPTOTIC FORMULÆ FOR THE DISTRIBUTION OF INTEGERS OF VARIOUS TYPES*

(*Proceedings of the London Mathematical Society*, 2, XVI, 1917, 112—132)

1. *Statement of the problem.*

1·1. We denote by q a number of the form

(1·11) $$2^{a_2} 3^{a_3} 5^{a_5} \dots p^{a_p},$$

where 2, 3, 5, ..., p are primes and

(1·111) $$a_2 \geqslant a_3 \geqslant a_5 \geqslant \dots \geqslant a_p;$$

and by $Q(x)$ the number of such numbers which do not exceed x: and our problem is that of determining the order of $Q(x)$. We prove that

(1·12) $$Q(x) = \exp\left[\{1 + o(1)\} \frac{2\pi}{\sqrt{3}} \sqrt{\left(\frac{\log x}{\log \log x}\right)}\right],$$

that is to say that to every positive ϵ corresponds an $x_0 = x_0(\epsilon)$, such that

(1·121)
$$\left(\frac{2\pi}{\sqrt{3}} - \epsilon\right) \sqrt{\left(\frac{\log x}{\log \log x}\right)} < \log Q(x) < \left(\frac{2\pi}{\sqrt{3}} + \epsilon\right) \sqrt{\left(\frac{\log x}{\log \log x}\right)},$$

for $x > x_0$. The function $Q(x)$ is thus of higher order than any power of $\log x$, but of lower order than any power of x.

The interest of the problem is threefold. In the first place the result itself, and the method by which it is obtained, are curious and interesting in themselves. Secondly, the method of proof is one which, as we shew at the end of the paper, may be applied to a whole class of problems in the analytic theory of numbers: it enables us, for example, to find asymptotic formulæ for the number of partitions of n into positive integers, or into different positive integers, or into primes. Finally, the class of numbers q includes as a sub-class the "highly composite" numbers recently studied by Mr Ramanujan in an elaborate memoir in these *Proceedings*†. The problem of determining, with any precision, the number $H(x)$ of highly composite numbers not exceeding x appears to be one of extreme difficulty. Mr Ramanujan has proved, by elementary methods, that the order of $H(x)$ is at any rate greater

* This paper was originally communicated under the title "A problem in the Analytic Theory of Numbers."

† Ramanujan, "Highly Composite Numbers," *Proc. London Math. Soc.*, Ser. 2, Vol. XIV 1915, pp. 347—409.

than that of $\log x$* : but it is still uncertain whether or no the order of $H(x)$ is greater than that of any power of $\log x$. In order to apply transcendental methods to this problem, it would be necessary to study the properties of the function

$$\mathfrak{H}(s) = \Sigma \frac{1}{h^s},$$

where h is a highly composite number, and we have not been able to make any progress in this direction. It is therefore very desirable to study the distribution of wider classes of numbers which include the highly composite numbers and possess some at any rate of their characteristic properties. The simplest and most natural such class is that of the numbers q; and here progress is comparatively easy, since the function

$$\mathfrak{Q}(s) = \Sigma \frac{1}{q^s} \tag{1·13}$$

possesses a product expression analogous to Euler's product expression for $\zeta(s)$, viz.

$$\mathfrak{Q}(s) = \prod_1^\infty \left(\frac{1}{1 - l_n^{-s}}\right), \tag{1·14}$$

where $l_n = 2 . 3 . 5 \ldots p_n$ is the product of the first n primes.

We have not been able to apply to this problem the methods, depending on the theory of functions of a complex variable, by which the Prime Number Theorem was proved. The function $\mathfrak{Q}(s)$ has the line $\sigma = 0$† as a line of essential singularities, and we are not able to obtain sufficiently accurate information concerning the nature of these singularities. But it is easy enough to determine the behaviour of $\mathfrak{Q}(s)$ as a function of the *real* variable s; and it proves sufficient for our purpose to determine an asymptotic formula for $\mathfrak{Q}(s)$ when $s \to 0$, and then to apply a "Tauberian" theorem similar to those proved by Messrs Hardy and Littlewood in a series of papers published in these *Proceedings* and elsewhere‡.

This "Tauberian" theorem is in itself of considerable interest as being (so far as we are aware) the first such theorem which deals with functions or sequences tending to infinity more rapidly than any power of the variable.

* As great as that of $$\frac{\log x \sqrt{(\log\log x)}}{(\log\log\log x)^{\frac{3}{2}}}:$$
see p. 385 of his memoir.

† We write as usual $s = \sigma + it$.

‡ See, in particular, Hardy and Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive," *Proc. London Math. Soc.*, Ser. 2, Vol. XIII, 1914, pp. 174—191; and "Some theorems concerning Dirichlet's series," *Messenger of Mathematics*, Vol. XLIII, 1914, pp. 134—147.

2. *Elementary results.*

2·1. Let us consider, before proceeding further, what information concerning the order of $Q(x)$ can be obtained by purely elementary methods.

Let

(2·11) $$l_n = 2 \cdot 3 \cdot 5 \ldots p_n = e^{\vartheta(p_n)},$$

where $\vartheta(x)$ is Tschebyschef's function

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

The class of numbers q is plainly identical with the class of numbers of the form

(2·12) $$l_1^{b_1} l_2^{b_2} \ldots l_n^{b_n},$$

where $$b_1 \geqslant 0, \quad b_2 \geqslant 0, \quad \ldots, \quad b_n \geqslant 0.$$

Now every b can be expressed in one and only one way in the form

(2·13) $$b_i = c_{i,m} 2^m + c_{i,m-1} 2^{m-1} + \ldots + c_{i,0},$$

where every c is equal to zero or to unity. We have therefore

(2·14) $$q = \prod_{i=1}^{n} \left(l_i^{\sum_{j=0}^{m} c_{i,j} 2^j} \right) = \prod_{j=0}^{m} \prod_{i=1}^{n} l_i^{c_{i,j} 2^j} = \prod_{j=0}^{m} r_j^{2^j},$$

say, where

(2·141) $$r_j = l_1^{c_{1,j}} l_2^{c_{2,j}} \ldots l_n^{c_{n,j}}.$$

Let r denote, generally, a number of the form

(2·15) $$r = l_1^{c_1} l_2^{c_2} \ldots l_n^{c_n},$$

where every c is zero or unity: and $R(x)$ the number of such numbers which do not exceed x. If $q \leqslant x$, we have

$$r_0 \leqslant x, \quad r_1^2 \leqslant x, \quad r_2^4 \leqslant x, \quad \ldots.$$

The number of possible values of r_0, in the formula (2·14), cannot therefore exceed $R(x)$; the number of possible values of r_1 cannot exceed $R(x^{\frac{1}{2}})$; and so on. The total number of values of q can therefore not exceed

(2·16) $$S(x) = R(x)\, R(x^{\frac{1}{2}})\, R(x^{\frac{1}{4}}) \ldots R(x^{2^{-\varpi}}),$$

where ϖ is the largest number such that

(2·161) $$x^{2^{-\varpi}} \geqslant 2, \quad x \geqslant 2^{2^{\varpi}}.$$

Thus

(2·17) $$Q(x) \leqslant S(x).$$

2·2. We denote by f and g the largest numbers such that

(2·211) $$l_f \leqslant x,$$

(2·212) $$l_1 l_2 \ldots l_g \leqslant x.$$

It is known* (and may be proved by elementary methods) that constants A and B exist, such that

$$(2{\cdot}221) \qquad \vartheta(x) \geqslant Ax \quad (x \geqslant 2),$$

and

$$(2{\cdot}222) \qquad p_n \geqslant Bn \log n \quad (n \geqslant 1).$$

We have therefore

$$e^{Ap_f} \leqslant x,$$

$$f \log f = O(\log x),$$

$$(2{\cdot}23) \qquad \log f = O(\log\log x);$$

and

$$\sum_1^g \vartheta(p_\nu) \leqslant \log x, \qquad \sum_1^g p_\nu = O(\log x),$$

$$\sum_1^g \nu \log \nu = O(\log x), \quad g^2 \log g = O(\log x),$$

$$(2{\cdot}24) \qquad g = O\sqrt{\left(\frac{\log x}{\log\log x}\right)}.$$

But it is easy to obtain an upper bound for $R(x)$ in terms of f and g. The number of numbers $l_1, l_2, \ldots$, not exceeding x, is not greater than f; the number of products, not exceeding x, of pairs of such numbers, is *a fortiori* not greater than $\frac{1}{2}f(f-1)$; and so on. Thus

$$R(x) \leqslant f + \frac{f(f-1)}{2!} + \frac{f(f-1)(f-2)}{3!} + \ldots,$$

where the summation need be extended to g terms only, since

$$l_1 l_2 \ldots l_g l_{g+1} > x.$$

A fortiori, we have

$$R(x) \leqslant 1 + f + \frac{f^2}{2!} + \ldots + \frac{f^g}{g!} < (1+f)^g = e^{g\log(1+f)}.$$

Thus

$$(2{\cdot}25) \qquad R(x) = e^{O(g\log f)} = e^{O\{\sqrt{(\log x \log\log x)}\}},$$

by (2·23) and (2·24). Finally, since

$$\log\sqrt{x}\log\log\sqrt{x} < \tfrac{1}{2}\log x \log\log x,$$

it follows from (2·16) and (2·17) that

$$(2{\cdot}26) \quad Q(x) = \exp\left[O\left\{\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^\varpi}\right)\sqrt{(\log x \log\log x)}\right\}\right]$$

$$= e^{O\{\sqrt{(\log x \log\log x)}\}}.$$

2·3. A *lower* bound for $Q(x)$ may be found as follows. If g is defined as in 2·2, we have

$$l_1 l_2 \ldots l_g \leqslant x < l_1 l_2 \ldots l_g l_{g+1}.$$

* See Landau, *Handbuch*, pp. 79, 83, 214.

It follows from the analysis of 2·2 that

$$l_{g+1} = e^{\vartheta(p_{g+1})} = e^{O(g \log g)},$$

and
$$l_1 l_2 \dots l_g = \exp\left\{\sum_1^g \vartheta(p_\nu)\right\} = e^{O(g^2 \log g)}.$$

Thus
$$x < e^{O(g^2 \log g)};$$

which is only possible if g is greater than a constant positive multiple of

$$\sqrt{\left(\frac{\log x}{\log\log x}\right)}.$$

Now the numbers $l_1, l_2, \dots, l_g$ can be combined in 2^g different ways, and each such combination gives a number q not greater than x. Thus

$$Q(x) \geqslant 2^g > \exp\left\{K\sqrt{\left(\frac{\log x}{\log\log x}\right)}\right\}, \tag{2·31}$$

where K is a positive constant. From (2·26) and (2·31) it follows that there are positive constants K and L such that

$$K\sqrt{\left(\frac{\log x}{\log\log x}\right)} < \log Q(x) < L\sqrt{(\log x \log\log x)}. \tag{2·32}$$

The inequalities (2·32) give a fairly accurate idea as to the order of magnitude of $Q(x)$. But they are much less precise than the inequalities (1·121). To obtain these requires the use of less elementary methods.

3. *The behaviour of* $\mathfrak{Q}(s)$ *when* $s \to 0$ *by positive values.*

3·1. From the fact, already used in 2·1, that the class of numbers q is identical with the class of numbers of the form (2·12), it follows at once that

$$\mathfrak{Q}(s) = \Sigma \frac{1}{q^s} = \prod_1^\infty \left(\frac{1}{1 - l_n^{-s}}\right). \tag{1·14}$$

Both series and product are absolutely convergent for $\sigma > 0$, and

$$\log \mathfrak{Q}(s) = \phi(s) + \tfrac{1}{2}\phi(2s) + \tfrac{1}{3}\phi(3s) + \dots, \tag{3·11}$$

where

$$\phi(s) = \sum_1^\infty l_n^{-s}. \tag{3·111}$$

We have also

(3·12)
$$\begin{aligned}
\phi(s) &= \frac{1-2^{-s}}{2^s-1} + 2^{-s}\frac{1-3^{-s}}{3^s-1} + 2^{-s}3^{-s}\frac{1-5^{-s}}{5^s-1} + \dots \\
&= \frac{1}{2^s-1} + 2^{-s}\left(\frac{1}{3^s-1} - \frac{1}{2^s-1}\right) + 2^{-s}3^{-s}\left(\frac{1}{5^s-1} - \frac{1}{3^s-1}\right) + \dots \\
&= \frac{1}{2^s-1} + \sum_1^\infty e^{-s\vartheta(p_n)} \int_{p_n}^{p_{n+1}} \frac{d}{dx}\left(\frac{1}{x^s-1}\right) dx \\
&= \frac{1}{2^s-1} - s\int_2^\infty \frac{x^{s-1}}{(x^s-1)^2} e^{-s\vartheta(x)}\, dx.
\end{aligned}$$

3·2. LEMMA.—*If* $x > 1$, $s > 0$, *then*

$$\frac{1}{(s\log x)^2} - \frac{1}{12} < \frac{x^s}{(x^s-1)^2} < \frac{1}{(s\log x)^2}. \tag{3·21}$$

Write $x^s = e^u$: then we have to prove that

$$\frac{1}{u^2} - \frac{1}{12} < \frac{e^u}{(e^u-1)^2} < \frac{1}{u^2} \tag{3·22}$$

for all positive values of u; or (writing w for $\frac{1}{2}u$) that

$$\frac{1}{w^2} - \frac{1}{3} < \frac{1}{\sinh^2 w} < \frac{1}{w^2} \tag{3·23}$$

for all positive values of w. But it is easy to prove that the function

$$f(w) = \frac{1}{w^2} - \frac{1}{\sinh^2 w}$$

is a steadily decreasing function of w, and that its limit when $w \to 0$ is $\frac{1}{3}$; and this establishes the truth of the lemma.

3·3. We have therefore

$$\phi(s) = \frac{1}{2^s-1} - \phi_1(s) = \frac{1}{s\log 2} - \phi_1(s) + O(1), \tag{3·31}$$

where

$$\frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-s\vartheta(x)} \frac{dx}{x} < \phi_1(s) < \frac{1}{s}\int_2^\infty \frac{e^{-s\vartheta(x)}}{(\log x)^2}\frac{dx}{x}. \tag{3·311}$$

From the second of these inequalities, and (2·221), it follows that

$$\begin{aligned}\phi_1(s) < \frac{1}{s}\int_2^\infty \frac{e^{-Asx}}{(\log x)^2}\frac{dx}{x} &= \frac{e^{-2As}}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}dx \\ &= \frac{1}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}dx + O(1);\end{aligned}$$

and so that

$$\phi(s) > A\int_2^\infty \frac{e^{-Asx}}{\log x}dx + O(1). \tag{3·32}$$

On the other hand there is a positive constant B, such that

$$\vartheta(x) < Bx \quad (x \geqslant 2)^*.$$

Thus

$$\begin{aligned}\phi_1(s) > \frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-Bsx}\frac{dx}{x} &= \frac{1}{s}\int_2^\infty \frac{e^{-Bsx}}{(\log x)^2}\frac{dx}{x} + O(1) \\ &= \frac{1}{s\log 2} - B\int_2^\infty \frac{e^{-Bsx}}{\log x}dx + O(1);\end{aligned}$$

and so

$$\phi(s) < B\int_2^\infty \frac{e^{-Bsx}}{\log x}dx + O(1). \tag{3·33}$$

* Landau, *Handbuch*, *loc. cit.*

3·4. LEMMA.—*If H is any positive number, then*

$$J(s) = H\int_2^\infty \frac{e^{-Hsx}}{\log x}\,dx \sim \frac{1}{s\log(1/s)},$$

when $s \to 0$.

Given any positive number ϵ, we can choose ξ and X, so that

$$\int_0^\xi He^{-Hx}\,dx < \epsilon, \quad \int_X^\infty He^{-Hx}\,dx < \epsilon.$$

Now
$$s\log\left(\frac{1}{s}\right)J(s) = \int_{2s}^\infty \frac{He^{-Hu}\log(1/s)}{\log u + \log(1/s)}\,du = \int_{2s}^{\sqrt{s}} + \int_{\sqrt{s}}^{\xi} + \int_\xi^X + \int_X^\infty$$
$$= j_1(s) + j_2(s) + j_3(s) + j_4(s),$$

say. And we have

$$0 < j_1(s) < \frac{\log(1/s)}{\log 2}\int_{2s}^{\sqrt{s}} He^{-Hu}\,du = O\{\sqrt{s}\log(1/s)\} = o(1),$$

$$0 < j_2(s) < 2\int_0^\xi He^{-Hu}\,du < 2\epsilon,$$

$$j_3(s) = \int_\xi^X He^{-Hu}\,du + o(1),$$

$$0 < j_4(s) < \int_X^\infty He^{-Hu}\,du < \epsilon;$$

and so

$$\left|1 - s\log\left(\frac{1}{s}\right)J(s)\right| = \left|\int_0^\infty He^{-Hu}\,du - j_1(s) - j_2(s) - j_3(s) - j_4(s)\right|$$
$$< 5\epsilon + o(1) < 6\epsilon,$$

for all sufficiently small values of s.

3·5. From (3·32), (3·33), and the lemma just proved, it follows that

(3·51) $$\phi(s) = \Sigma l_n^{-s} \sim \frac{1}{s\log(1/s)}.$$

From this formula we can deduce an asymptotic formula for $\log \mathfrak{Q}(s)$. We choose N so that

(3·52) $$\sum_{N<n} \frac{1}{n^2} < \epsilon,$$

and we write

(3·53) $$\log \mathfrak{Q}(s) = \Sigma \frac{1}{n}\phi(ns) = \sum_{1\leq n\leq N} + \sum_{N<n<1/\sqrt{s}} + \sum_{1/\sqrt{s}\leq n\leq 1/s} + \sum_{1/s<n}$$
$$= \Phi_1(s) + \Phi_2(s) + \Phi_3(s) + \Phi_4(s),$$

say.

In the first place

(3·541) $$\Phi_1(s) = \frac{1+o(1)}{s\log(1/s)}\sum_1^N \frac{1}{n^2}.$$

In the second place

$$\phi(ns) = O\left\{\frac{1}{ns\log(1/ns)}\right\},$$

and $$\log(1/ns) > \tfrac{1}{2}\log(1/s),$$

if $N < n < 1/\sqrt{s}$. It follows that a constant K exists such that

(3·542) $$\Phi_2(s) < \frac{K}{s\log(1/s)}\sum_{N<n}\frac{1}{n^2} < \frac{K\epsilon}{s\log(1/s)}.$$

Thirdly, $\sqrt{s} \leqslant ns \leqslant 1$ in $\Phi_3(s)$, and a constant L exists such that

$$\phi(ns) < \frac{L}{\sqrt{s}\log(1/s)}.$$

Thus

(3·543) $$\Phi_3(s) < \frac{L}{\sqrt{s}\log(1/s)}\sum_1^{1/s}\frac{1}{n} < \frac{2L}{\sqrt{s}},$$

for all sufficiently small values of s.

Finally, in $\Phi_4(s)$ we have $ns > 1$, and a constant M exists such that

$$\phi(ns) < M2^{-ns}.$$

Thus

(3·544) $$\Phi_4(s) < M\sum_{1/s<n}\frac{2^{-ns}}{n} < sM\sum_{1/s<n}2^{-ns} < \frac{sM}{1-2^{-s}} = O(1).$$

From (3·53), (3·541)—(3·544), and (3·52) it follows that

(3·55) $$\log \mathfrak{Q}(s) = \frac{1}{s\log(1/s)}\left[\{1+o(1)\}\left(\frac{\pi^2}{6}+\rho\right)+\rho'\right] + O\left\{\frac{1}{\sqrt{s}\log(1/s)}\right\} + O(1),$$

where $$|\rho| < \epsilon, \quad |\rho'| < K\epsilon.$$

Thus

(3·56) $$\log \mathfrak{Q}(s) \sim \frac{\pi^2}{6s\log(1/s)},$$

or

(3·57) $$\mathfrak{Q}(s) = \exp\left[\{1+o(1)\}\frac{\pi^2}{6s\log(1/s)}\right].$$

4. *A Tauberian theorem.*

4·1. The passage from (3·57) to (1·12) depends upon a theorem of the "Tauberian" type.

THEOREM A. *Suppose that*

(1) $\lambda_1 \geqslant 0, \quad \lambda_n > \lambda_{n-1}, \quad \lambda_n \to \infty$;

(2) $\lambda_n/\lambda_{n-1} \to 1$;

(3) $a_n \geqslant 0$;

(4) $A > 0, \quad \alpha > 0$;

(5) $\Sigma a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$;

(6) $f(s) = \Sigma a_n e^{-\lambda_n s} = \exp\left[\{1 + o(1)\} A s^{-\alpha} \left\{\log\left(\frac{1}{s}\right)\right\}^{-\beta}\right]$,

when $s \to 0$. *Then*

$$A_n = a_1 + a_2 + \ldots + a_n = \exp\left[\{1 + o(1)\} B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}\right],$$

where

$$B = A^{1/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} (1 + \alpha)^{1+\{\beta/(1+\alpha)\}},$$

when $n \to \infty$.

We are given that

(4·11) $$(1-\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} < \log f(s) < (1 + \delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta},$$

for every positive δ and all sufficiently small values of s; and we have to shew that

(4·12)
$$(1 - \epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)} < \log A_n < (1 + \epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)},$$

for every positive ϵ and all sufficiently large values of n.

In the argument which follows we shall be dealing with three variables, δ, s, and n (or m), the two latter variables being connected by an equation or by inequalities, and with an auxiliary parameter ζ. We shall use the letter η, without a suffix, to denote generally a function of δ, s, and n (or m)*, which is not the same in different formulæ, but in all cases tends to zero when δ and s tend to zero and n (or m) to infinity; so that, given any positive ϵ, we have

$$0 < |\eta| < \epsilon,$$

for $$0 < \delta < \delta_0, \quad 0 < s < s_0, \quad n > n_0.$$

We shall use the symbol η_ζ to denote a function *of* ζ *only* which tends to zero with ζ, so that

$$0 < |\eta_\zeta| < \epsilon,$$

if ζ is small enough. It is to be understood that the choice of a ζ to satisfy certain conditions is in all cases prior to that of δ, s, and n (or m). Finally, we use the letters $H, K, \ldots$ to denote positive numbers independent of these variables *and of* ζ.

The *second* of the inequalities (4·12) is very easily proved. For

(4·131) $$A_n e^{-\lambda_n s} < a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s} + \ldots + a_n e^{-\lambda_n s} < f(s) < \exp\left\{(1 + \delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\},$$

(4·132) $$A_n < \exp \chi,$$

where

(4·1321) $$\chi = (1 + \delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} + \lambda_n s.$$

* η may, of course, in some cases be a function of some of these variables only.

We can choose a value of s, corresponding to every large value of n, such that

$$(4{\cdot}14)\qquad (1-\delta)\,A\alpha s^{-1-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \lambda_n < (1+\delta)\,A\alpha s^{-1-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}.$$

From these inequalities we deduce, by an elementary process of approximation,

$$(4{\cdot}151)\qquad (1-\eta)\,(A\alpha)^{-1/(1+\alpha)}\lambda_n^{1/(1+\alpha)}\left(\log\frac{1}{s}\right)^{\beta/(1+\alpha)}$$
$$< \frac{1}{s} < (1+\eta)\,(A\alpha)^{-1/(1+\alpha)}\lambda_n^{1/(1+\alpha)}\left(\log\frac{1}{s}\right)^{\beta/(1+\alpha)},$$

$$(4{\cdot}152)\qquad \frac{1-\eta}{1+\alpha}\log\lambda_n < \log\frac{1}{s} < \frac{1+\eta}{1+\alpha}\log\lambda_n,$$

$$(4{\cdot}153)\qquad (1-\eta)\frac{\alpha B}{1+\alpha}\lambda_n^{-1/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}$$
$$< s < (1+\eta)\frac{\alpha B}{1+\alpha}\lambda_n^{-1/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)},$$

$$(4{\cdot}154)\qquad \chi < (1+\eta)\,B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}.$$

We have therefore

$$(4{\cdot}16)\qquad \log A_n < (1+\epsilon)\,B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)},$$

for every positive ϵ and all sufficiently large values of n*.

4·2. We have

$$(4{\cdot}21)\qquad f(s) = \Sigma a_n e^{-\lambda_n s} = \Sigma A_n\left(e^{-\lambda_n s} - e^{-\lambda_{n+1}s}\right)$$
$$= s\sum_1^\infty A_n\int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}dx = s\int_0^\infty \mathscr{A}(x)\,e^{-sx}dx,$$

where $\mathscr{A}(x)$ is the discontinuous function defined by

$$\mathscr{A}(x) = A_n \quad (\lambda_n \leqslant x < \lambda_{n+1})\dagger,$$

so that, by (4·16),

$$(4{\cdot}22)\qquad \log\mathscr{A}(x) < (1+\epsilon)\,Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)}$$

for every positive ϵ and all sufficiently large values of x. We have therefore

$$(4{\cdot}23)\qquad \exp\left\{(1-\delta)\,As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} < s\int_0^\infty \mathscr{A}(x)\,e^{-sx}dx$$
$$< \exp\left\{(1+\delta)\,As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}$$

for every positive δ and all sufficiently small values of s.

* We use the second inequality (4·12) in the proof of the first. It would be sufficient for our purpose to begin by proving a result cruder than (4·16), with any constant K on the right-hand side instead of $(1+\epsilon)B$. But it is equally easy to obtain the more precise inequality. Compare the argument in the second of the two papers by Hardy and Littlewood quoted on p. 114 [p. 246 of this volume] (pp. 143 *et seq.*).

† Compare Hardy and Riesz, "The General Theory of Dirichlet's Series," *Cambridge Tracts in Mathematics*, No. 18, 1915, p. 24.

We define λ_x, a steadily increasing and continuous function of the continuous variable x, by the equation

$$\lambda_x = \lambda_n + (x-n)(\lambda_{n+1} - \lambda_n) \quad (n \leqslant x \leqslant n+1).$$

We can then choose m so that

$$\frac{1}{s} = \frac{1+\alpha}{\alpha B} \lambda_m^{1/(1+\alpha)} (\log \lambda_m)^{\beta/(1+\alpha)}. \tag{4·24}$$

We shall now shew that the limits of the integral in (4·23) may be replaced by $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$, where ζ is an arbitrary positive number less than unity.

We write.

(4·25)

$$J(s) = s\int_0^\infty \mathscr{A}(x)\, e^{-sx}\, dx = s\left\{\int_0^{\lambda_m/H} + \int_{\lambda_m/H}^{(1-\zeta)\lambda_m} + \int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} + \int_{(1+\zeta)\lambda_m}^{H\lambda_m} + \int_{H\lambda_m}^{\infty}\right\}$$
$$= J_1 + J_2 + J_3 + J_4 + J_5,$$

where H is a constant, in any case greater than 1, and large enough to satisfy certain further conditions which will appear in a moment; and we proceed to shew that J_1, J_2, J_4, and J_5 are negligible in comparison with the exponentials which occur in (4·23), and so in comparison with J_3.

4·3. The integrals J_1 and J_5 are easily disposed of. In the first place we have

$$J_1 = s\int_0^{\lambda_m/H} \mathscr{A}(x)\, e^{-sx}\, dx < \mathscr{A}\left(\frac{\lambda_m}{H}\right) \tag{4·31}$$
$$< \exp\left\{(1+\delta) B \left(\frac{\lambda_m}{H}\right)^{\alpha/(1+\alpha)} \left(\log\frac{\lambda_m}{H}\right)^{-\beta/(1+\alpha)}\right\},$$

by (4·22)*. It will be found, by a straightforward calculation, that this expression is less than

$$\exp\left\{(1+\eta) A (1+\alpha) H^{-\alpha/(1+\alpha)} s^{-\alpha} \left(\log\frac{1}{s}\right)^{-\beta}\right\}, \tag{4·32}$$

and is therefore certainly negligible if H is sufficiently large.

Thus J_1 is negligible. To prove that J_5 is negligible we prove first that

$$sx > 4Bx^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)},$$

if $x > H\lambda_m$ and H is large enough†. It follows that

$$J_5 = s\int_{H\lambda_m}^\infty \mathscr{A}(x)\, e^{-sx}\, dx < s\int_{H\lambda_m}^\infty \exp\{(1+\delta) Bx^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)} - sx\}\, dx$$
$$< s\int_0^\infty e^{-\frac{1}{2}sx}\, dx = \tfrac{1}{2},$$

and is therefore negligible.

* With δ in the place of ϵ.

† We suppress the details of the calculation, which is quite straightforward.

4·4. The integrals J_2 and J_4 may be discussed in practically the same way, and we may confine ourselves to the latter.

We have

$$(4\cdot41) \qquad J_4(s) = s\int_{(1+\zeta)\lambda_m}^{H\lambda_m} \mathscr{A}(x)\, e^{-sx}\, dx < s\int_{(1+\zeta)\lambda_m}^{H\lambda_m} e^{\psi}\, dx,$$

where

$$(4\cdot411) \qquad \psi = (1+\delta)\, B x^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)} - sx.$$

The maximum of the function ψ occurs for $x = x_0$, where

$$(4\cdot42) \qquad \frac{1}{s} = (1+\eta)\frac{1+\alpha}{\alpha B} x_0^{1/(1+\alpha)} (\log x_0)^{\beta/(1+\alpha)}.$$

From this equation, and (4·24), it plainly results that

$$(4\cdot43) \qquad (1-\eta)\lambda_m < x_0 < (1+\eta)\lambda_m,$$

and that x_0 falls (when δ and s are small enough) between $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$.

Let us write
$$x = x_0 + \xi$$
in J_4. Then

$$\psi(x) = \psi(x_0) + \tfrac{1}{2}(1+\delta) B\xi^2 \frac{d^2}{dx_1^2}\{x_1^{\alpha/(1+\alpha)} (\log x_1)^{-\beta/(1+\alpha)}\},$$

where $x_0 < x_1 < x$ and *a fortiori*

$$(1-\zeta)\lambda_m < x_1 < H\lambda_m.$$

It follows that

$$(4\cdot44) \qquad \frac{d^2}{dx_1^2}\{x_1^{\alpha/(1+\alpha)} (\log x_1)^{-\beta/(1+\alpha)}\} < -K\lambda_m^{\alpha/(1+\alpha)-2} (\log \lambda_m)^{-\beta/(1+\alpha)}.$$

On the other hand, an easy calculation shews that

$$(4\cdot45) \qquad (1-\eta) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \psi(x_0) < (1+\eta) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}.$$

Thus

$$(4\cdot46) \qquad J_4 < \exp\left\{(1+\eta) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}$$
$$\times \int_{(\zeta-\eta)\lambda_m}^{\infty} \exp\{-L\xi^2\lambda_m^{\alpha/(1+\alpha)-2} (\log\lambda_m)^{-\beta/(1+\alpha)}\}\, d\xi$$
$$< \exp\left\{(1+\eta) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} - M\zeta^2\lambda_m^{\alpha/(1+\alpha)} (\log\lambda_m)^{-\beta/(1+\alpha)}\right\}$$
$$< \exp\left\{(1+\eta - N\zeta^2) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}.$$

Since ζ is independent of δ and s, this inequality shews that J_4 is negligible; and a similar argument may be applied to J_2.

4·5. We may therefore replace the inequalities (4·23) by

(4·51) $$\exp\left\{(1-\delta)\,A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}$$
$$< s\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m}\mathscr{A}(x)\,e^{-sx}\,dx < \exp\left\{(1+\delta)\,A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}.$$

Since $\mathscr{A}(x)$ is a steadily increasing function of x, it follows that

(4·521)
$$\exp\left\{(1-\delta)\,A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} < s\mathscr{A}\{(1+\zeta)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} e^{-sx}\,dx,$$

(4·522)
$$\exp\left\{(1+\delta)\,A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} > s\mathscr{A}\{(1-\zeta)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} e^{-sx}\,dx;$$

or

(4·531)
$$(e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m})\,\mathscr{A}\{(1-\zeta)\lambda_m\} < \exp\left\{(1+\delta)\,A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}+\lambda_m s\right\},$$

(4·532)
$$(e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m})\,\mathscr{A}\{(1+\zeta)\lambda_m\} > \exp\left\{(1-\delta)\,A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}+\lambda_m s\right\}.$$

If we substitute for s, in terms of λ_m, in the right-hand sides of (4·531) and (4·532), we obtain expressions of the form
$$\exp\{(1+\eta)\,B\lambda_m^{\alpha/(1+\alpha)}\,(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$
On the other hand
$$e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m}$$
is of the form
$$\exp\{\eta_\zeta\lambda_m^{\alpha/(1+\alpha)}\,(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$
We have thus

(4·541)
$$A\{(1-\zeta)\lambda_m\} < \exp\{(1+\eta_\zeta+\eta)\,B\lambda_m^{\alpha/(1+\alpha)}\,(\log\lambda_m)^{-\beta/(1+\alpha)}\},$$

(4·542)
$$A\{(1+\zeta)\lambda_m\} > \exp\{(1-\eta_\zeta-\eta)\,B\lambda_m^{\alpha/(1+\alpha)}\,(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

Now let ν be any number such that

(4·55) $$(1-\zeta)\lambda_m \leqslant \lambda_\nu \leqslant (1+\zeta)\lambda_m.$$

Since $\lambda_n/\lambda_{n-1}\to 1$, it is clear that *all* numbers n from a certain point onwards will fall among the numbers ν. It follows from (4·541) and (4·542) that

(4·56)
$$\exp\{(1-\eta_\zeta-\eta)(1-\eta_\zeta)\,B\lambda_\nu^{\alpha/(1+\alpha)}\,(\log\lambda_\nu)^{-\beta/(1+\alpha)}\} < A(\lambda_\nu)$$
$$< \exp\{(1+\eta_\zeta+\eta)(1+\eta_\zeta)\,B\lambda_\nu^{\alpha/(1+\alpha)}\,(\log\lambda_\nu)^{-\beta/(1+\alpha)}\};$$

and therefore that, given ϵ, we can choose first ζ and then n_0 so that

(4·57) $$\exp\{(1-\epsilon)\,B\lambda_n^{\alpha/(1+\alpha)}\,(\log\lambda_n)^{-\beta/(1+\alpha)}\} < A(\lambda_n)$$
$$< \exp\{(1+\epsilon)\,B\lambda_n^{\alpha/(1+\alpha)}\,(\log\lambda_n)^{-\beta/(1+\alpha)}\},$$

for $n \geqslant n_0$. This completes the proof of the theorem.

4·6. There is of course a corresponding "Abelian" theorem, which we content ourselves with enunciating. This theorem is naturally not limited by the restriction that the coefficients a_n are positive.

THEOREM B. *Suppose that*

(1) $\lambda_1 \geqslant 0, \quad \lambda_n > \lambda_{n-1}, \quad \lambda_n \to \infty$;

(2) $\lambda_n/\lambda_{n-1} \to 1$;

(3) $A > 0, \quad 0 < \alpha < 1$;

(4) $A_n = a_1 + a_2 + \dots + a_n = \exp[\{1 + o(1)\} A \lambda_n^\alpha (\log \lambda_n)^{-\beta}]$,

when $n \to \infty$. *Then the series* $\Sigma a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$, *and*

$$f(s) = \Sigma a_n e^{-\lambda_n s} = \exp\left[\{1 + o(1)\} B s^{-\alpha/(1-\alpha)} \left(\log \frac{1}{s}\right)^{-\beta/(1-\alpha)}\right],$$

where $$B = A^{1/(1-\alpha)} \alpha^{\alpha/(1-\alpha)} (1-\alpha)^{1+[\beta/(1-\alpha)]},$$

when $s \to 0$.

The proof of this theorem, which is naturally easier than that of the correlative Tauberian theorem, should present no difficulty to anyone who has followed the analysis which precedes.

4·7. The simplest and most interesting cases of Theorems A and B are those in which

$$\lambda_n = n, \quad \beta = 0.$$

It is then convenient to write x for e^{-s}. We thus obtain

THEOREM C. *If* $A > 0$, $0 < \alpha < 1$, *and*

$$\log A_n = \log(a_1 + a_2 + \dots + a_n) \sim A n^\alpha,$$

then the series $\Sigma a_n x^n$ *is convergent for* $|x| < 1$, *and*

$$\log f(x) = \log(\Sigma a_n x^n) \sim B(1-x)^{-\alpha/(1-\alpha)},$$

where $$B = (1-\alpha)\alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)},$$

when $x \to 1$ *by real values.*

If the coefficients are positive the converse inference is also correct. That is to say, if

$$A > 0, \quad \alpha > 0,$$

and $$\log f(x) \sim A(1-x)^{-\alpha},$$

then $$\log A_n \sim B n^{\alpha/(1+\alpha)},$$

where $$B = (1+\alpha)\alpha^{-\alpha/(1+\alpha)} A^{1/(1+\alpha)}.$$

5. *Application to our problem, and to other problems in the Theory of Numbers.*

5·1. We proved in 3 that

$$\log \mathfrak{Q}(s) \sim \frac{\pi^2}{6s \log(1/s)}. \tag{3·56}$$

In Theorem A take

$$\lambda_n = \log n, \quad A = \frac{\pi^2}{6}, \quad \alpha = 1, \quad \beta = 1.$$

Then all the conditions of the theorem are satisfied. And A_n is $Q(n)$, the number of numbers q not exceeding n. We have therefore

$$\log Q(n) \sim B \sqrt{\left(\frac{\log n}{\log\log n}\right)}, \tag{5·11}$$

where

$$B = 2^{\frac{1}{2}} \sqrt{\left(\frac{\pi^2}{6}\right)} = \frac{2\pi}{\sqrt{3}}. \tag{5·12}$$

5·2. The method which we have followed in solving this problem is one capable of many other interesting applications.

Suppose, for example, that $R_r(n)$ is the number of ways in which n can be represented as the sum of any number of rth powers of positive integers*. We shall prove that

$$\log R_r(n) \sim (r+1)\left\{\frac{1}{r}\Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)}. \tag{5·21}$$

In particular, if $P(n) = R_1(n)$ is the number of partitions of n, then

$$\log P(n) \sim \pi \sqrt{\left(\frac{2n}{3}\right)}. \tag{5·22}$$

We need only sketch the proof, which is in principle similar to the main proof of this paper. We have

$$\sum_1^\infty R_r(n) e^{-ns} = \prod_1^\infty \left(\frac{1}{1-e^{-s\nu^r}}\right),$$

and so

$$f(s) = \sum_1^\infty \{R_r(n) - R_r(n-1)\} e^{-ns} = \prod_2^\infty \left(\frac{1}{1-e^{-s\nu^r}}\right)\dagger. \tag{5·23}$$

It is obvious that $R_r(n)$ increases with n and that all the coefficients in $f(s)$ are positive. Again,

$$\log f(s) = \sum_2^\infty \log\left(\frac{1}{1-e^{-s\nu^r}}\right) = \sum_2^\infty \left(e^{-s\nu^r} + \tfrac{1}{2}e^{-2s\nu^r} + \dots\right) \tag{5·24}$$

$$= \sum_{k=1}^\infty \frac{1}{k}\phi(ks),$$

where

$$\phi(s) = \sum_2^\infty e^{-s\nu^r}. \tag{5·241}$$

* Thus $28 = 3^3 + 1^3 = 3 . 2^3 + 4 . 1^3 = 2 . 2^3 + 12 . 1^3 = 2^3 + 20 . 1^3 = 28 . 1^3$:

and $R_3(28) = 5$.

The *order* of the powers is supposed to be indifferent, so that (e.g.) $3^3 + 1^3$ and $1^3 + 3^3$ are not reckoned as separate representations.

† $R_r(0)$ is to be interpreted as zero.

But

$$\phi(s) \sim \Gamma\left(\frac{1}{r}+1\right) s^{-1/r}, \tag{5·25}$$

when $s \to 0$; and we can deduce, by an argument similar to that of 3·5, that

$$\log f(s) \sim \Gamma\left(\frac{1}{r}+1\right) \zeta\left(\frac{1}{r}+1\right) s^{-1/r}. \tag{5·26}$$

We now obtain (5·21) by an application of Theorem A, taking

$$\lambda_n = n, \quad \alpha = \frac{1}{r}, \quad \beta = 0, \quad A = \Gamma\left(\frac{1}{r}+1\right) \zeta\left(\frac{1}{r}+1\right).$$

In a similar manner we can shew that, if $S(n)$ is the number of partitions of n into *different* positive integers, so that

$$\begin{aligned}\Sigma S(n) e^{-ns} &= (1+e^{-s})(1+e^{-2s})(1+e^{-3s}) + \ldots \\ &= \frac{1}{(1-e^{-s})(1-e^{-3s})(1-e^{-5s})\ldots},\end{aligned}$$

then

$$\log S(n) \sim \pi \sqrt{\left(\frac{n}{3}\right)}; \tag{5·27}$$

that if $T_r(n)$ is the number of representations of n as the sum of rth powers of *primes*, then

(5·28)

$$\log T_r(n) \sim (r+1)\left\{\Gamma\left(\frac{1}{r}+2\right) \zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)} (\log n)^{-r/(r+1)};$$

and, in particular, that if $T(n) = T_1(n)$ is the number of partitions of n into primes, then

$$\log T(n) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\left(\frac{n}{\log n}\right)}. \tag{5·281}$$

Finally, we can shew that if r and s are positive integers, $a > 0$, and $0 \leqslant b \leqslant 1$, and

$$\Sigma \phi(n) x^n = \frac{\{(1+ax)(1+ax^2)(1+ax^3)\ldots\}^r}{\{(1-bx)(1-bx^2)(1-bx^3)\ldots\}^s}, \tag{5·291}$$

then

$$\log \phi(n) \sim 2\sqrt{(cn)}, \tag{5·292}$$

where

$$c = r\int_0^a \frac{\log(1+t)}{t}\,dt - s\int_0^b \frac{\log(1-t)}{t}\,dt. \tag{5·2921}$$

In particular, if $a = 1$, $b = 1$, and $r = s$, we have

$$\Sigma \phi(n) x^n = (1 - 2x + 2x^4 - 2x^9 + \ldots)^{-r}, \tag{5·293}$$

$$\log \phi(n) \sim \pi \sqrt{(rn)}. \tag{5·294}$$

[*Added March 28th,* 1917.—Since this paper was written M. G. Valiron ("Sur la croissance du module maximum des séries entières," *Bulletin de la*

Société mathématique de France, Vol. XLIV, 1916, pp. 45—64) has published a number of very interesting theorems concerning power-series which are more or less directly related to ours. M. Valiron considers power-series only, and his point of view is different from ours, in some respects more restricted and in others more general.

He proves in particular that *the necessary and sufficient conditions that*

$$\log M(r) \sim \frac{A}{(1-r)^{\alpha}},$$

where $M(r)$ *is the maximum modulus of* $f(x) = \Sigma a_n x^n$ *for* $|x| = r$, *are that*

$$\log |a_n| < (1 + \epsilon)(1 + \alpha) A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n > n_0(\epsilon)$, *and*

$$\log |a_n| > (1 - \epsilon_p)(1 + \alpha) A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n = n_p$ $(p = 1, 2, 3, \ldots)$, *where* $n_{p+1}/n_p \to 1$ *and* $\epsilon_p \to 0$ *as* $p \to \infty$.

M. Valiron refers to previous, but less general or less precise, results given by Borel (*Leçons sur les séries à termes positifs*, 1902, Ch. V) and by Wiman ("Über dem Zusammenhang zwischen dem Maximal-betrage einer analytischen Funktion und dem grössten Gliede der zugehörigen Taylor'schen Reihe," *Acta Mathematica*, Vol. XXXVII, 1914, pp. 305—326). We may add a reference to Le Roy, "Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle," *Bulletin des sciences mathématiques*, Ser. 2, Vol. XXIV, 1900, pp. 245—268.

We have more recently obtained results concerning $P(n)$, the number of partitions of n, far more precise than (5·22).]

ON THE COEFFICIENTS IN THE EXPANSIONS OF CERTAIN MODULAR FUNCTIONS

(*Proceedings of the Royal Society*, A, xcv, 1919, 144—155)

1. A very large proportion of the most interesting arithmetical functions —of the functions, for example, which occur in the theory of partitions, the theory of the divisors of numbers, or the theory of the representation of numbers by sums of squares—occur as the coefficients in the expansions of elliptic modular functions in powers of the variable $q = e^{\pi i \tau}$. All of these functions have a restricted region of existence, the unit circle $|q| = 1$ being a "natural boundary" or line of essential singularities. The most important of them, such as the functions*

$$(\omega_1/\pi)^{12}\Delta = q^2\{(1-q^2)(1-q^4)\ldots\}^{24}, \tag{1·1}$$

$$\vartheta_3(0) = 1 + 2q + 2q^4 + 2q^9 + \ldots, \tag{1·2}$$

$$12\left(\frac{\omega_1}{\pi}\right)^4 g_2 = 1 + 240\left(\frac{1^3q^2}{1-q^2} + \frac{2^3q^4}{1-q^4} + \ldots\right), \tag{1·3}$$

$$216\left(\frac{\omega_1}{\pi}\right)^6 g_3 = 1 - 504\left(\frac{1^5q^2}{1-q^2} + \frac{2^5q^4}{1-q^4} + \ldots\right), \tag{1·4}$$

are regular inside the unit circle; and many, such as the functions (1·1) and (1·2), have the additional property of having no zeros inside the circle, so that their reciprocals are also regular.

In a series of recent papers† we have applied a new method to the study of these arithmetical functions. Our aim has been to express them as series which exhibit explicitly their order of magnitude, and the genesis of their irregular variations as n increases. We find, for example, for $p(n)$, the number

* We follow, in general, the notation of Tannery and Molk's *Éléments de la théorie des fonctions elliptiques*. Tannery and Molk, however, write $16G$ in place of the more usual Δ.

† (1) G. H. Hardy and S. Ramanujan, "Une formule asymptotique pour le nombre des partitions de n," *Comptes Rendus*, January 2, 1917; (2) G. H. Hardy and S. Ramanujan, "Asymptotic Formulæ in Combinatory Analysis," *Proc. London Math. Soc.*, Ser. 2, Vol. xvii, 1918, pp. 75—115; (3) S. Ramanujan, "On Certain Trigonometrical Sums and their Applications in the Theory of Numbers," *Trans. Camb. Phil. Soc.*, Vol. xxii, 1918, pp. 259—276; (4) G. H. Hardy, "On the Expression of a Number as the Sum of any Number of Squares, and in Particular of Five or Seven," *Proc. National Acad. of Sciences*, Vol. iv, 1918, pp. 189—193: [and G. H. Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five," *Trans. American Math. Soc.*, Vol. xxi, 1920, pp. 255—284].

1918, 2 (with S. Ramanujan) ***Proceedings of the Royal Society*, A, 95, 144–55.**

of unrestricted partitions of n, and for $r_s(n)$, the number of representations of n as the sum of an even number s of squares, the series

(1·5)
$$\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+\frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right)$$
$$+\pi\sqrt{\left(\frac{3}{2}\right)}\cos\left(\frac{2}{3}n\pi-\frac{1}{18}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)+\ldots,$$

where $\lambda_n=\sqrt{\left(n-\frac{1}{24}\right)}$ and $C=\pi\sqrt{\left(\frac{2}{3}\right)}$, and

(1·6)
$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}n^{\frac{1}{2}s-1}\{1^{-\frac{1}{2}s}+2\cos(\tfrac{1}{2}n\pi-\tfrac{1}{4}s\pi)2^{-\frac{1}{2}s}+2\cos(\tfrac{2}{3}n\pi-\tfrac{1}{2}s\pi)3^{-\frac{1}{2}s}+\ldots\};$$

and our methods enable us to write down similar formulæ for a very large variety of other arithmetical functions.

The study of series such as (1·5) and (1·6) raises a number of interesting problems, some of which appear to be exceedingly difficult. The first purpose for which they are intended is that of obtaining approximations to the functions with which they are associated. Sometimes they give also an exact representation of the functions, and sometimes they do not. Thus the sum of the series (1·6) is equal to $r_s(n)$ if s is 4, 6, or 8, but not in any other case. The series (1·5) enables us, by stopping after an appropriate number of terms, to find approximations to $p(n)$ of quite startling accuracy; thus six terms of the series give $p(200)=3972999029388$, a number of 13 figures, with an error of 0·004. But we have never been able to prove that the sum of the series is $p(n)$ exactly, nor even that it is convergent.

There is one class of series, of the same general character as (1·5) or (1·6), which lends itself to comparatively simple treatment. These series arise when the generating modular function $f(q)$ or $\phi(\tau)$ satisfies an equation

$$\phi(\tau)=(a+b\tau)^n\phi\left(\frac{c+d\tau}{a+b\tau}\right),$$

where n is a positive integer, and behaves, inside the unit circle, like a rational function; that is to say, possesses no singularities but poles. The simplest examples of such functions are the reciprocals of the functions (1·3) and (1·4). The coefficients in their expansions are integral, but possess otherwise no particular arithmetical interest. The results, however, are very remarkable from the point of view of approximation; and it is, in any case, well worth while, in view of the many arithmetical applications of this type of series, to study in detail any example in which the results can be obtained by comparatively simple analysis.

We begin by proving a general theorem (Theorem 1) concerning the expression of a modular function with poles as a series of partial fractions.

This series is (as appears in Theorem 2) a "Poincaré's series": what our theorem asserts is, in effect, that the sum of a certain Poincaré's series is the only function which satisfies certain conditions. It would, no doubt, be possible to obtain this result as a corollary from propositions in the general theory of automorphic functions; but we thought it best to give an independent proof, which is interesting in itself and demands no knowledge of this theory.

2. THEOREM 1. *Suppose that*

$$f(q) = f(e^{\pi i \tau}) = \phi(\tau) \tag{2·1}$$

is regular for $q=0$, *has no singularities save poles within the unit circle, and satisfies the functional equation*

$$\phi(\tau) = (a+b\tau)^n \phi\left(\frac{c+d\tau}{a+b\tau}\right) = (a+b\tau)^n \phi(T), \tag{2·2}$$

n being a positive integer and a, b, c, d any integers such that $ad-bc=1$. *Then*

$$f(q) = \Sigma R, \tag{2·3}$$

where R is a residue of $\qquad f(x)/(q-x)$

at a pole of $f(x)$, *if* $|q|<1$; *while if* $|q|>1$ *the sum of the series on the right-hand side of* (2·3) *is zero.*

The proof requires certain geometrical preliminaries.

3. The half-plane $\mathbf{I}(\tau)>0$, which corresponds to the inside of the unit circle in the plane of q, is divided up, by the substitutions of the modular group, into a series of triangles whose sides are arcs of circles and whose angles are $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, and 0*. One of these, which is called the *fundamental polygon* (P)†, has its vertices at the points ρ, ρ^2, and $i\infty$, where $\rho = e^{\frac{1}{3}\pi i}$, and its sides are parts of the unit circle $|\tau|=1$ and the lines $\mathbf{R}(\tau) = \pm\frac{1}{2}$.

Suppose that F_m is the "Farey's series" of order m, that is to say the aggregate of the rational fractions between 0 and 1, whose denominators are not greater than m, arranged in order of magnitude‡, and that h'/k' and h/k, where $0 < h'/k' < h/k < 1$, are two adjacent terms in the series. We shall consider what regions in the τ-plane correspond to P in the T-plane, when

$$T = -\frac{h'-k'\tau}{h-k\tau}, \tag{3·1}$$ $$T = \frac{h-k\tau}{h'-k'\tau}. \tag{3·2}$$

Both of these substitutions belong to the modular group, since $hk'-h'k=1$. The points $i\infty$, $\frac{1}{2}$, $-\frac{1}{2}$, in the T-plane correspond to h/k, $(h+2h')/(k+2k')$,

* It is for many purposes necessary to divide each triangle into two, whose angles are $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, and 0; but this further subdivision is not required for our present purpose. For the detailed theory of the modular group, see Klein-Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunktionen*, 1890—1892.

† See Fig. 1.

‡ The first and last terms are 0/1 and 1/1. A brief account of the properties of Farey's series is given in § 4·2 of our paper (2).

$(h-2h')/(k-2k')$ in the τ-plane. Thus the lines $\mathbf{R}(T)=\frac{1}{2}$, $\mathbf{R}(T)=-\frac{1}{2}$ correspond to semicircles described on the segments

$$\left(\frac{h}{k}, \frac{h+2h'}{k+2k'}\right), \quad \left(\frac{h}{k}, \frac{h-2h'}{k-2k'}\right)$$

respectively as diameters. Similarly the upper half of the unit circle corresponds to a semicircle on the segment

$$\left(\frac{h+h'}{k+k'}, \frac{h-h'}{k-k'}\right).$$

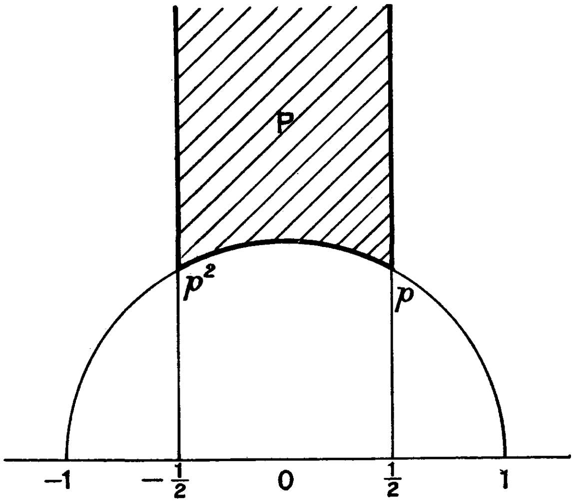


Fig. 1.

The polygon P corresponds to the region bounded by these three semicircles. In particular, the right-hand edge of P corresponds to a circular arc stretching from h/k (where it cuts the real axis at right angles) to the point

$$\text{(3·3)} \qquad \frac{h'k'+hk+\frac{1}{2}(hk'+h'k)+\frac{1}{2}i\sqrt{3}}{k^2+kk'+k'^2}$$

corresponding to $\tau=\rho$.

Similarly we find that the substitution (3·2) correlates to P a triangle bounded by semicircles on the segments

$$\left(\frac{h'}{k'}, \frac{h'-2h}{k'-2k}\right), \quad \left(\frac{h'}{k'}, \frac{h'+2h}{k'+2k}\right), \quad \left(\frac{h'-h}{k'-k}, \frac{h'+h}{k'+k}\right).$$

In particular, the left-hand edge of P corresponds to a circular arc from h'/k' to the point (3·3). These two arcs, meeting at the point (3·3), form a continuous path ω, connecting h/k and h'/k', every point of which corresponds, in virtue of one or other of the substitutions (3·1) and (3·2), to a point on one of the rectilinear boundaries of P*.

* Fig. 2 illustrates the case in which $h/k=\frac{2}{5}$, $h'/k'=\frac{1}{2}$. These fractions are adjacent in F_5 and F_6, but not in F_7.

Performing a similar construction for every pair of adjacent fractions of F_m, we obtain a continuous path from $\tau=0$ to $\tau=1$. This path, and its reflexion in the imaginary axis, give a continuous path from $\tau=-1$ to $\tau=1$, which we shall denote by Ω_m. To Ω_m corresponds a path in the q-plane, which we call H_m; H_m is a closed path, formed entirely by arcs of circles which cut the unit circle at right angles.

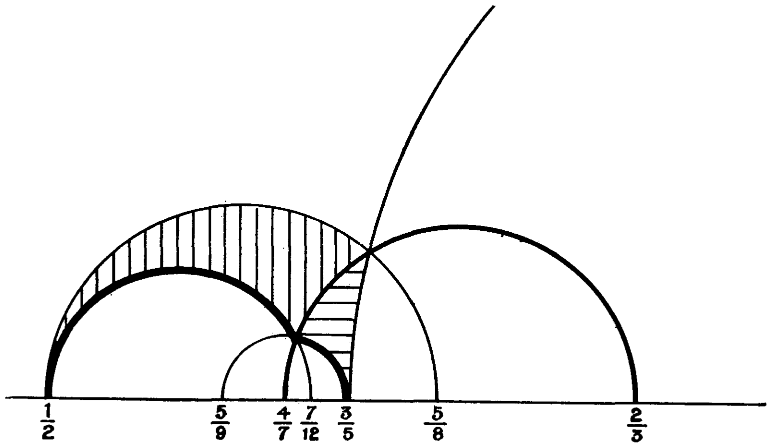


Fig. 2.

The region shaded horizontally corresponds to P for the substitution (3·1), that shaded vertically for the substitution (3·2). The thickest lines shew the path ω; the line of medium thickness shews the semicircle which corresponds (for either substitution) to the unit semicircle in the plane of T. The large incomplete semicircle passes through $\tau=1$.

Since $$\frac{h'}{k'}<\frac{h'+2h}{k'+2k}, \qquad \frac{h+2h'}{k+2k'}<\frac{h}{k},$$

the path ω from h'/k' to h/k is always passing from left to right, and its length is less than twice that of the semicircle on $(h'/k', h/k)$, i.e., than π/kk'. The total length of Ω_m is less than 2π; and, since

$$\left|\frac{dq}{d\tau}\right| = \left|\pi i e^{\pi i\tau}\right| \leqslant \pi,$$

the length of H_m is less than $2\pi^2$. Finally, we observe that the maximum distance of Ω_m from the real axis is less than half the maximum distance between two adjacent terms of F_m, and so less than $1/2m$*. Hence Ω_m tends uniformly to the real axis, and H_m to the unit circle, when $m\to\infty$.

4. The function $\phi(\tau)$ can have but a finite number of poles in P; we suppose, for simplicity, that none of them lie on the boundary. There is then a constant K such that $|f(q)|<K$ on the boundary of P.

* See Lemma 4·22 of our paper (2).

We now consider the integral

$$(4\cdot 1) \qquad \frac{1}{2\pi i}\int \frac{f(x)}{x-q}\,dx,$$

where $|q|<1$ and the contour of integration is H_m*. By Cauchy's Theorem, the integral is equal to

$$f(q)-\Sigma R,$$

where R is a residue of $f(x)/(q-x)$ at a pole of $f(x)$ inside H_m†. To prove our theorem, then, we have merely to shew that the integral (4·1) tends to zero when $m\to\infty$.

Let ω_1' and ω_1 be the left- and right-hand parts of ω, and ζ_1', ζ_1 and ζ the corresponding arcs of H_m. The length of ω_1 is, as we have seen, less than $\frac{1}{2}\pi/kk'$, and that of ζ_1 than $\frac{1}{2}\pi^2/kk'$. Further, we have, on ζ_1,

$$\left|f(x)\right| = \left|\phi(\tau)\right| = \left|h-k\tau\right|^n \left|\phi(T)\right| < K\left\{k\left(\frac{h}{k}-\frac{h'}{k'}\right)\right\}^n = \frac{K}{k'^n}.$$

Thus the contribution of ζ_1 to the integral is numerically less than $C/(kk'^{n+1})$, where C is independent of m; and the whole integral (4·1) is numerically less than

$$(4\cdot 2) \qquad 2C\Sigma\frac{1}{kk'}\left(\frac{1}{k^n}+\frac{1}{k'^n}\right),$$

where the summation extends to all pairs of adjacent terms of F_m.

When ν is fixed and $m>\nu$, the number of terms of F_m whose denominators are less than ν is a function of ν only, say $N(\nu)$. If h/k is one of these, and h'/k' is adjacent to it, $k+k'>m$‡, and so $k'>m-\nu$. Thus the terms of (4·2) in which either k or k' is less than ν contribute less than $8CN(\nu)/(m-\nu)$. The remaining terms contribute less than

$$\frac{4C}{\nu^n}\Sigma\frac{1}{kk'}=\frac{4C}{\nu^n}.$$

Hence the sum (4·3) is less than

$$\frac{8CN(\nu)}{m-\nu}+\frac{4C}{\nu^n},$$

and it is plain that, by choice of first ν and then m, this may be made as small as we please. Thus (4·1) tends to zero and the theorem is proved. It should be observed that ΣR must, for the present at any rate, be interpreted as meaning the limit of the sum of terms corresponding to poles inside H_m; we have not established the absolute convergence of the series.

* Strictly speaking, $f(x)$ is not defined at the points where H_m meets the unit circle, and we should integrate round a path just inside H_m and proceed to the limit. The point is trivial, as $f(x)$, in virtue of the functional equation, tends to zero when we approach a cusp of H_m from inside.

† We suppose m large enough to ensure that $x=q$ lies inside H_m.

‡ See our paper (2), *loc. cit.*

We supposed that no pole of $\phi(\tau)$ lies on the boundary of P. This restriction, however, is in no way essential; if it is not satisfied, we have only to select our "fundamental polygon" somewhat differently. The theorem is consequently true independently of any such restriction.

So far we have supposed $|q|<1$. It is plain that, if $|q|>1$, the same reasoning proves that

$$\Sigma R = 0. \tag{4·3}$$

5. Suppose in particular that $\phi(\tau)$ has one pole only, and that a simple pole at $\tau=\alpha$, with residue A. The complete system of poles is then given by

$$\tau = \mathbf{a} = \frac{c+d\alpha}{a+b\alpha} \quad (ad-bc=1). \tag{5·1}$$

If a and b are fixed, and (c, d) is one pair of solutions of $ad-bc=1$, the complete system of solutions is $(c+ma, d+mb)$, where m is an integer. To each pair (a, b) correspond an infinity of poles in the plane of τ; but these poles correspond to two different poles only in the plane of q, viz.,

$$q = \pm \mathbf{q} = \pm e^{\pi i \mathbf{a}}, \tag{5·2}$$

the positive and negative signs corresponding to even and odd values of m respectively. It is to be observed, moreover, that different pairs (a, b) may give rise to the same pole $\mathbf{q}$.

The residue of $\phi(\tau)$ for $\tau=\mathbf{a}$ is, in virtue of the functional equation (2·2),

$$\frac{A}{(a+b\alpha)^{n+2}};$$

and the residue of $f(q)$ for $q=\mathbf{q}$ is

$$\frac{A}{(a+b\alpha)^{n+2}}\left(\frac{dq}{d\tau}\right)_{\tau=\mathbf{a}} = \frac{\pi i A \mathbf{q}}{(a+b\alpha)^{n+2}}.$$

Thus the sum of the terms of our series which correspond to the poles (5·2) is

$$\frac{\pi i A}{(a+b\alpha)^{n+2}}\left(\frac{\mathbf{q}}{q-\mathbf{q}} - \frac{\mathbf{q}}{q+\mathbf{q}}\right) = \frac{2\pi i A}{(a+b\alpha)^{n+2}}\frac{\mathbf{q}^2}{q^2-\mathbf{q}^2}.$$

We thus obtain:

THEOREM 2. *If $\phi(\tau)$ has one pole only in P, viz., a simple pole at $\tau=\alpha$, with residue A, and $|q|<1$, then*

$$f(q) = 2\pi i A \Sigma \frac{1}{(a+b\alpha)^{n+2}}\frac{\mathbf{q}^2}{q^2-\mathbf{q}^2}, \tag{5·3}$$

where

$$\mathbf{q} = \exp\left(\frac{c+d\alpha}{a+b\alpha}\right)\pi i;$$

c, d being any pair of solutions of $ad-bc=1$, and the summation extending over all pairs a, b, which give rise to distinct values of $\mathbf{q}$. If $|q|>1$, the sum of the series on the right-hand side of (5·3) *is zero.*

If $\phi(\tau)$ has several poles in P, $f(q)$, of course, will be the sum of a number of series such as (5·3). Incidentally, we may observe that it now appears that the series in question are absolutely convergent.

6. As an example, we select the function

$$(6{\cdot}1) \qquad f(q) = \frac{\pi^6}{216\omega_1{}^6 g_3} = 1 \Big/ \left(1 - 504 \sum_1^\infty \frac{r^5 q^{2r}}{1 - q^{2r}}\right) = \sum_0^\infty p_n x^n,$$

say, where $x = q^2$. It is evident that p_n is always an integer; the values of the first 13 coefficients are

$$p_0 = 1, \quad p_1 = 504, \quad p_2 = 270648, \quad p_3 = 144912096,$$
$$p_4 = 77599626552, \quad p_5 = 41553943041744, \quad p_6 = 22251789971649504,$$
$$p_7 = 11915647845248387520, \quad p_8 = 6380729991419236488504,$$
$$p_9 = 3416827666558895485479576,$$
$$p_{10} = 1829682703808504464920468048,$$
$$p_{11} = 979779820147442370107345764512,$$
$$p_{12} = 524663917940510191509934144603104;$$

so that p_{12} is a number of 33 figures.

By means of the formulæ*

$$g_3 = \tfrac{8}{27}(e_1 - e_3)^2 (1 + k^2)(1 - \tfrac{1}{2}k^2)(1 - 2k^2),$$
$$e_1 - e_3 = \left(\frac{\pi}{2\omega_1}\right)^2 \{\vartheta_3(0)\}^4, \quad \frac{2K}{\pi} = \{\vartheta_3(0)\}^2,$$

we find that
$$\frac{1}{f(q)} = \left(\frac{2K}{\pi}\right)^6 (1 + k^2)(1 - \tfrac{1}{2}k^2)(1 - 2k^2).$$

The value of n is 6. The poles of $f(q)$ correspond to the values of τ which make $K = k^2$ equal to -1, 2, or $\frac{1}{2}$. It is easily verified† that these values are given by the general formula

$$\tau = \frac{c + di}{a + bi} \quad (ad - bc = 1),$$

so that

$$(6{\cdot}2) \qquad q = \exp\left(\frac{c + di}{a + bi}\pi i\right) = \exp\left(\frac{ac + bd}{a^2 + b^2}\pi i - \frac{\pi}{a^2 + b^2}\right).$$

The value of α is i‡. In order to determine A we observe that

$$-504 \frac{d}{dq}\left(\frac{1^5 q^2}{1 - q^2} + \frac{2^5 q^4}{1 - q^4} + \dots\right) = -\frac{1008}{q}\left\{\frac{1^6 q^2}{(1 - q^2)^2} + \frac{2^6 q^4}{(1 - q^4)^2} + \dots\right\}.$$

* All the formulæ which we quote are given in Tannery and Molk's Tables; see in particular Tables XXXVI (3), LXXI (3), XCVI, CX (3).

† A full account of the problem of finding τ when κ is given will be found in Tannery and Molk, *loc. cit.*, Vol. III, ch. 7 ("On donne k^2 ou g_2, g_3; trouver τ ou ω_1, ω_3").

‡ It will be observed that in this case α is on the boundary of P; see the concluding remarks of § 4. As it happens, $\tau = i$ lies on that edge of P (the circular edge) which was not used in the construction of H_m, so that our analysis is applicable as it stands.

The series in curly brackets is the function called by Ramanujan* $\Phi_{1,6}$ and†

$$1008\Phi_{1,6} = Q^2 - PR,$$

where $$P = \frac{12\eta_1\omega_1}{\pi^2}, \quad Q = 12g_2\left(\frac{\omega_1}{\pi}\right)^4, \quad R = 216g_3\left(\frac{\omega_1}{\pi}\right)^6.$$

Here $R = 0$, so that

$$1008\Phi_{1,6} = Q^2 = 1 + 480\Phi_{0,7}\ddagger = 1 + 480\left(\frac{1^7q^2}{1-q^2} + \frac{2^7q^4}{1-q^4} + \dots\right).$$

Hence we find that $\quad A = i/\pi C, \quad 2\pi i A = -2/C,$

where

(6·3) $$C = 1 + 480\left(\frac{1^7}{e^{2\pi}-1} + \frac{2^7}{e^{4\pi}-1} + \dots\right).$$

Another expression for C is

(6·4) $$C = 144\left(\frac{K_0}{\pi}\right)^8,$$

where

(6·41) $$K_0 = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{\pi}}.$$

We have still to consider more closely the values of a and b, over which the summation is effected. Let us fix k, and suppose that (a, b) is a pair of positive solutions of the equation $a^2 + b^2 = k$. This pair gives rise to a system of eight solutions, viz.,

$$(\pm a, \pm b), \quad (\pm b, \pm a).$$

But it is obvious that, if we change the signs of both a and b, we do not affect the aggregate of values of $\mathbf{a}$. Thus we need only consider the four pairs

$$(a, b), \quad (a, -b), \quad (b, a), \quad (b, -a).$$

If a or b is zero, or if $a = b$, these four pairs reduce to two.

It is easily verified that, if (a, b) leads to the pair of poles

$$q = \pm \mathbf{q} = \pm \exp\left(\frac{ac+bd}{a^2+b^2}\pi i - \frac{\pi}{a^2+b^2}\right),$$

then $(a, -b)$ and (b, a) each lead to $q = \pm\bar{\mathbf{q}}$, where $\bar{\mathbf{q}}$ is the conjugate of $\mathbf{q}$. Thus, in general (a, b) and the solutions derived from it lead to four distinct poles, viz., $\pm\mathbf{q}$ and $\pm\bar{\mathbf{q}}$. These four reduce to two in two cases, when $\mathbf{q}$ is real, so that $\mathbf{q} = \bar{\mathbf{q}}$, and when $\mathbf{q}$ is purely imaginary, so that $\mathbf{q} = -\bar{\mathbf{q}}$. It is

* S. Ramanujan, "On Certain Arithmetical Functions," *Trans. Camb. Phil. Soc.*, Vol. XXII, pp. 159—184 (p. 163).

† Ramanujan, *loc. cit.*, p. 164.

‡ Ramanujan, *loc. cit.*, p. 163.

easy to see that the first case can occur only when $k=1$, and the second when $k=2$*.

If $k=1$ we take $a=1$, $b=0$, $c=0$, $d=1$; and $\mathfrak{q}=\bar{\mathfrak{q}}=e^{-\pi}$. If $k=2$ we take $a=1$, $b=1$, $c=0$, $d=1$; and $\mathfrak{q}=-\bar{\mathfrak{q}}=ie^{-\frac{1}{2}\pi}$. The corresponding terms in our series are

$$\frac{1}{1-q^2e^{2\pi}},\quad \frac{1}{2^4(1+qe^{\pi})}.$$

If k is greater than 2, and is the sum of two coprime squares a^2 and b^2, it gives rise to terms

$$\frac{1}{(a+bi)^8}\frac{1}{1-(q/\mathfrak{q})^2}+\frac{1}{(a-bi)^8}\frac{1}{1-(q/\bar{\mathfrak{q}})^2}.$$

There is, of course, a similar pair of terms corresponding to every other distinct representation of k as a sum of coprime squares. Thus finally we obtain the following result:

THEOREM 3. *If*

$$f(q)=\frac{\pi^6}{216\omega_1^6 g_3}=1\Big/\left(1-504\sum_1^\infty\frac{r^5q^{2r}}{1-q^{2r}}\right)=\sum_0^\infty p_n q^{2n},$$

and $|q|<1$, *then*

$$(6\cdot5)\qquad \tfrac{1}{2}Cf(q)=\frac{1}{1-q^2e^{2\pi}}+\frac{1}{2^4(1+q^2e^{\pi})}$$
$$+\sum\left\{\frac{1}{(a+bi)^8}\frac{1}{1-(q/\mathfrak{q})^2}+\frac{1}{(a-bi)^8}\frac{1}{1-(q/\bar{\mathfrak{q}})^2}\right\};$$

where
$$C=1+480\left(\frac{1^7}{e^{2\pi}-1}+\frac{2^7}{e^{4\pi}-1}+\dots\right)=\frac{9\pi^4}{16\{\Gamma(\frac{3}{4})\}^{16}},$$

$$\mathfrak{q}=\exp\left(\frac{c+di}{a+bi}\pi i\right)=\exp\left(\frac{ac+bd}{a^2+b^2}\pi i-\frac{\pi}{a^2+b^2}\right),$$

and $\bar{\mathfrak{q}}$ *is the conjugate of* $\mathfrak{q}$. *The summation applies to every pair of coprime positive numbers* a *and* b, *such that* $k=a^2+b^2\geqslant5$, *such pairs, however, only being counted as distinct if they correspond to independent representations of* k *as a sum of squares. If* $|q|>1$, *then the sum of the series on the right-hand side of* (6·5) *is zero.*

* When a and b are given, we can always choose c and d so that $|ac+bd|\leqslant\frac{1}{2}(a^2+b^2)$. If $\mathfrak{q}$ is real, we have $ad-bc=1$ and $ac+bd=0$ simultaneously: whence

$$(a^2+b^2)(c^2+d^2)=1.$$

If $\mathfrak{q}$ is purely imaginary, we have

$$ad-bc=1,\quad 2|ac+bd|=a^2+b^2,$$

whence
$$(c^2+d^2)^2=(|ac+bd|-c^2-d^2)^2+1.$$

This is possible only if $c^2+d^2=1$ and $|ac+bd|=1$, whence $a^2+b^2=2$.

7. It follows that

$$(7\cdot1)\quad \tfrac{1}{2}Cp_n = e^{2n\pi} + \frac{(-1)^n}{2^4}e^{n\pi} + \Sigma\left\{\frac{1}{(a+bi)^8}\mathfrak{q}^{-2n} + \frac{1}{(a-bi)^8}\bar{\mathfrak{q}}^{-2n}\right\} = \sum_{(\lambda)}\frac{c_\lambda(n)}{\lambda^4}e^{2n\pi/\lambda},$$

say. Here λ is the sum of two coprime squares, so that

$$\lambda = 2^{a_2}5^{a_5}13^{a_{13}}17^{a_{17}}\ldots,$$

where a_2 is 0 or 1 and 5, 13, 17, ... are the primes of the form $4k+1$; and the first few values of $c_\lambda(n)$ are

$$c_1(n)=1,\quad c_2(n)=(-1)^n,\quad c_5(n)=2\cos\left(\tfrac{4}{5}n\pi + 8\arctan 2\right),$$

$$c_{10}(n)=2\cos\left(\tfrac{3}{5}n\pi - 8\arctan 2\right),\quad c_{13}(n)=2\cos\left(\tfrac{10}{13}n\pi + 8\arctan 5\right).$$

The approximations to the coefficients given by the formula (7·1) are exceedingly remarkable. Dividing by $\frac{1}{2}C$, and taking $n=0$, 1, 2, 3, 6, and 12, we find the following results:

(0)	(1)	(2)
0·944	505·361	270616·406
+0·059	−1·365	+31·585
−0·003	+0·004	+0·009
$p_0=1\cdot000$	$p_1=504\cdot000$	$p_2=270648\cdot000$

(3)	(6)
144912827·002	22251789962592450·237
−730·900	+9057051·688
−0·101	+2·081
−0·001	−0·006
$p_3=144912096\cdot000$	$p_6=22251789971649504\cdot000$

(12)
5246639179405101901191972719383395·329
+1390736872662028·140
+2680·418
+0·130
−0·014
−0·003
$p_{12}=52466391794051019150993414460310 4\cdot000$

An alternative expression for C is

$$C = 96^2 e^{-8\pi/3}\{(1-e^{-4\pi})(1-e^{-8\pi})\ldots\}^{16},$$

by means of which C may be calculated with great accuracy*. To five places we have $2/C = 0\cdot94373$, which is very nearly equal to $352/373 = 0\cdot94370$.

* Gauss, *Werke*, Vol. III, pp. 418—419, gives the values of various powers of $e^{-\pi}$ to a large number of figures.

It is easy to see directly that p_n lies between the coefficients of x^n in the expansions of

$$\frac{1}{(1-535x)(1+31x)}, \qquad \frac{1-7{\cdot}5x}{(1-535{\cdot}5x)(1+24x)},$$

and so that

$$\frac{(535)^{n+1}-(-31)^{n+1}}{566} \leqslant p_n \leqslant \frac{352\,(535{\cdot}5)^n+21\,(-24)^n}{373}.$$

The function
$$\Omega(x)=\sum_{(\lambda)}\frac{c_\lambda(x)}{\lambda^4}e^{2x\pi/\lambda}$$

has very remarkable properties. It is an integral function of x, whose maximum modulus is less than a constant multiple of $e^{2\pi|x|}$. It is equal to p_n, an integer, when $x=n$, a positive integer; and to zero when $x=-n$. But we must reserve the discussion of these peculiarities for some other occasion.

ASYMPTOTIC FORMULÆ IN COMBINATORY ANALYSIS*

(*Proceedings of the London Mathematical Society*, 2, XVII, 1918, 75—115)

1. Introduction and summary of results.

1·1. The present paper is the outcome of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory of analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represent certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$(1·11) \qquad f(x) = 1 + \sum_1^\infty p(n)\, x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}\dagger.$$

If we write

$$(1·12) \qquad x = q^2 = e^{2\pi i\tau},$$

where the imaginary part of τ is positive, we see that $f(x)$ is substantially the reciprocal of the modular function called by Tannery and Molk‡ $h(\tau)$; that, in fact,

$$(1·13) \qquad h(\tau) = q^{\frac{1}{12}} q_0 = q^{\frac{1}{12}} \prod_1^\infty (1-q^{2n}) = \frac{x^{\frac{1}{24}}}{f(x)}.$$

The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities—many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formulæ, one may fairly say, there are none§. So true is this, in fact, that we have

* A short abstract of the contents of part of this paper appeared under the title "Une formule asymptotique pour le nombre des partitions de n," in the *Comptes Rendus*, January 2nd, 1917.

† P. A. MacMahon, *Combinatory Analysis*, Vol. II, 1916, p. 1.

‡ J. Tannery and J. Molk, *Fonctions elliptiques*, Vol. II, 1896, pp. 31 *et seq.* We shall follow the notation of this work whenever we have to quote formulæ from the theory of elliptic functions.

§ We should mention one exception to this statement, to which our attention was called by Major MacMahon. The number of partitions of n *into parts none of which exceed* r is the coefficient $p_r(n)$ in the series

$$1 + \sum_1^\infty p_r(n)\, x^n = \frac{1}{(1-x)(1-x^2)\dots(1-x^r)}.$$

This function has been studied in much detail, for various special values of r, by Cayley,

1918, 5 (with S. Ramanujan) *Proceedings of the London Mathematical Society*, (2) 17, 75–115.

been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of $p(n)$.

1·2. The function $p(n)$ may, of course, be expressed in the form of an integral

$$p(n) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(x)}{x^{n+1}}\,dx, \tag{1·21}$$

by means of Cauchy's theorem, the path Γ enclosing the origin and lying entirely inside the unit circle. The idea which dominates this paper is that of obtaining asymptotic formulæ for $p(n)$ by a detailed study of the integral (1·21). This idea is an extremely obvious one; it is the idea which has dominated nine-tenths of modern research in the analytic theory of numbers: and it may seem very strange that it should never have been applied to this particular problem before. Of this there are no doubt two explanations. The first is that the theory of partitions has received its most important developments, since its foundation by Euler, at the hands of a series of mathematicians whose interests have lain primarily in algebra. The second and more fundamental reason is to be found in the extreme complexity of the behaviour of the generating function $f(x)$ near a point of the unit circle.

It is instructive to contrast this problem with the corresponding problems which arise for the arithmetical functions $\pi(n)$, $\vartheta(n)$, $\psi(n)$, $\mu(n)$, $d(n)$, ... which have their genesis in Riemann's Zeta-function and the functions allied

Sylvester, and Glaisher: we may refer in particular to J. J. Sylvester, "On a discovery in the theory of partitions," *Quarterly Journal*, Vol. I, 1857, pp. 81—85, and "On the partition of numbers," *ibid.*, pp. 141—152 (Sylvester's *Works*, Vol. II, pp. 86—89 and 90—99); J. W. L. Glaisher, "On the number of partitions of a number into a given number of parts," *Quarterly Journal*, Vol. XL, 1909, pp. 57—143; "Formulæ for partitions into given elements, derived from Sylvester's Theorem," *ibid.*, pp. 275—348; "Formulæ for the number of partitions of a number into the elements 1, 2, 3, ..., n up to $n=9$," *ibid.*, Vol. XLI, 1910, pp. 94—112: and further references will be found in MacMahon, *loc. cit.*, pp. 59—71, and E. Netto, *Lehrbuch der Combinatorik*, 1901, pp. 146—158. Thus, for example, the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

is

$$p_3(n) = \tfrac{1}{12}(n+3)^2 - \tfrac{7}{72} + \tfrac{1}{8}(-1)^n + \tfrac{2}{9}\cos\frac{2n\pi}{3};$$

as is easily found by separating the function into partial fractions. This function may also be expressed in the forms

$$\tfrac{1}{12}(n+3)^2 + (\tfrac{1}{2}\cos\tfrac{1}{2}\pi n)^2 - (\tfrac{2}{3}\sin\tfrac{1}{3}\pi n)^2,$$

$$1 + [\tfrac{1}{12}n(n+6)], \quad \{\tfrac{1}{12}(n+3)^2\},$$

where $[n]$ and $\{n\}$ denote the greatest integer contained in n and the integer nearest to n. These formulæ do, of course, furnish incidentally asymptotic formulæ for the functions in question. But they are, from this point of view, of a very trivial character: the interest which they possess is algebraical.

to it. In the latter problems we are dealing with functions defined by Dirichlet's series. The study of such functions presents difficulties far more fundamental than any which confront us in the theory of the modular functions. These difficulties, however, relate to the distribution of the zeros of the functions and their general behaviour at infinity: no difficulties whatever are occasioned by the crude singularities of the functions in the finite part of the plane. The single finite singularity of $\zeta(s)$, for example, the pole at $s=1$, is a singularity of the simplest possible character. It is this pole which gives rise to the *dominant* terms in the asymptotic formulæ for the arithmetical functions associated with $\zeta(s)$. To prove such a formula rigorously is often exceedingly difficult; to determine precisely the order of the error which it involves is in many cases a problem which still defies the utmost resources of analysis. But to write down the dominant terms involves, as a rule, no difficulty more formidable than that of deforming a path of integration over a pole of the subject of integration and calculating the corresponding residue.

In the theory of partitions, on the other hand, we are dealing with functions which do not exist at all outside the unit circle. Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a "dominant term."

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulæ of the theory of the linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behaviour of $f(x)$ near any assigned point of the unit circle*. It is to an appropriate use of these formulæ that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

1·3. It is very important, in dealing with such a problem as this, to distinguish clearly the various stages to which we can progress by arguments of a progressively "deeper" and less elementary character. The earlier results are naturally (so far as the particular problem is concerned) superseded by the later. But the more elementary methods are likely to be applicable to other problems in which the more subtle analysis is impracticable.

We have attacked this particular problem by a considerable number of different methods, and cannot profess to have reached any very precise conclusions as to the possibilities of each. A detailed comparison of the results

* See G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation (II: The trigonometrical series associated with the elliptic Theta-functions)," *Acta Mathematica*, Vol. XXXVII, 1914, pp. 193—238, for applications of the formulæ to different but not unrelated problems.

to which they lead would moreover expand this paper to a quite unreasonable length. But we have thought it worth while to include a short account of two of them. The first is quite elementary; it depends only on Euler's identity

(1·31)
$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \dots$$

—an identity capable of wide generalisation—and on elementary algebraical reasoning. By these means we shew, in section 2, that

(1·32)
$$e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}},$$

where A and B are positive constants, for all sufficiently large values of n.

It follows that

(1·33)
$$A\sqrt{n} < \log p(n) < B\sqrt{n};$$

and the next question which arises is the question whether a constant C exists such that

(1·34)
$$\log p(n) \sim C\sqrt{n}.$$

We prove that this is so in section 3. Our proof is still, in a sense, "elementary." It does not appeal to the theory of analytic functions, depending only on a general arithmetic theorem concerning infinite series; but this theorem is of the difficult and delicate type which Messrs Hardy and Littlewood have called "Tauberian." The actual theorem required was proved by us in a paper recently printed in these *Proceedings**. It shews that

(1·35)
$$C = \frac{2\pi}{\sqrt{6}};$$

in other words that

(1·36)
$$p(n) = \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}(1+\epsilon)\right\},$$

where ϵ is small when n is large. This method is one of very wide application. It may be used, for example, to prove that, if $p^{(s)}(n)$ denotes the number of partitions of n into perfect s-th powers, then

$$\log p^{(s)}(n) \sim (s+1)\left\{\frac{1}{s}\Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right)\right\}^{s/(s+1)} n^{1/(s+1)}.$$

It is certainly possible to obtain, by means of arguments of this general character, information about $p(n)$ more precise than that furnished by the formula (1·36). And it is equally possible to prove (1·36) by reasoning of a more elementary, though more special, character: we have a proof, for example, based on the identity

$$np(n) = \sum_{\nu=1}^{n} \sigma(\nu)\, p(n-\nu),$$

* G. H. Hardy and S. Ramanujan, "Asymptotic formulæ for the distribution of integers of various types," *Proc. London Math. Soc.*, Ser. 2, Vol. XVI, 1917, pp. 112—132.

where $\sigma(\nu)$ is the sum of the divisors of ν, and a process of induction. But we are at present unable to obtain, by any method which does not depend upon Cauchy's theorem, a result as precise as that which we state in the next paragraph, a result, that is to say, which is "vraiment asymptotique."

1·4. Our next step was to replace (1·36) by the much more precise formula

$$(1\cdot41)\qquad p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}.$$

The proof of this formula appears to necessitate the use of much more powerful machinery, Cauchy's integral (1·21) and the functional relation

$$(1\cdot42)\qquad f(x) = \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)} \exp\left\{\frac{\pi^2}{6\log(1/x)}\right\} f(x'),$$

where

$$(1\cdot43)\qquad x' = \exp\left\{-\frac{4\pi^2}{\log(1/x)}\right\}.$$

This formula is merely a statement in a different notation of the relation between $h(\tau)$ and $h(\mathrm{T})$, where

$$\mathrm{T} = \frac{c+d\tau}{a+b\tau}, \quad a=d=0, \quad b=1, \quad c=-1;$$

viz.

$$h(\tau) = \sqrt{\left(\frac{i}{\tau}\right)}\, h(\mathrm{T})^*.$$

It is interesting to observe the correspondence between (1·41) and the results of numerical computation. Numerical data furnished to us by Major MacMahon gave the following results: we denote the right-hand side of (1·41) by $\varpi(n)$.

n	$p(n)$	$\varpi(n)$	ϖ/p
10	42	48·104	1·145
20	627	692·385	1·104
50	204226	217590·499	1·065
80	15796476	16606781·567	1·051

It will be observed that the progress of ϖ/p towards its limit unity is not very rapid, and that $\varpi - p$ is always positive and appears to tend rapidly to infinity.

* Tannery and Molk, *loc. cit.*, p. 265 (Table XLV, 5).

1·5. In order to obtain more satisfactory results it is necessary to construct some auxiliary function $F(x)$ which is regular at all points of the unit circle save $x=1$, and has there a singularity of a type as near as possible to that of the singularity of $f(x)$. We may then hope to obtain a much more precise approximation by applying Cauchy's theorem to $f-F$ instead of to f. For although every point of the circle is a singular point of f, $x=1$ is, to put it roughly, much the *heaviest* singularity. When $x \rightarrow 1$ by real values, $f(x)$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6(1-x)}\right\};$$

when

$$x = re^{2p\pi i/q},$$

p and q being co-prime integers, and $r \rightarrow 1$, $|f(x)|$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6q^2(1-r)}\right\};$$

while, if

$$x = re^{2\theta\pi i},$$

where θ is irrational, $|f(x)|$ can become infinite at most like an exponential of the type

$$\exp\left\{o\left(\frac{1}{1-r}\right)\right\}*.$$

The function required is

$$F(x) = \frac{1}{\pi\sqrt{2}} \sum_1^\infty \psi(n) x^n, \tag{1·51}$$

where

$$\psi(n) = \frac{d}{dn}\left\{\frac{\cosh C\lambda_n - 1}{\lambda_n}\right\}, \tag{1·52}$$

$$C = 2\pi/\sqrt{6} = \pi\sqrt{(\tfrac{2}{3})}, \quad \lambda_n = \sqrt{(n - \tfrac{1}{24})}. \tag{1·53}$$

This function may be transformed into an integral by means of a general formula given by Lindelöf†; and it is then easy to prove that the "principal branch" of $F(x)$ is regular all over the plane except at $x=1$‡; and that

$$F(x) - \chi(x),$$

* The statements concerning the "rational" points are corollaries of the formulæ of the transformation theory, and proofs of them are contained in the body of the paper. The proposition concerning "irrational" points may be proved by arguments similar to those used by Hardy and Littlewood in their memoir already quoted. It is not needed for our present purpose. As a matter of fact it is *generally* true that $f(x) \rightarrow 0$ when θ is irrational, and very nearly as rapidly as $\sqrt[4]{(1-r)}$. It is in reality owing to this that our final method is so successful.

† E. Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions* (Gauthier-Villars, Collection Borel, 1905), p. 111.

‡ We speak, of course, of the principal branch of the function, viz. that represented by the series (1·51) when x is small. The other branches are singular at the origin.

where

$$(1{\cdot}54) \qquad \chi(x) = \frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\left[\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\} - 1\right]$$

is regular for $x = 1$. If we compare (1·42) and (1·54), and observe that $f(x')$ tends to unity with extreme rapidity when x tends to 1 along any regular path which does not touch the circle of convergence, we can see at once the very close similarity between the behaviour of f and F inside the unit circle and in the neighbourhood of $x = 1$.

It should be observed that the term -1 in (1·52) and (1·54) is—so far as our present assertions are concerned—otiose: all that we have said remains true if it is omitted; the resemblance between the singularities of f and F becomes indeed even closer. The term is inserted merely in order to facilitate some of our preliminary analysis, and will prove to be without influence on the final result.

Applying Cauchy's theorem to $f - F$, we obtain

$$(1{\cdot}55) \qquad p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}),$$

where D is any number greater than

$$\tfrac{1}{2}C = \tfrac{1}{2}\pi\sqrt{(\tfrac{2}{3})}.$$

1·6. The formula (1·55) is an asymptotic formula of a type far more precise than that of (1·41). The error term is, however, of an exponential type, and may be expected ultimately to increase with very great rapidity. It was therefore with considerable surprise that we found what exceedingly good results the formula gives for fairly large values of n. For $n = 61, 62, 63$ it gives

1121538·972, 1300121·359, 1505535·606,

while the correct values are

1121505, 1300156, 1505499.

The errors 33·972, $-34{\cdot}641$, 36·606

are relatively very small, and alternate in sign.

The next step is naturally to direct our attention to the singular point of $f(x)$ next in importance after that at $x = 1$, viz., that at $x = -1$; and to subtract from $f(x)$ a second auxiliary function, related to this point as $F(x)$ is to $x = 1$. No new difficulty of principle is involved, and we find that

$$(1{\cdot}61) \qquad p(n) = \frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right) + \frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right) + O(e^{D\sqrt{n}}),$$

where D is now any number greater than $\frac{1}{3}C$. It now becomes obvious why our earlier approximation gave errors alternately of excess and of defect.

It is obvious that this process may be repeated indefinitely. The singularities next in importance are those at $x = e^{\frac{2}{3}\pi i}$ and $x = e^{\frac{4}{3}\pi i}$; the next those at $x=i$ and $x=-i$; and so on. The next two terms in the approximate formula are found to be

$$\frac{\sqrt{3}}{\pi\sqrt{2}} \cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{18}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)$$

and

$$\frac{\sqrt{2}}{\pi} \cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{8}\pi\right) \frac{d}{dn}\left(\frac{e^{\frac{1}{4}C\lambda_n}}{\lambda_n}\right).$$

As we proceed further, the complexity of the calculations increases. The auxiliary function associated with the point $x = e^{2p\pi i/q}$ involves a certain $24q$-th root of unity, connected with the linear transformation which must be used in order to elucidate the behaviour of $f(x)$ near the point; and the explicit expression of this root in terms of p and q, though known, is somewhat complex. But it is plain that, by taking a sufficient number of terms, we can find a formula in which the error is

$$O\left(e^{C\lambda_n/\nu}\right),$$

where ν is a fixed but arbitrarily large integer.

1·7. A final question remains. We have still the resource of making ν a function of n, that is to say of making the number of terms in our approximate formula itself a function of n. In this way we may reasonably hope, at any rate, to find a formula in which the error is of order less than that of any exponential of the type e^{an}; of the order of a power of n, for example, or even bounded.

When, however, we proceeded to test this hypothesis by means of the numerical data most kindly provided for us by Major MacMahon, we found a correspondence between the real and the approximate values of such astonishing accuracy as to lead us to hope for even more. Taking $n = 100$, we found that the first six terms of our formula gave

$$\begin{array}{r} 190568944{\cdot}783 \\ +348{\cdot}872 \\ -2{\cdot}598 \\ +{\cdot}685 \\ +{\cdot}318 \\ -{\cdot}064 \\ \hline 190569291{\cdot}996, \end{array}$$

while

$$p(100) = 190569292;$$

so that the error after six terms is only ·004. We then proceeded to calculate $p(200)$, and found

$$\begin{array}{r} 3,\ 972,\ 998,\ 993,\ 185{\cdot}896 \\ +\,36,\ 282{\cdot}978 \\ -\,87{\cdot}555 \\ +\ \ 5{\cdot}147 \\ +\ \ 1{\cdot}424 \\ +\ \ 0{\cdot}071 \\ +\ \ 0{\cdot}000^{*} \\ +\ \ 0{\cdot}043 \\ \hline 3,\ 972,\ 999,\ 029,\ 388{\cdot}004, \end{array}$$

and Major MacMahon's subsequent calculations shewed that $p(200)$ is, in fact,

3, 972, 999, 029, 388.

These results suggest very forcibly that it is possible to obtain a formula for $p(n)$, which not only exhibits its order of magnitude and structure, but may be used to calculate its *exact* value for any value of n. That this is in fact so is shewn by the following theorem.

Statement of the main theorem.

THEOREM. *Suppose that*

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right), \tag{1·71}$$

where C and λ_n are defined by the equations (1·53), *for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula*

(1·721)

$$\omega_{p,q} = \left(\frac{-q}{p}\right)\exp\left[-\left\{\tfrac{1}{4}(2-pq-p)+\tfrac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

(1·722)

$$\omega_{p,q} = \left(\frac{-p}{q}\right)\exp\left[-\left\{\tfrac{1}{4}(q-1)+\tfrac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

where (a/b) is the symbol of Legendre and Jacobi†, *and p' is any positive integer such that $1+pp'$ is divisible by q; that*

$$A_q(n) = \sum_{(p)} \omega_{p,q}\, e^{-2np\pi i/q}; \tag{1·73}$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.

* This term vanishes identically.

† See Tannery and Molk, *loc. cit.*, pp. 104—106, for a complete set of rules for the calculation of the value of (a/b), which is, of course, always 1 or -1. When *both* p and q are odd it is indifferent which formula is adopted.

Then

$$p(n) = \sum_1^{\nu} A_q \phi_q + O(n^{-\frac{1}{4}}), \tag{1·74}$$

so that $p(n)$ *is, for all sufficiently large values of* n*, the integer nearest to*

$$\sum_1^{\nu} A_q \phi_q. \tag{1·75}$$

It should be observed that all the numbers A_q are real. A table of A_q from $q = 1$ to $q = 18$ is given at the end of the paper (Table II).

The proof of this theorem is given in section 5; section 4 being devoted to a number of preliminary lemmas. The proof is naturally somewhat intricate; and we trust that we have arranged it in such a form as to be readily intelligible. In section 6 we draw attention to one or two questions which our theorem, in spite of its apparent completeness, still leaves open. In section 7 we indicate some other problems in combinatory analysis and the analytic theory of numbers to which our method may be applied; and we conclude by giving some functional and numerical tables: for the latter we are indebted to Major MacMahon and Mr H. B. C. Darling. To Major MacMahon in particular we owe many thanks for the amount of trouble he has taken over very tedious calculations. It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated.

2. Elementary proof that $e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}$ for sufficiently large values of n.

2·1. In this section we give the elementary proof of the inequalities (1·32). We prove, in fact, rather more, viz., that positive constants H and K exist such that

$$\frac{H}{n} e^{2\sqrt{n}} < p(n) < \frac{K}{n} e^{2\sqrt{(2n)}} \tag{2·11}$$

for $n \geqslant 1$*. We shall use in our proof only Euler's formula (1·31) and a debased form of Stirling's theorem, easily demonstrable by quite elementary methods: the proposition that

$$n!\, e^n / n^{n+\frac{1}{2}}$$

lies between two positive constants for all positive integral values of n.

* Somewhat inferior inequalities, of the type

$$2^{A[\sqrt{n}]} < p(n) < n^{B[\sqrt{n}]},$$

may be proved by *entirely* elementary reasoning; by reasoning, that is to say, which depends only on the arithmetical definition of $p(n)$ and on elementary finite algebra, and does not presuppose the notion of a limit or the definitions of the logarithmic or exponential functions.

2·2. The proof of the first of the two inequalities is slightly the simpler. It is obvious that if

$$\Sigma p_r(n)\, x^n = \frac{1}{(1-x)(1-x^2)\dots(1-x^r)}$$

so that $p_r(n)$ is the number of partitions of n into parts not exceeding r, then

(2·21) $$p_r(n) = p_{r-1}(n) + p_{r-1}(n-r) + p_{r-1}(n-2r) + \dots.$$

We shall use this equation to prove, by induction, that

(2·22) $$p_r(n) \geqslant \frac{rn^{r-1}}{(r!)^2}.$$

It is obvious that (2·22) is true for $r=1$. Assuming it to be true for $r=s$, and using (2·21), we obtain

$$p_{s+1}(n) \geqslant \frac{s}{(s!)^2}\{n^{s-1} + (n-s-1)^{s-1} + (n-2s-2)^{s-1} + \dots\}$$

$$\geqslant \frac{s}{(s!)^2}\left\{\frac{n^s-(n-s-1)^s}{s(s+1)} + \frac{(n-s-1)^s-(n-2s-2)^s}{s(s+1)} + \dots\right\}$$

$$= \frac{n^s}{(s+1)(s!)^2} = \frac{(s+1)\,n^s}{\{(s+1)!\}^2}.$$

This proves (2·22). Now $p(n)$ is obviously not less than $p_r(n)$, whatever the value of r. Take $r=[\surd n]$: then

$$p(n) \geqslant p_{[\surd n]}(n) \geqslant \frac{[\surd n]}{n}\frac{n^{[\surd n]}}{\{[\surd n]!\}^2} > \frac{H}{n}e^{2\surd n},$$

by a simple application of the degenerate form of Stirling's theorem mentioned above.

2·3. The proof of the second inequality depends upon Euler's identity. If we write

$$\Sigma q_r(n)\, x^n = \frac{1}{(1-x)^2(1-x^2)^2\dots(1-x^r)^2},$$

we have

(2·31) $$q_r(n) = q_{r-1}(n) + 2q_{r-1}(n-r) + 3q_{r-1}(n-2r) + \dots,$$

and

(2·32) $$p(n) = q_1(n-1) + q_2(n-4) + q_3(n-9) + \dots.$$

We shall first prove by induction that

(2·33) $$q_r(n) \leqslant \frac{(n+r^2)^{2r-1}}{(2r-1)!\,(r!)^2}.$$

This is obviously true for $r=1$. Assuming it to be true for $r=s$, and using (2·31), we obtain

$$q_{s+1}(n) \leqslant \frac{1}{(2s-1)!\,(s!)^2}\{(n+s^2)^{2s-1} + 2(n+s^2-s-1)^{2s-1} + 3(n+s^2-2s-2)^{2s-1} + \dots\}.$$

Now
$$m(m-1)\,a^{m-2}b^2 \leqslant (a+b)^m - 2a^m + (a-b)^m,$$
if m is a positive integer, and a, b, and $a-b$ are positive, while if $a-b \leqslant 0$, and m is odd, the term $(a-b)^m$ may be omitted. In this inequality write
$$m = 2s+1, \quad a = n+s^2-ks-k \ (k=0,1,2,\ldots), \quad b = s+1,$$
and sum with respect to k. We find that
$$(2s+1)\,2s\,(s+1)^2\{(n+s^2)^{2s-1} + 2(n+s^2-s-1)^{2s-1}+\ldots\} \leqslant (n+s^2+s+1)^{2s+1};$$
and so
$$q_{s+1}(n) \leqslant \frac{(n+s^2+s+1)^{2s+1}}{(2s+1)\,2s\,(s+1)^2(2s-1)!\,(s!)^2} \leqslant \frac{\{n+(s+1)^2\}^{2s+1}}{(2s+1)!\,\{(s+1)!\}^2}.$$
Hence (2·33) is true generally.

It follows from (2·32) that
$$p(n) = q_1(n-1) + q_2(n-4) + \ldots \leqslant \sum_1^\infty \frac{n^{2r-1}}{(2r-1)!\,(r!)^2}.$$
But, using the degenerate form of Stirling's theorem once more, we find without difficulty that
$$\frac{1}{(2r-1)!\,(r!)^2} < \frac{2^{6r}K}{4r!},$$
where K is a constant. Hence
$$p(n) < 8K\sum_1^\infty \frac{(8n)^{2r-1}}{4r!} < 8K\sum_1^\infty \frac{(8n)^{\frac{1}{2}r-1}}{r!} < \frac{K}{n}e^{2\sqrt{(2n)}}.$$
This is the second of the inequalities (2·11).

3. Application of a Tauberian theorem to the determination of the constant C.

3·1. The value of the constant
$$C = \lim \frac{\log p(n)}{\sqrt{n}},$$
is most naturally determined by the use of the following theorem.

If $g(x) = \Sigma a_n x^n$ *is a power-series with positive coefficients, and*
$$\log g(x) \sim \frac{A}{1-x}$$
when $x \to 1$, *then*
$$\log s_n = \log(a_0 + a_1 + \ldots + a_n) \sim 2\sqrt{(An)}$$
when $n \to \infty$.

This theorem is a special case* of Theorem C in our paper already referred to.

Now suppose that
$$g(x) = (1-x)f(x) = \Sigma\{p(n)-p(n-1)\}x^n = \frac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots}.$$

* *Loc. cit.*, p. 129 (with $a=1$).

Then $$a_n = p(n) - p(n-1)$$

is plainly positive. And

(3·11)
$$\log g(x) = \sum_2^\infty \log \frac{1}{1-x^\mu} = \sum_1^\infty \frac{1}{\nu}\frac{x^{2\nu}}{1-x^\nu} \sim \frac{1}{1-x}\sum_1^\infty \frac{1}{\nu^2} = \frac{\pi^2}{6(1-x)},$$

when $x \to 1$*. Hence

(3·12) $$\log p(n) = a_0 + a_1 + \ldots + a_n \sim C\sqrt{n},$$

where $C = 2\pi/\sqrt{6} = \pi\sqrt{(\frac{2}{3})}$, as in (1·53).

3·2. There is no doubt that it is possible, by "Tauberian" arguments, to prove a good deal more about $p(n)$ than is asserted by (3·12). The functional equation satisfied by $f(x)$ shews, for example, that

$$g(x) \sim \frac{(1-x)^{\frac{3}{2}}}{\sqrt{(2\pi)}} \exp\left\{\frac{\pi^2}{6(1-x)}\right\},$$

a relation far more precise than (3·11). From this relation, and the fact that the coefficients in $g(x)$ are positive, it is certainly possible to deduce more than (3·12). But it hardly seems likely that arguments of this character will lead us to a proof of (1·41). It would be exceedingly interesting to know exactly how far they will carry us, since the method is comparatively elementary, and has a much wider range of application than the more powerful methods employed later in this paper. We must, however, reserve the discussion of this question for some future occasion.

4. Lemmas preliminary to the proof of the main theorem.

4·1. We proceed now to the proof of our main theorem. The proof is somewhat intricate, and depends on a number of subsidiary theorems which we shall state as lemmas.

Lemmas concerning Farey's series.

4·21. The *Farey's series of order m* is the aggregate of irreducible rational fractions

$$p/q \quad (0 \leqslant p \leqslant q \leqslant m),$$

* This is a special case of much more general theorems: see K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," *Inaugural-Dissertation*, Berlin, 1907, pp. 25 *et seq.*; K. Knopp, "Über Lambertsche Reihen," *Journal für Math.*, Vol. CXLII, 1913, pp. 283—315; G. H. Hardy, "Theorems connected with Abel's Theorem on the continuity of power series," *Proc. London Math. Soc.*, Ser. 2, Vol. IV, 1906, pp. 247—265 (pp. 252, 253); G. H. Hardy, "Some theorems concerning infinite series," *Math. Ann.*, Vol. LXIV, 1907, pp. 77—94; G. H. Hardy, "Note on Lambert's series," *Proc. London Math. Soc.*, Ser. 2, Vol. XIII, 1913, pp. 192—198.

A direct proof is very easy: for

$$\nu x^{\nu-1}(1-x) < 1 - x^\nu < \nu(1-x),$$
$$\frac{1}{1-x}\Sigma\frac{x^{2\nu}}{\nu^2} < \log g(x) < \frac{1}{1-x}\Sigma\frac{x^{\nu+1}}{\nu^2}.$$

arranged in ascending order of magnitude. Thus

$$\tfrac{0}{1},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2},\ \tfrac{4}{7},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{5}{7},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{5}{6},\ \tfrac{6}{7},\ \tfrac{1}{1}$$

is the Farey's series of order 7.

LEMMA 4·21. *If p/q, p'/q' are two successive terms of a Farey's series, then*

$$p'q - pq' = 1. \tag{4·211}$$

This is, of course, a well-known theorem, first observed by Farey and first proved by Cauchy*. The following exceedingly simple proof is due to Hurwitz†.

The result is plainly true when $m = 1$. Let us suppose it true for $m = k$; and let p/q, p'/q' be two consecutive terms in the series of order k.

Suppose now that p''/q'' is a term of the series of order $k+1$ which falls between p/q and p'/q'. Let

$$p''q - pq'' = \lambda > 0, \quad p'q'' - p''q' = \mu > 0.$$

Solving these equations for p'', q'', and observing that $p'q - pq' = 1$, we obtain

$$p'' = \mu p + \lambda p', \quad q'' = \mu q + \lambda q'.$$

Consider now the aggregate of fractions

$$(\mu p + \lambda p')/(\mu q + \lambda q'),$$

where λ and μ are positive integers without common factor. All of these fractions lie between p/q and p'/q'; and all are in their lowest terms, since a factor common to numerator and denominator would divide

$$\lambda = q(\mu p + \lambda p') - p(\mu q + \lambda q'),$$

and

$$\mu = p'(\mu q + \lambda q') - q'(\mu p + \lambda p').$$

Each of them first makes its appearance in the Farey's series of order $\mu q + \lambda q'$, and the *first* of them to make its appearance must be that for which $\lambda = 1$, $\mu = 1$. Hence

$$p'' = p + p', \quad q'' = q + q',$$
$$p''q - pq'' = p'q'' - p''q' = 1.$$

The lemma is consequently proved by induction.

LEMMA 4·22. *Suppose that p/q is a term of the Farey's series of order m, and p''/q'', p'/q' the adjacent terms on the left and the right: and let $j_{p,q}$ denote the interval*

$$\frac{p}{q} - \frac{1}{q(q+q'')}, \quad \frac{p}{q} + \frac{1}{q(q+q')}‡.$$

* J. Farey, "On a curious property of vulgar fractions," *Phil. Mag.*, Ser. 1, Vol. XLVII, 1816, pp. 385, 386; A. L. Cauchy, "Démonstration d'un théorème curieux sur les nombres," *Exerciçes de mathématiques*, Vol. I, 1826, pp. 114—116. Cauchy's proof was first published in the *Bulletin de la Société Philomatique* in 1816.

† A. Hurwitz, "Ueber die angenäherte Darstellung der Zahlen durch rationale Brüche," *Math. Ann.*, Vol. XLIV, 1894, pp. 417—436.

‡ When p/q is 0/1 or 1/1, only the part of this interval inside (0, 1) is to be taken; thus $j_{0,1}$ is 0, $1/(m+1)$ and $j_{1,1}$ is $1-1/(m+1)$, 1.

Then (i) *the intervals* $j_{p,q}$ *exactly fill up the continuum* $(0, 1)$, *and* (ii) *the length of each of the parts into which* $j_{p,q}$ *is divided by* p/q* *is greater than* $1/2mq$ *and less than* $1/mq$.

(i) Since

$$\frac{1}{q(q+q')}+\frac{1}{q'(q'+q)}=\frac{1}{qq'}=\frac{p'q-pq'}{qq'}=\frac{p'}{q'}-\frac{p}{q},$$

the intervals just fill up the continuum.

(ii) Since neither q nor q' exceeds m, and one at least must be less than m, we have

$$\frac{1}{q(q+q')}>\frac{1}{2mq}.$$

Also $q+q'>m$, since otherwise $(p+p')/(q+q')$ would be a term in the series between p/q and p'/q'. Hence

$$\frac{1}{q(q+q')}<\frac{1}{mq}.$$

Standard dissection of a circle.

4·23. The following mode of dissection of a circle, based upon Lemma 4·22, is of fundamental importance for our analysis.

Suppose that the circle is defined by

$$x=Re^{2\pi i\theta}\quad(0\leqslant\theta\leqslant 1).$$

Construct the Farey's series of order m, and the corresponding intervals $j_{p,q}$. When these intervals are considered as intervals of variation of θ, and the two extreme intervals, which correspond to abutting arcs on the circle, are regarded as constituting a single interval $\xi_{1,1}$, the circle is divided into a number of arcs

$$\xi_{p,q},$$

where q ranges from 1 to m and p through the numbers not exceeding and prime to q†. We call this dissection of the circle *the dissection* Ξ_m.

Lemmas from the theory of the linear transformation of the elliptic modular functions.

4·3. LEMMA 4·31. *Suppose that* q *is a positive integer; that* p *is a positive integer not exceeding and prime to* q; *that* p' *is a positive integer such that* $1+pp'$ *is divisible by* q; *that* $\omega_{p,q}$ *is defined by the formulæ* (1·721) *or* (1·722); *that*

$$x=\exp\left(-\frac{2\pi z}{q}+\frac{2p\pi i}{q}\right),\quad x'=\exp\left(-\frac{2\pi}{qz}+\frac{2p'\pi i}{q}\right),$$

* See the preceding footnote [footnote ‡ of p. 289].
† $p=0$ occurring with $q=1$ only.

where the real part of z is positive; and that

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

Then $$f(x) = \omega_{p,q}\sqrt{z}\exp\left(\frac{\pi}{12qz} - \frac{\pi z}{12q}\right) f(x').$$

This lemma is merely a restatement in a different notation of well-known formulæ in the transformation theory.

Suppose, for example, that p is odd. If we take

$$a = p, \quad b = -q, \quad c = \frac{1+pp'}{q}, \quad d = -p',$$

so that $ad - bc = 1$; and write

$$x = q^2 = e^{2\pi i\tau}, \quad x' = Q^2 = e^{2\pi i \mathrm{T}},$$

so that $$\tau = \frac{p}{q} + \frac{iz}{q}, \quad \mathrm{T} = \frac{p'}{q} + \frac{i}{qz};$$

then we can easily verify that

$$\mathrm{T} = \frac{c + d\tau}{a + b\tau}.$$

Also, in the notation of Tannery and Molk, we have

$$f(x) = \frac{q^{\frac{1}{12}}}{h(\tau)}, \quad f(x') = \frac{Q^{\frac{1}{12}}}{h(\mathrm{T})};$$

and the formula for the linear transformation of $h(\tau)$ is

$$h(\mathrm{T}) = \left(\frac{b}{a}\right)\exp\left[\{\tfrac{1}{4}(a-1) - \tfrac{1}{12}[a(b-c) + bd(a^2-1)]\}\pi i\right]\sqrt{(a+b\tau)}\, h(\tau),$$

where $\sqrt{(a+b\tau)}$ has its real part positive*. A little elementary algebra will shew the equivalence of this result and ours.

The other formula for $\omega_{p,q}$ may be verified similarly, but in this case we must take

$$a = -p, \quad b = q, \quad c = -\frac{1+pp'}{q}, \quad d = p'.$$

We have included in the Appendix (Table I) a short table of some values of $\omega_{p,q}$, or rather of $(\log \omega_{p,q})/\pi i$.

LEMMA 4·32. *The function $f(x)$ satisfies the equation*

(4·321)

$$f(x) = \omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}}\exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\} f(x'_{p,q}),$$

where

(4·322)

$$x_{p,q} = xe^{-2p\pi i/q}, \quad x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})} + \frac{2p'\pi i}{q}\right\}.$$

* Tannery and Molk, *loc. cit.*, pp. 113, 267.

This is an immediate corollary from Lemma 4·31, since

$$z = \frac{q}{2\pi} \log \left(\frac{1}{x_{p,q}}\right), \qquad e^{-\pi z/12q} = x_{p,q}^{\frac{1}{24}},$$

$$\frac{\pi}{12qz} = \frac{\pi^2}{6q^2 \log (1/x_{p,q})}, \qquad x' = \exp\left(-\frac{2\pi}{qz} + \frac{2p'\pi i}{q}\right) = x'_{p,q}.$$

If we observe that

$$f(x'_{p,q}) = 1 + p(1)\, x'_{p,q} + \ldots,$$

we see that, if x tends to $e^{2p\pi i/q}$ along a radius vector, or indeed any regular path which does not touch the circle of convergence, the difference

$$f(x) - \omega_{p,q} \sqrt{\left\{\frac{q}{2\pi} \log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}} \exp\left\{\frac{\pi^2}{6q^2 \log (1/x_{p,q})}\right\}$$

tends to zero with great rapidity. It is on this fact that our analysis is based.

Lemmas concerning the auxiliary function $F_a(x)$.

4·41. The auxiliary function $F_a(x)$ is defined by the equation

$$F_a(x) = \sum_1^\infty \psi_a(n)\, x^n,$$

where

$$\psi_a(n) = \frac{d}{dn} \frac{\cosh a\lambda_n - 1}{\lambda_n},$$

$$\lambda_n = \sqrt{(n - \tfrac{1}{24})}, \quad a > 0.$$

LEMMA 4·41. *Suppose that a cut is made along the segment* $(1, \infty)$ *in the plane of* x. *Then* $F_a(x)$ *is regular at all points inside the region thus defined.*

This lemma is an immediate corollary of a general theorem proved by Lindelöf on pp. 109 *et seq.* of his *Calcul des résidus**.

The function
$$\psi_a(z) = \frac{d}{dz} \frac{\cosh a\sqrt{(z - \frac{1}{24})} - 1}{\sqrt{(z - \frac{1}{24})}}$$

satisfies the conditions imposed upon it by Lindelöf, if the number which he calls α is greater than $\frac{1}{24}$; and

$$(4·411) \qquad F_a(x) = \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{x^z}{1 - e^{2\pi i z}}\, \phi(z)\, dz,$$

if
$$x = re^{i\theta}, \quad 0 < \theta < 2\pi, \quad x^z = \exp\{z(\log r + i\theta)\}.$$

4·42. LEMMA 4·42. *Suppose that D is the region defined by the inequalities*

$$-\pi < -\theta_0 < \theta < \theta_0 < \pi, \quad r_0 < r, \quad 0 < r_0 < 1,$$

* Lindelöf gives references to Mellin and Le Roy, who had previously established the theorem in less general forms.

and that $\log(1/x)$ *has its principal value, so that* $\log(1/x)$ *is one-valued, and its square root two-valued, in D. Further, let*

$$\chi_a(x) = \sqrt{\{\pi \log(1/x)\}}\, x^{\frac{1}{24}} \left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\} - 1\right],$$

that value of the square root being chosen which is positive when $0 < x < 1$. *Then*

$$F_a(x) - \chi_a(x)$$

*is regular inside D**.

We observe first that, when θ has a fixed value between 0 and 2π, the integral on the right-hand side of (4·411) is uniformly convergent for $\frac{1}{24} \leqslant \alpha \leqslant \alpha_0$. Hence we may take $\alpha = \frac{1}{24}$ in (4·411). We thus obtain

$$F_a(x) = ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{1 - e^{\frac{1}{12}\pi i - 2\pi t}} \psi_a(\tfrac{1}{24} + it)\, dt + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1 - e^{\frac{1}{12}\pi i + 2\pi t}} \psi_a(\tfrac{1}{24} - it)\, dt,$$

where the $\sqrt{(it)}$ and $\sqrt{(-it)}$ which occur in $\psi_a(\frac{1}{24} + it)$ and $\psi_a(\frac{1}{24} - it)$ are to be interpreted as $e^{\frac{1}{4}\pi i}\sqrt{t}$ and $e^{-\frac{1}{4}\pi i}\sqrt{t}$ respectively. We write this in the form

$$(4{\cdot}421)\quad F_a(x) = X_a(x) + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{e^{-\frac{1}{12}\pi i + 2\pi t} - 1} \psi_a(\tfrac{1}{24} + it)\, dt$$

$$+ ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1 - e^{\frac{1}{12}\pi i + 2\pi t}} \psi_a(\tfrac{1}{24} - it)\, dt$$

$$= X_a(x) + \Theta_1(x) + \Theta_2(x),$$

say, where

$$X_a(x) = ix^{\frac{1}{24}} \int_0^\infty x^{it} \psi_a(\tfrac{1}{24} + it)\, dt.$$

Now, since

$$|x^{it}| = e^{-\theta t}, \quad |x^{-it}| = e^{\theta t},$$

the functions Θ are regular throughout the angle of Lemma 4·42. And

$$X_a(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}} \int_0^\infty e^{-\lambda t} \frac{d}{dt}\left(\frac{\cosh \mu\sqrt{t} - 1}{\sqrt{t}}\right) dt,$$

where

$$\lambda = i \log\frac{1}{x}, \quad \mu = a\sqrt{i}.$$

The form of this integral may be calculated by supposing λ and μ positive, when we obtain

$$\int_0^\infty e^{-\lambda w^2} \frac{d}{dw}\left(\frac{\cosh \mu w - 1}{w}\right) dw = 2\lambda \int_0^\infty e^{-\lambda w^2} (\cosh \mu w - 1)\, dw$$

$$= \sqrt{(\lambda\pi)}\,(e^{\mu^2/4\lambda} - 1).$$

Hence

$$(4{\cdot}422)\quad X_a(x) = \sqrt{\{\pi \log(1/x)\}}\, x^{\frac{1}{24}} \left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\} - 1\right] = \chi_a(x),$$

and the proof of the lemma is completed.

* Both $F_a(x)$ and $\chi_a(x)$ are two-valued in D. The value of $F_a(x)$ contemplated is naturally that represented by the power-series.

Lemmas 4·41 and 4·42 shew that $x=1$ is the sole finite singularity of the principal branch of $F_a(x)$.

4·43. LEMMA 4·43. *Suppose that* P, θ_1, *and* A *are positive constants,* θ_1 *being less than* π. *Then*

$$|F_a(x)| < K = K(\mathrm{P}, \theta_1, A),$$

for $$0 \leqslant r \leqslant \mathrm{P}, \quad \theta_1 \leqslant \theta \leqslant 2\pi - \theta_1, \quad 0 < a \leqslant A.$$

We use K generally to denote a positive number independent of x and of a. We may employ the formula (4·411). It is plain that

$$\left|\frac{x^z}{1-e^{2\pi i z}}\right| < Ke^{-\theta_1|\eta|},$$

$$|\psi_a(z)| = \left|\frac{d}{dz}\left\{\frac{\cosh a\sqrt{(z-\frac{1}{24})}-1}{\sqrt{(z-\frac{1}{24})}}\right\}\right| < Ke^{K\sqrt{|\eta|}},$$

where η is the imaginary part of z. Hence

$$|F_a(x)| < K\int_{-\infty}^{\infty} e^{K\sqrt{|\eta|}-\theta_1|\eta|}\,d\eta < K.$$

4·44. LEMMA 4·44. *Let* c *be a circle whose centre is* $x=1$, *and whose radius* δ *is less than unity. Then*

$$|F_a(x) - \chi_a(x)| < Ka^2,$$

if x *lies in* c *and* $0 < a \leqslant A$, $K = K(\delta, A)$ *being as before independent of* x *and of* a.

If we refer back to (4·421) and (4·422), we see that it is sufficient to prove that

$$|\Theta_1(x)| < Ka^2, \quad |\Theta_2(x)| < Ka^2;$$

and we may plainly confine ourselves to the first of these inequalities. We have

$$\Theta_1(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}}\int_0^{\infty}\frac{x^{it}}{e^{-\frac{1}{12}\pi i+2\pi t}-1}\,\frac{d}{dt}\left\{\frac{\cosh a\sqrt{(it)}-1}{\sqrt{t}}\right\}dt.$$

Rejecting the extraneous factor, which is plainly without importance, and integrating by parts, we obtain

$$\Theta(x) = \int_0^{\infty}\Phi(t)\,\frac{\cosh a\sqrt{(it)}-1}{\sqrt{t}}\,dt,$$

where $$\Phi(t) = -\frac{ix^{it}\log x}{e^{-\frac{1}{12}\pi i+2\pi t}-1} + \frac{2\pi x^{it}e^{-\frac{1}{12}\pi i+2\pi t}}{(e^{-\frac{1}{12}\pi i+2\pi t}-1)^2}.$$

Now $|\theta| < \frac{1}{2}\pi$ and $|x^{it}| < Ke^{\frac{1}{2}\pi t}$. It follows that

$$|\Phi(t)| < Ke^{-\pi t};$$

and
$$|\Theta(x)| < K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}|\sinh^2 \tfrac{1}{2}a\sqrt{(it)}|\,dt$$
$$< K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\{\cosh a\sqrt{(\tfrac{1}{2}t)} - \cos a\sqrt{(\tfrac{1}{2}t)}\}\,dt$$
$$< K\int_0^\infty e^{-\pi w^2}\left(\cosh\frac{aw}{\sqrt{2}} - \cos\frac{aw}{\sqrt{2}}\right)dw$$
$$= K(e^{a^2/8\pi} - e^{-a^2/8\pi}) < Ka^2.$$

5. Proof of the main theorem.

5·1. We write

(5·11)
$$F_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}F_{C/q}(x_{p,q}),$$

where $C = \pi\sqrt{\tfrac{2}{3}}$, $x_{p,q} = xe^{-2p\pi i/q}$; and

(5·12)
$$\Phi(x) = f(x) - \sum_q\sum_p F_{p,q}(x),$$

where the summation applies to all values of p not exceeding q and prime to q, and to all values of q such that

(5·13)
$$1 \leqslant q \leqslant \nu = [\alpha\sqrt{n}],$$

α being positive and independent of n. If then

(5·14)
$$F_{p,q}(x) = \sum c_{p,q,n}x^n,$$

we have

(5·15)
$$p(n) - \sum_q\sum_p c_{p,q,n} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}dx,$$

where Γ is a circle whose centre is the origin and whose radius R is less than unity. We take

(5·16)
$$R = 1 - \frac{\beta}{n},$$

where β also is positive and independent of n.

Our object is to shew that the integral on the right-hand side of (5·15) is of the form $O(n^{-\frac{1}{4}})$; the constant implied in the O will of course be a function of α and β. It is to be understood throughout that O's are used in this sense; $O(1)$, for instance, stands for *a function of x, n, p, q, α, and β* (or of some only of these variables) *which is less in absolute value than a number $K = K(\alpha, \beta)$ independent of x, n, p, and q.*

We divide up the circle Γ, by means of the dissection Ξ_ν of 4·23, into arcs $\xi_{p,q}$ each associated with a point $Re^{2p\pi i/q}$; and we denote by $\eta_{p,q}$ the arc of Γ complementary to $\xi_{p,q}$. This being so, we have

(5·17)
$$\int_\Gamma \frac{\Phi(x)}{x^{n+1}}dx = \sum\int_{\xi_{p,q}}\frac{f(x) - F_{p,q}(x)}{x^{n+1}}dx - \sum\int_{\eta_{p,q}}\frac{F_{p,q}(x)}{x^{n+1}}dx$$
$$= \sum J_{p,q} - \sum j_{p,q},$$

say. We shall prove that each of these sums is of the form $O(n^{-\frac{1}{4}})$; and we shall begin with the second sum, which only involves the auxiliary functions F.

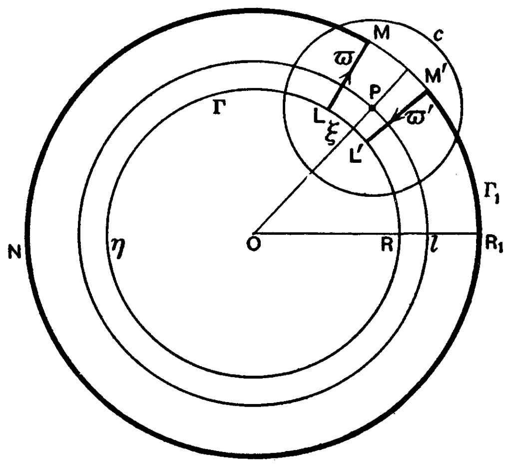
Proof that $\Sigma j_{p,q} = O(n^{-\frac{1}{4}})$.

5·21. We have, by Cauchy's theorem,

$$(5{\cdot}211)\qquad j_{p,q} = \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx = \int_{\zeta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx,$$

where $\zeta_{p,q}$ consists of the contour $LMNM'L'$ shewn in the figure. Here L and L' are the ends of $\xi_{p,q}$, LM and $M'L'$ are radii vectores, and MNM' is part of a circle Γ_1 whose radius R_1 is greater than 1. P is the point $e^{2p\pi i/q}$; and we suppose that R_1 is small enough to ensure that all points of LM and $M'L'$ are at a distance from P less than $\frac{1}{2}$. The other circle c shewn in the figure has P as its centre and radius $\frac{1}{2}$. We denote LM by $\varpi_{p,q}$, $M'L'$ by $\varpi'_{p,q}$, and MNM' by $\gamma_{p,q}$: and we write

$$(5{\cdot}212)\qquad j_{p,q} = \int_{\zeta_{p,q}} = \int_{\gamma_{p,q}} + \int_{\varpi_{p,q}} + \int_{\varpi'_{p,q}} = j^1_{p,q} + j^2_{p,q} + j^3_{p,q}.$$



The contribution of $\Sigma j^1_{p,q}$.

5·22. Suppose first that x lies on $\gamma_{p,q}$ and outside c. Then, in virtue of (5·11) and Lemma 4·43, we have

$$(5{\cdot}221)\qquad F_{p,q}(x) = O(\sqrt{q}).$$

If on the other hand x lies on $\gamma_{p,q}$, but inside c, we have, by (5·11) and Lemma 4·44,

$$(5{\cdot}222)\qquad F_{p,q}(x) - \chi_{p,q}(x) = O(q^{-\frac{3}{2}}),$$

where

$$(5{\cdot}2221)\qquad \chi_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\chi_{C/q}(x_{p,q}).$$

But, if we recur to the definition of $\chi_a(x)$ in Lemma 4·42, and observe that

$$\left|\exp\frac{a^2}{4\log(1/x)}\right| = \exp\frac{a^2\log(1/r)}{4[\{\log(1/r)\}^2+\theta^2]} < 1$$

if $x = re^{i\theta}$ and $r > 1$, we see that

(5·223) $$\chi_{p,q}(x) = O(\surd q)$$

on the part of $\gamma_{p,q}$ in question. Hence (5·221) holds for all $\gamma_{p,q}$. It follows that

$$j^1_{p,q} = O(R_1^{-n}\surd q),$$

(5·224) $$\Sigma j_{p,q} = O(R_1^{-n}\sum_q q^{\frac{3}{2}}) = O(n^{\frac{5}{4}}R_1^{-n})^*.$$

This sum tends to zero more rapidly than any power of n, and is therefore completely trivial.

The contributions of $\Sigma j^2_{p,q}$ and $\Sigma j^3_{p,q}$.

5·231. We must now consider the sums which arise from the integrals along $\varpi_{p,q}$ and $\varpi'_{p,q}$; and it is evident that we need consider in detail only the first of these two lines. We write

(5·2311)
$$j^2_{p,q} = \int_{\varpi_{p,q}}\frac{F_{p,q}(x)-\chi_{p,q}(x)}{x^{n+1}}dx + \int_{\varpi_{p,q}}\frac{\chi_{p,q}(x)}{x^{n+1}}dx = j'_{p,q} + j''_{p,q},$$

say.

In the first place we have, from (5·222),

$$j'_{p,q} = O\left(q^{-\frac{3}{2}}\int_R^{R_1}\frac{dr}{r^{n+1}}\right) = O(q^{-\frac{3}{2}}n^{-1}),$$

since

(5·2312) $$R^{-n} = \left(1-\frac{\beta}{n}\right)^{-n} = O(1).$$

Thus

(5·2313) $$\Sigma j'_{p,q} = O\{n^{-1}\sum_{q<O(\surd n)} q^{-\frac{1}{2}}\} = O(n^{-\frac{3}{4}}).$$

5·232. In the second place we have

$$j''_{p,q} = \omega_{p,q}\frac{\surd q}{\pi\surd 2}\int_{\varpi_{p,q}}\frac{\chi_{C/q}(x_{p,q})}{x^{n+1}}dx.$$

It is plain that, if we substitute y for $xe^{-2p\pi i/q}$, then write x again for y, and finally substitute for $\chi_{C/q}$ its explicit expression as an elementary function, given in Lemma 4·42, we obtain

(5·2321)
$$j''_{p,q} = O(\surd q)\int\{E(x)-1\}\sqrt{\left(\log\frac{1}{x}\right)}\,x^{-n-\frac{23}{24}}dx = O(\surd q)\,J,$$

* Here, and in many passages in our subsequent argument, it is to be remembered that the number of values of p, corresponding to a given q, is less than q, and that the number of values of q is of order $\surd n$. Thus we have generally

$$\Sigma O(q^s) = O(\sum_{q<O(\surd n)} q^{s+1}) = O(n^{\frac{1}{2}s+1}).$$

say, where

(5·23211) $$E(x) = \exp\left\{\frac{\pi^2}{6q^2 \log(1/x)}\right\},$$

and the path of integration is now a line related to $x = 1$ as $\varpi_{p,q}$ is to $x = e^{2p\pi i/q}$: the line defined by $x = re^{i\theta}$, where $R \leqslant r \leqslant R_1$, and θ is fixed and (by Lemma 4·22) lies between $1/2q\nu$ and $1/q\nu$.

Integrating J by parts, we find

(5·2322)
$$(n - \tfrac{1}{24})\,J = -\left[\{E(x) - 1\} \sqrt{\left(\log\frac{1}{x}\right)}\, x^{-n+\frac{1}{24}}\right]_{r=R}^{r=R_1}$$
$$- \tfrac{1}{2}\int \{E(x) - 1\}\left(\log\frac{1}{x}\right)^{-\frac{1}{2}} x^{-n-\frac{23}{24}}\,dx$$
$$+ \frac{\pi^2}{6q^2}\int E(x)\left(\log\frac{1}{x}\right)^{-\frac{3}{2}} x^{-n-\frac{23}{24}}\,dx = J_1 + J_2 + J_3,$$

say.

5·233. In estimating J_1, J_2, and J_3, we must bear the following facts in mind.

(1) Since $|x| \geqslant R$, it follows from (5·2312) that $|x|^{-n} = O(1)$ throughout the range of integration.

(2) Since $1 - R = \beta/n$ and $1/2q\nu < \theta < 1/q\nu$, where $\nu = [\alpha\sqrt{n}]$, we have

$$\log\left(\frac{1}{x}\right) = O\left(\frac{1}{q\sqrt{n}}\right),$$

when $r = R$, and $$\frac{1}{\log(1/x)} = O(q\sqrt{n}),$$

throughout the range of integration.

(3) Since $$|E(x)| = \exp\frac{\pi^2 \log(1/r)}{6q^2\left[\{\log(1/r)\}^2 + \theta^2\right]},$$

$E(x)$ is less than 1 in absolute value when $r > 1$. And, on the part of the path for which $r < 1$, it is of the form

$$\exp O\left(\frac{1}{q^2 n\theta^2}\right) = \exp O(1) = O(1).$$

It is accordingly of the form $O(1)$ throughout the range of integration.

5·234. Thus we have, first

(5·2341)
$$J_1 = O(1)\,O(1)\,O(R_1^{-n}) + O(1)\,O(q^{-\frac{1}{2}} n^{-\frac{1}{4}})\,O(1) = O(q^{-\frac{1}{2}} n^{-\frac{1}{4}}),$$

secondly

(5·2342) $$J_2 = O(1)\,O(q^{\frac{1}{2}} n^{\frac{1}{4}}) \int_R^{R_1} \frac{dr}{r^{n+\frac{23}{24}}} = O(q^{\frac{1}{2}} n^{-\frac{3}{4}}),$$

and thirdly

$$(5\cdot2343)\qquad J_3 = O(q^{-2})\,O(1)\,O(q^{\frac{3}{2}}n^{\frac{3}{4}})\int_R^{R_1}\frac{dr}{r^{n+\frac{23}{24}}} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}).$$

From (5·2341), (5·2342), (5·2343), and (5·2322), we obtain

$$J = O(q^{-\frac{1}{2}}n^{-\frac{5}{4}}) + O(q^{\frac{1}{2}}n^{-\frac{7}{4}});$$

and, from (5·2321), $\qquad j''_{p,q} = O(n^{-\frac{5}{4}}) + O(qn^{-\frac{7}{4}}).$

Summing, we obtain

$$(5\cdot2344)\qquad \Sigma j''_{p,q} = O\Big(n^{-\frac{5}{4}}\sum_{q<O(\sqrt{n})} q\Big) + O\Big(n^{-\frac{7}{4}}\sum_{q<O(\sqrt{n})} q^2\Big)$$
$$= O(n^{-\frac{1}{4}}) + O(n^{-\frac{1}{4}}) = O(n^{-\frac{1}{4}}).$$

5·235. From (5·2311), (5·2313), and (5·2344), we obtain

$$(5\cdot2351)\qquad \Sigma j^2_{p,q} = O(n^{-\frac{1}{4}});$$

and in exactly the same way we can prove

$$(5\cdot2352)\qquad \Sigma j^3_{p,q} = O(n^{-\frac{1}{4}}).$$

And from (5·212), (5·224), (5·2351), and (5·2352), we obtain, finally,

$$(5\cdot2353)\qquad \Sigma j_{p,q} = O(n^{-\frac{1}{4}}).$$

Proof that $\Sigma J_{p,q} = O(n^{-\frac{1}{4}})$.

5·31. We turn now to the discussion of

$$(5\cdot311)\qquad J_{p,q} = \int_{\xi_{p,q}}\frac{f(x)-F_{p,q}(x)}{x^{n+1}}\,dx$$
$$= \int_{\xi_{p,q}}\frac{f(x)-X_{p,q}(x)}{x^{n+1}}\,dx - \int_{\xi_{p,q}}\frac{F_{p,q}(x)-\chi_{p,q}(x)}{x^{n+1}}\,dx + \int_{\xi_{p,q}}\frac{\rho_{p,q}(x)}{x^{n+1}}\,dx$$
$$= J^1_{p,q} + J^2_{p,q} + J^3_{p,q},$$

say, where $\qquad \rho_{p,q}(x) = \omega_{p,q}\sqrt{\left(\frac{q}{2\pi}\log\frac{1}{x_{p,q}}\right)}\,x_{p,q}^{\frac{1}{24}},$

$$X_{p,q}(x) = \chi_{p,q}(x) + \rho_{p,q}(x) = \rho_{p,q}(x)\,E(x_{p,q}),$$

$E(x)$ being defined as in (5·23211).

Discussion of $\Sigma J^2_{p,q}$ *and* $\Sigma J^3_{p,q}$.

5·32. The discussion of these two sums is, after the analysis which precedes, a simple matter. The arc $\xi_{p,q}$ is less than a constant multiple of $1/q\sqrt{n}$; and $x^{-n} = O(1)$ on $\xi_{p,q}$. Also

$$|F_{p,q}(x) - \chi_{p,q}(x)| = O(q^{-\frac{3}{2}}),$$

by (5·222); and

$$(5\cdot321)\qquad \sqrt{\left(\log\frac{1}{x_{p,q}}\right)} = O(q^{-\frac{1}{2}}n^{-\frac{1}{4}}),$$

since $\qquad |x_{p,q}| = R = 1 - (\beta/n), \quad |am\ x_{p,q}| < 1/q\nu.$

Hence
$$J^2_{p,q} = O(q^{-\frac{5}{2}} n^{-\frac{1}{2}}),$$

(5·322)
$$\Sigma J^2_{p,q} = O(n^{-\frac{1}{2}} \sum_{q < O(\surd n)} q^{-\frac{3}{2}}) = O(n^{-\frac{1}{2}});$$

and
$$J^3_{p,q} = O(q^{-1} n^{-\frac{3}{4}}),$$

(5·323)
$$\Sigma J^3_{p,q} = O(n^{-\frac{3}{4}} \sum_{q < O(\surd n)} 1) = O(n^{-\frac{1}{4}}).$$

Discussion of $\Sigma J^1_{p,q}$.

5·33. From (4·321) and (5·2221), we have

(5·331)
$$f(x) - X_{p,q}(x) = \omega_{p,q} \sqrt{\left\{\frac{q}{2\pi} \log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}} E(x_{p,q})\, \Omega(x'_{p,q}),$$

where
$$\Omega(z) = f(z) - 1 = \prod_1^\infty \left(\frac{1}{1 - z^\nu}\right) - 1 = \sum_1^\infty p(\nu) z^\nu,$$

if $|z| < 1$, and
$$x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2 \log(1/x_{p,q})} + \frac{2\pi i p'}{q}\right\}.$$

Now
$$|x'_{p,q}| = \exp\left[-\frac{4\pi^2 \log(1/R)}{q^2 \{[\log(1/R)]^2 + \theta^2\}}\right],$$

where θ is the amplitude of $x_{p,q}$. Also
$$q^2 \{[\log(1/R)]^2 + \theta^2\} = O\left\{q^2\left(\frac{1}{n^2} + \frac{1}{q^2 n}\right)\right\} = O\left(\frac{1}{n}\right),$$

while $\log(1/R)$ is greater than a constant multiple of $1/n$. There is therefore a positive number δ, less than unity and independent of n and of q, such that
$$|x'_{p,q}| < \delta;$$

and we may write
$$\Omega(x'_{p,q}) = O(|x'_{p,q}|).$$

We have therefore
$$E(x_{p,q})\, \Omega(x'_{p,q}) = O(|x'_{p,q}|^{-\frac{1}{24}})\, O(|x'_{p,q}|) = O(|x'_{p,q}|^{\frac{23}{24}}) = O(1);$$

and so, by (5·321),
$$f(x) - \chi_{p,q}(x) = O(\surd q)\, O\left(\sqrt{\left|\log\frac{1}{x_{p,q}}\right|}\right) O(1) = O(n^{-\frac{1}{4}}).$$

And hence, as the length of $\xi_{p,q}$ is of the form $O(1/q\surd n)$, we obtain
$$J^1_{p,q} = O(q^{-1} n^{-\frac{3}{4}}),$$

(5·332)
$$\Sigma J^1_{p,q} = O(n^{-\frac{3}{4}} \sum_{q<O(\surd n)} 1) = O(n^{-\frac{1}{4}}).$$

5·34. From (5·311), (5·322), (5·323), and (5·332), we obtain

(5·341)
$$\Sigma J_{p,q} = O(n^{-\frac{1}{4}}).$$

Completion of the proof.

5·4. From (5·15), (5·17), (5·2353), and (5·341), we obtain

(5·41) $$p(n) - \sum_q \sum_p c_{p,q,n} = O(n^{-\frac{1}{4}}).$$

But $$\sum_p c_{p,q,n} = \frac{\sqrt{q}}{\pi\sqrt{2}} A_q \frac{d}{dn} \frac{\cosh(C\lambda_n/q) - 1}{\lambda_n},$$

where $$A_q = \sum_p \omega_{p,q} e^{-2np\pi i/q}.$$

All that remains, in order to complete the proof of the theorem, is to shew that

$$\cosh(C\lambda_n/q) - 1$$

may be replaced by $$\tfrac{1}{2} e^{C\lambda_n/q};$$

and in order to prove this it is only necessary to shew that

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \frac{d}{dn} \frac{\frac{1}{2}e^{C\lambda_n/q} - \cosh(C\lambda_n/q) + 1}{\lambda_n} = O(n^{-\frac{1}{4}}).$$

On differentiating we find that the sum is of the form

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \left\{ O\left(\frac{1}{qn}\right) + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \right\} = O\left\{ \frac{1}{n} \sum_{q<O(\sqrt{n})} q^{\frac{1}{2}} \right\} = O(n^{-\frac{1}{4}}).$$

Thus the theorem is proved.

6. Additional remarks on the theorem.

6·1. The theorem which we have proved gives information about $p(n)$ which is in some ways extraordinarily exact. We are for this reason the more anxious to point out explicitly two respects in which the results of our analysis are incomplete.

6·21. We have proved that

$$p(n) = \sum A_q \phi_q + O(n^{-\frac{1}{4}}),$$

where the summation extends over the values of q specified in the theorem, for every fixed value of α; that is to say that, when α is given, a number $K = K(\alpha)$ can be found such that

$$|\, p(n) - \sum A_q \phi_q \,| < K n^{-\frac{1}{4}}$$

for every value of n. It follows that

(6·211) $$p(n) = \{\sum A_q \phi_q\},$$

where $\{x\}$ denotes the integer nearest to x, for $n \geqslant n_0$, where $n_0 = n_0(\alpha)$ is a certain function of α.

The question remains whether we can, by an appropriate choice of α, secure the truth of (6·211) for *all* values of n, and not merely for all sufficiently large values. Our opinion is that this is possible, and that it could be proved to be possible without any fundamental change in our analysis. Such a proof

would however involve a very careful revision of our argument. It would be necessary to replace all formulæ involving O's by inequalities, containing only numbers expressed explicitly as functions of the various parameters employed. This process would certainly add very considerably to the length and the complexity of our argument. It is, as it stands, sufficient to prove what is, from our point of view, of the greatest interest; and we have not thought it worth while to elaborate it further.

6·22. The second point of incompleteness of our results is of much greater interest and importance. We have not proved either that the series

$$\sum_{1}^{\infty} A_q \phi_q$$

is convergent, or that, if it is convergent, it represents $p(n)$. Nor does it seem likely that our method is one intrinsically capable of proving these results, if they are true—a point on which we are not prepared to express any definite opinion.

It should be observed in this connection that we have not even discovered anything definite concerning the order of magnitude of A_q for large values of q. We can prove nothing better than the absolutely trivial equation $A_q = O(q)$. On the other hand we cannot assert that A_q is, for an infinity of values of q, effectively of an order as great as q, or indeed even that it does not tend to zero (though of course this is most unlikely).

6·3. Our formula directs us, if we wish to obtain the exact value of $p(n)$ for a large value of n, to take a number of terms of order $\sqrt{n}$. The numerical data suggest that a considerably smaller number of terms will be equally effective; and it is easy to see that this conjecture is correct.

Let us write $$\beta = 4\pi\sqrt{(\tfrac{2}{3})} = 4C, \qquad \mu = \left[\frac{\beta\sqrt{n}}{\log n}\right],$$
and let us suppose that $\alpha < 2$. Then

$$\sum_{\mu+1}^{\nu} A_q\phi_q = \sum_{\mu+1}^{\nu} O(q^{\frac{3}{2}})\, O\left(\frac{1}{qn}\right) O\left(e^{C\sqrt{n}/q}\right) = O\left(\frac{1}{n}\sum_{\mu+1}^{\nu} \sqrt{q}\, e^{C\sqrt{n}/q}\right)$$
$$= O\left(\frac{1}{n}\int_{\mu}^{\nu} \sqrt{x}\, e^{C\sqrt{n}/x}\, dx\right),$$

since $\sqrt{q}\,e^{C\sqrt{n}/q}$ decreases steadily throughout the range of summation*.

Writing $\sqrt{n}/y$ for x, we obtain

$$O\left(n^{-\frac{1}{4}}\int_{1/\alpha}^{\sqrt{n}/\mu} y^{-\frac{5}{2}}\, e^{Cy}\, dy\right) = O\left\{n^{-\frac{1}{4}}\left(\frac{\sqrt{n}}{\mu}\right)^{-\frac{3}{2}} e^{C\sqrt{n}/\mu}\right\} = O\left\{n^{-\frac{1}{4}}(\log n)^{-\frac{3}{2}} e^{\frac{1}{4}\log n}\right\}$$
$$= O(\log n)^{-\frac{3}{2}} = o(1).$$

* The minimum occurs when q is about equal to $2C\sqrt{n}$.

It follows that it is enough, when n is sufficiently large, to take

$$\left[\frac{\beta\sqrt{n}}{\log n}\right]$$

terms of the series. It is probably also *necessary* to take a number of terms of order $\sqrt{n}/(\log n)$; but it is not possible to prove this rigorously without a more exact knowledge of the properties of A_q than we possess.

6·4. We add a word on certain simple approximate formulæ for $\log p(n)$ found empirically by Major MacMahon and by ourselves. Major MacMahon found that if

$$\log_{10} p(n) = \sqrt{(n+4)} - a_n, \tag{6·41}$$

then a_n is approximately equal to 2 within the limits of his table of values of $p(n)$ (Table IV). This suggested to us that we should endeavour to find more accurate formulæ of the same type. The most striking that we have found is

$$\log_{10} p(n) = \tfrac{10}{9}\{\sqrt{(n+10)} - a_n\}; \tag{6·42}$$

the mode of variation of a_n is shewn in Table III.

In this connection it is interesting to observe that the function

$$13^{-\sqrt{n}} p(n)$$

(which ultimately tends to infinity with exponential rapidity) is equal to ·973 for $n = 30000000000$.

7. Further applications of the method.

7·1. We shall conclude with a few remarks concerning actual or possible applications of our method to other problems in Combinatory Analysis or the Analytic Theory of Numbers.

The class of problems in which the method gives the most striking results may be defined as follows. Suppose that $q(n)$ is the coefficient of x^n in the expansion of $F(x)$, where $F(x)$ is a function of the form

$$\frac{\{f(\pm x^a)\}^{\alpha}\{f(\pm x^{a'})\}^{\alpha'}\dots}{\{f(\pm x^b)\}^{\beta}\{f(\pm x^{b'})\}^{\beta'}\dots}\,{}^*; \tag{7·11}$$

$f(x)$ being the function considered in this paper, the a's, b's, α's, and β's being positive integers, and the number of factors in numerator and denominator being finite; and suppose that $|F(x)|$ tends exponentially to infinity when x tends in an appropriate manner to some or all of the points $e^{2p\pi i/q}$. Then our method may be applied in its full power to the asymptotic study of $q(n)$, and yields results very similar to those which we have found concerning $p(n)$.

* Since
$$f(-x) = \frac{\{f(x^2)\}^3}{f(x)f(x^4)},$$
the arguments with a negative sign may be eliminated if this is desired.

Thus, if

$$F(x)=\frac{f(x)}{f(x^2)}=(1+x)(1+x^2)(1+x^3)\ldots=\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots},$$

so that $q(n)$ is the number of partitions of n into odd parts, or into unequal parts*, we find that

$$q(n)=\frac{1}{\sqrt{2}}\frac{d}{dn}J_0[i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}]$$
$$+\sqrt{2}\cos(\tfrac{2}{3}n\pi-\tfrac{1}{9}\pi)\frac{d}{dn}J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}]+\ldots.$$

The error after $[\alpha\sqrt{n}]$ terms is of the form $O(1)$. We are not in a position to assert that the *exact* value of $q(n)$ can always be obtained from the formula (though this is probable); but the error is certainly bounded.

If
$$F(x)=\frac{f(x^2)}{f(-x)}=\frac{f(x)f(x^4)}{\{f(x^2)\}^2}=(1+x)(1+x^3)(1+x^5)\ldots,$$

so that $q(n)$ is the number of partitions of n into parts which are both odd and unequal, then

$$q(n)=\frac{d}{dn}J_0[i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}]$$
$$+2\cos(\tfrac{2}{3}n\pi-\tfrac{2}{9}\pi)\frac{d}{dn}J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}]+\ldots.$$

The error is again bounded (and probably tends to zero).

If
$$F(x)=\frac{\{f(x)\}^2}{f(x^2)}=\frac{1}{1-2x+2x^4-2x^9+\ldots},$$

$q(n)$ has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of a simple ϑ-function, of particular interest. In this case we find

$$q(n)=\frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\sqrt{n}}+\frac{\sqrt{3}}{2\pi}\cos(\tfrac{2}{3}n\pi-\tfrac{1}{6}\pi)\frac{d}{dn}\frac{e^{\frac{1}{3}\pi\sqrt{n}}}{\sqrt{n}}+\ldots.$$

The error here is (as in the partition problem) of order $O(n^{-\frac{1}{4}})$, and the exact value can always be found from the formula.

7·2. The method may also be applied to products of the form (7·11) which have (to put the matter roughly) no exponential infinities. In such cases the approximation is of a much less exact character. On the other hand the problems of this character are of even greater arithmetical interest.

The standard problem of this category is that of the representation of a number as the sum of s squares, s being any positive integer odd or even†. We must reserve the application of our method to this problem for another occasion; but we can indicate the character of our main result as follows.

* Cf. MacMahon, *loc. cit.*, p. 11. We give at the end of the paper a table (Table V) of the values of $q(n)$ up to $n=100$. This table was calculated by Mr Darling.

† As is well known, the arithmetical difficulties of the problem are much greater when s is odd.

If $r_s(n)$ is the number of representations of n as the sum of s squares, we have

$$F(x) = \Sigma r_s(n)\, x^n = (1 + 2x + 2x^4 + \ldots)^s = \frac{\{f(x^2)\}^s}{\{f(-x)\}^{2s}} = \frac{\{f(x)\}^{2s}\{f(x^4)\}^{2s}}{\{f(x^2)\}^{5s}}.$$

We find that

$$(7{\cdot}21) \qquad r_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \Sigma \frac{c_q}{q^{\frac{1}{2}s}} + O(n^{\frac{1}{4}s}),$$

where c_q is a function of q and of n of the same general type as the function A_q of this paper. The series

$$(7{\cdot}22) \qquad \Sigma \frac{c_q}{q^{\frac{1}{2}s}}$$

is absolutely convergent for sufficiently large values of s, and the summation in (7·21) may be regarded indifferently as extended over all values of q or only over a range $1 \leqslant q \leqslant \alpha\sqrt{n}$. It should be observed that the series (7·22) is quite different in form from any of the infinite series which are already known to occur in connection with this problem.

7·3. There is also a wide range of problems to which our methods are *partly* applicable. Suppose, for example, that

$$F(x) = \Sigma p^2(n)\, x^n = \frac{1}{(1-x)(1-x^4)(1-x^9)\ldots},$$

so that $p^2(n)$ is the number of partitions of n into *squares*. Then $F(x)$ is not an elliptic modular function; it possesses no general transformation theory: and the full force of our method cannot be applied. We can still, however, apply some of our preliminary methods. Thus the "Tauberian" argument shews that

$$\log p^2(n) \sim 2^{-\frac{4}{3}} 3\pi^{\frac{1}{3}} \{\zeta(\tfrac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}.$$

And although there is no general transformation theory, there is a formula which enables us to specify the nature of the singularity at $x = 1$. This formula is

$$\frac{1}{f(e^{-\pi z})} = 2\sqrt{\left(\frac{\pi}{z}\right)} \exp\left\{\frac{2\pi}{\sqrt{z}} \zeta(-\tfrac{1}{2})\right\}$$
$$\times \prod_1^\infty \{1 - 2e^{-2\pi\sqrt{(n/z)}} \cos 2\pi\sqrt{(n/z)} + e^{-4\pi\sqrt{(n/z)}}\}.$$

By the use of this formula, in conjunction with Cauchy's theorem, it is certainly possible to obtain much more precise information about $p^2(n)$, and in particular the formula

$$p^2(n) \sim 3^{-\frac{1}{2}} (4\pi n)^{-\frac{7}{6}} \{\zeta(\tfrac{3}{2})\}^{\frac{2}{3}} e^{2^{-\frac{4}{3}} 3\pi^{\frac{1}{3}} \{\zeta(\frac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}}.$$

The corresponding formula for $p^s(n)$, the number of partitions of n into perfect s-th powers, is

$$p^s(n) \sim (2\pi)^{-\frac{1}{2}(s+1)} \sqrt{\left(\frac{s}{s+1}\right)}\, k n^{\frac{1}{s+1} - \frac{3}{2}} e^{(s+1)\, k n^{1/(s+1)}},$$

where
$$k = \left\{\frac{1}{s} \Gamma\left(1 + \frac{1}{s}\right) \zeta\left(1 + \frac{1}{s}\right)\right\}^{\frac{s}{s+1}}.$$

The series (7·21) may be written in the form

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \sum_{p,q} \frac{\omega^s_{p,q}}{q^{\frac{1}{2}s}} e^{-np\pi i/q},$$

where $\omega_{p,q}$ is always one of the five numbers 0, $e^{\frac{1}{4}\pi i}$, $e^{-\frac{1}{4}\pi i}$, $-e^{\frac{1}{4}\pi i}$, $-e^{-\frac{1}{4}\pi i}$. When s is even it begins

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \{1^{-\frac{1}{2}s} + 2\cos(\tfrac{1}{2}n\pi - \tfrac{1}{4}s\pi)\, 2^{-\frac{1}{2}s} + 2\cos(\tfrac{2}{3}n\pi - \tfrac{1}{2}s\pi)\, 3^{-\frac{1}{2}s} + \dots\}.$$

It has been proved by Ramanujan that the series gives an *exact* representation of $r_s(n)$ when $s=4, 6, 8$; and by Hardy that this is also true when $s=3, 5, 7$. See Ramanujan, "On certain trigonometrical sums and their applications in the Theory of Numbers"; Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five or seven."

TABLE I $\omega_{p,q}$.

p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$
1	1	0	3	11	3/22	8	15	7/18
1	2	0	4	"	3/22	11	"	−19/90
1	3	1/18	5	"	−5/22	13	"	−7/18
2	"	−1/18	6	"	5/22	14	"	−1/90
1	4	1/8	7	"	−3/22	1	16	−29/32
3	"	−1/8	8	"	−3/22	3	"	−27/32
1	5	1/5	9	"	−5/22	5	"	−5/32
2	"	0	10	"	−15/22	7	"	−3/32
3	"	0	1	12	55/72	9	"	3/32
4	"	−1/5	5	"	−1/72	11	"	5/32
1	6	5/18	7	"	1/72	13	"	27/32
5	"	−5/18	11	"	−55/72	15	"	29/32
1	7	5/14	1	13	11/13	1	17	−14/17
2	"	1/14	2	"	4/13	2	"	8/17
3	"	−1/14	3	"	1/13	3	"	5/17
4	"	1/14	4	"	−1/13	4	"	0
5	"	−1/14	5	"	0	5	"	1/17
6	"	−5/14	6	"	−4/13	6	"	5/17
1	8	7/16	7	"	4/13	7	"	1/17
3	"	1/16	8	"	0	8	"	−8/17
5	"	−1/16	9	"	1/13	9	"	8/17
7	"	−7/16	10	"	−1/13	10	"	−1/17
1	9	14/27	11	"	−4/13	11	"	−5/17
2	"	4/27	12	"	−11/13	12	"	−1/17
4	"	−4/27	1	14	13/14	13	"	0
5	"	4/27	3	"	3/14	14	"	−5/17
7	"	−4/27	5	"	3/14	15	"	−8/17
8	"	−14/27	9	"	−3/14	16	"	14/17
1	10	3/5	11	"	−3/14	1	18	−20/27
3	"	0	13	"	−13/14	5	"	2/27
7	"	0	1	15	1/90	7	"	−2/27
9	"	−3/5	2	"	7/18	11	"	2/27
1	11	15/22	4	"	19/90	13	"	−2/27
2	"	5/22	7	"	−7/18	17	"	20/27

TABLE II: A_q.

$A_1 = 1.$

$A_2 = \cos n\pi.$

$A_3 = 2\cos(\frac{2}{3}n\pi - \frac{1}{18}\pi).$

$A_4 = 2\cos(\frac{1}{2}n\pi - \frac{1}{8}\pi).$

$A_5 = 2\cos(\frac{2}{5}n\pi - \frac{1}{5}\pi) + 2\cos\frac{4}{5}n\pi.$

$A_6 = 2\cos(\frac{1}{3}n\pi - \frac{5}{18}\pi).$

$A_7 = 2\cos(\frac{2}{7}n\pi - \frac{5}{14}\pi) + 2\cos(\frac{4}{7}n\pi - \frac{1}{14}\pi) + 2\cos(\frac{6}{7}n\pi + \frac{1}{14}\pi).$

$A_8 = 2\cos(\frac{1}{4}n\pi - \frac{7}{16}\pi) + 2\cos(\frac{3}{4}n\pi - \frac{1}{16}\pi).$

$A_9 = 2\cos(\frac{2}{9}n\pi - \frac{14}{27}\pi) + 2\cos(\frac{4}{9}n\pi - \frac{4}{27}\pi) + 2\cos(\frac{8}{9}n\pi + \frac{4}{27}\pi).$

$A_{10} = 2\cos(\frac{1}{5}n\pi - \frac{3}{5}\pi) + 2\cos\frac{3}{5}n\pi.$

$A_{11} = 2\cos(\frac{2}{11}n\pi - \frac{15}{22}\pi) + 2\cos(\frac{4}{11}n\pi - \frac{5}{22}\pi) + 2\cos(\frac{6}{11}n\pi - \frac{3}{22}\pi) + 2\cos(\frac{8}{11}n\pi - \frac{3}{22}\pi)$
$+ 2\cos(\frac{10}{11}n\pi + \frac{5}{22}\pi).$

$A_{12} = 2\cos(\frac{1}{6}n\pi - \frac{55}{72}\pi) + 2\cos(\frac{5}{6}n\pi + \frac{1}{72}\pi).$

$A_{13} = 2\cos(\frac{2}{13}n\pi - \frac{11}{13}\pi) + 2\cos(\frac{4}{13}n\pi - \frac{4}{13}\pi) + 2\cos(\frac{6}{13}n\pi - \frac{1}{13}\pi) + 2\cos(\frac{8}{13}n\pi + \frac{1}{13}\pi)$
$+ 2\cos\frac{10}{13}n + 2\cos(\frac{12}{13}n\pi + \frac{4}{13}\pi).$

$A_{14} = 2\cos(\frac{1}{7}n\pi - \frac{13}{14}\pi) + 2\cos(\frac{3}{7}n\pi - \frac{3}{14}\pi) + 2\cos(\frac{5}{7}n\pi - \frac{3}{14}\pi).$

$A_{15} = 2\cos(\frac{2}{15}n\pi - \frac{1}{90}\pi) + 2\cos(\frac{4}{15}n\pi - \frac{7}{18}\pi) + 2\cos(\frac{8}{15}n\pi - \frac{19}{90}\pi) + 2\cos(\frac{14}{15}n\pi + \frac{7}{18}\pi).$

$A_{16} = 2\cos(\frac{1}{8}n\pi + \frac{29}{32}\pi) + 2\cos(\frac{3}{8}n\pi + \frac{27}{32}\pi) + 2\cos(\frac{5}{8}n\pi + \frac{5}{32}\pi) + 2\cos(\frac{7}{8}n\pi + \frac{3}{32}\pi).$

$A_{17} = 2\cos(\frac{2}{17}n\pi + \frac{14}{17}\pi) + 2\cos(\frac{4}{17}n\pi - \frac{8}{17}\pi) + 2\cos(\frac{6}{17}n\pi - \frac{5}{17}\pi) + 2\cos\frac{8}{17}n\pi$
$+ 2\cos(\frac{10}{17}n\pi - \frac{1}{17}\pi) + 2\cos(\frac{12}{17}n\pi - \frac{5}{17}n\pi) + 2\cos(\frac{14}{17}n\pi - \frac{1}{17}\pi) + 2\cos(\frac{16}{17}n\pi + \frac{8}{17}\pi).$

$A_{18} = 2\cos(\frac{1}{9}n\pi + \frac{20}{27}\pi) + 2\cos(\frac{5}{9}n\pi - \frac{2}{27}\pi) + 2\cos(\frac{7}{9}n\pi + \frac{2}{27}\pi).$

It may be observed that

$A_5 = 0\ (n \equiv 1, 2 \pmod 5)$, $\quad A_7 = 0\ (n \equiv 1, 3, 4 \pmod 7)$,

$A_{10} = 0\ (n \equiv 1, 2 \pmod 5)$, $\quad A_{11} = 0\ (n \equiv 1, 2, 3, 5, 7 \pmod{11})$,

$A_{13} = 0\ (n \equiv 2, 3, 5, 7, 9, 10 \pmod{13})$, $\quad A_{14} = 0\ (n \equiv 1, 3, 4 \pmod 7)$,

$A_{16} = 0\ (n \equiv 0 \pmod 2)$, $\quad A_{17} = 0\ (n \equiv 1, 3, 4, 6, 7, 9, 13, 14 \pmod{17})$;

while A_1, A_2, A_3, A_4, A_6, A_8, A_9, A_{12}, A_{15}, and A_{18} never vanish.

TABLE III: $\log_{10} p(n) = \frac{10}{9}\{\sqrt{(n+10)} - a_n\}$.

n	a_n	n	a_n
1	3·317	10000	4·148
3	3·176	30000	4·364
10	3·011	100000	4·448
30	2·951	300000	4·267
100	3·036	1000000	3·554
300	3·237	3000000	2·072
1000	3·537	10000000	−1·188
3000	3·838	30000000	−6·796
		∞	$-\infty$

TABLE IV*: $p(n)$.

1...	1	51...	239943	101...	214481126	151...	45060624582
2...	2	52...	281589	102...	241265379	152...	49686288421
3...	3	53...	329931	103...	271248950	153...	54770336324
4...	5	54...	386155	104...	304801365	154...	60356673280
5...	7	55...	451276	105...	342325709	155...	66493182097
6...	11	56...	526823	106...	384276336	156...	73232243759
7...	15	57...	614154	107...	431149389	157...	80630964769
8...	22	58...	715220	108...	483502844	158...	88751778802
9...	30	59...	831820	109...	541946240	159...	97662728555
10...	42	60...	966467	110...	607163746	160...	107438159466
11...	56	61...	1121505	111...	679903203	161...	118159068427
12...	77	62...	1300156	112...	761002156	162...	129913904637
13...	101	63...	1505499	113...	851376628	163...	142798995930
14...	135	64...	1741630	114...	952050665	164...	156919475295
15...	176	65...	2012558	115...	1064144451	165...	172389800255
16...	231	66...	2323520	116...	1188908248	166...	189334822579
17...	297	67...	2679689	117...	1327710076	167...	207890420102
18...	385	68...	3087735	118...	1482074143	168...	228204732751
19...	490	69...	3554345	119...	1653668665	169...	250438925115
20...	627	70...	4087968	120...	1844349560	170...	274768617130
21...	792	71...	4697205	121...	2056148051	171...	301384802048
22...	1002	72...	5392783	122...	2291320912	172...	330495499613
23...	1255	73...	6185689	123...	2552338241	173...	362326859895
24...	1575	74...	7089500	124...	2841940500	174...	397125074750
25...	1958	75...	8118264	125...	3163127352	175...	435157697830
26...	2436	76...	9289091	126...	3519222692	176...	476715857290
27...	3010	77...	10619863	127...	3913864295	177...	522115831195
28...	3718	78...	12132164	128...	4351078600	178...	571701605655
29...	4565	79...	13848650	129...	4835271870	179...	625846753120
30...	5604	80...	15796476	130...	5371315400	180...	684957390936
31...	6842	81...	18004327	131...	5964539504	181...	749474411781
32...	8349	82...	20506255	132...	6620830889	182...	819876908323
33...	10143	83...	23338469	133...	7346629512	183...	896684817527
34...	12310	84...	26543660	134...	8149040695	184...	980462880430
35...	14883	85...	30167357	135...	9035836076	185...	1071823774337
36...	17977	86...	34262962	136...	10015581680	186...	1171432692373
37...	21637	87...	38887673	137...	11097645016	187...	1280011042268
38...	26015	88...	44108109	138...	12292341831	188...	1398341745571
39...	31185	89...	49995925	139...	13610949895	189...	1527273599625
40...	37338	90...	56634173	140...	15065878135	190...	1667727404093
41...	44583	91...	64112359	141...	16670689208	191...	1820701100652
42...	53174	92...	72533807	142...	18440293320	192...	1987276856363
43...	63261	93...	82010177	143...	20390982757	193...	2168627105469
44...	75175	94...	92669720	144...	22540654445	194...	2366022741845
45...	89134	95...	104651419	145...	24908858009	195...	2580840212973
46...	105558	96...	118114304	146...	27517052599	196...	2814570987591
47...	124754	97...	133230930	147...	30388671978	197...	3068829878530
48...	147273	98...	150198136	148...	33549419497	198...	3345365983698
49...	173525	99...	169229875	149...	37027355200	199...	3646072432125
50...	204226	100...	190569292	150...	40853235313	200...	3972999029388

* The numbers in this table were calculated by Major MacMahon, by means of the recurrence formulæ obtained by equating coefficients in the identity

$$(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots)\sum_0^\infty p(n)\,x^n=1.$$

We have verified the table by direct calculation up to $n=158$. Our calculation of $p(200)$ from the asymptotic formula then seemed to render further verification unnecessary.

TABLE V*: $q(n)$.

n	c_n	n	c_n	n	c_n	n	c_n
1	1	26	165	51	4097	76	53250
2	1	27	192	52	4582	77	58499
3	2	28	222	53	5120	78	64234
4	2	29	256	54	5718	79	70488
5	3	30	296	55	6378	80	77312
6	4	31	340	56	7108	81	84756
7	5	32	390	57	7917	82	92864
8	6	33	448	58	8808	83	101698
9	8	34	512	59	9792	84	111322
10	10	35	585	60	10880	85	121792
11	12	36	668	61	12076	86	133184
12	15	37	760	62	13394	87	145578
13	18	38	864	63	14848	88	159046
14	22	39	982	64	16444	89	173682
15	27	40	1113	65	18200	90	189586
16	32	41	1260	66	20132	91	206848
17	38	42	1426	67	22250	92	225585
18	46	43	1610	68	24576	93	245920
19	54	44	1816	69	27130	94	267968
20	64	45	2048	70	29927	95	291874
21	76	46	2304	71	32992	96	317788
22	89	47	2590	72	36352	97	345856
23	104	48	2910	73	40026	98	376256
24	122	49	3264	74	44046	99	409174
25	142	50	3658	75	48446	100	444793

* We are indebted to Mr Darling for this table.

ON THE REPRESENTATION OF A NUMBER AS THE SUM OF ANY NUMBER OF SQUARES, AND IN PARTICULAR OF FIVE OR SEVEN

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Communicated by E. H. Moore, May 21, 1918

1. The formulae concerning the representation of a number as the sum of 5 or 7 squares belong to one of the most unfamiliar and difficult chapters in the Theory of Numbers, and only one proof of them has been given. The proof depends on the general arithmetic theory of quadratic forms, initiated by Eisenstein and perfected by Smith and Minkowski. This theory, of which a systematic account will be found in the fourth volume of Bachmann's *Zahlentheorie* gives a complete solution of the problem of any number s of squares not exceeding 8. Beyond $s = 8$ it fails.

When s is *even* there is an alternative method. This method, which depends on the theory of the elliptic modular functions, is much simpler in idea than the method of Smith and Minkowski; and it has another very important merit, that it can be used—within the limits of human capacity for calculation—for *any* even value of s. Thus Jacobi solved the problem for 2, 4, 6 and 8. In these cases the number of representations can be expressed in terms of the divisors of n. Suppose, *e.g.*, that $s = 8$; and let us write, generally,

$$1+\sum_{1}^{\infty} r_s(n)q^n = (1+2q+2q^4+\ \dots\ )^s = \{\vartheta_3(0,\tau)\}^s = \vartheta^s,$$

where $q = e^{\pi i\tau}$. Then

$$\vartheta^8 = 1+16\left(\frac{1^3q}{1+q}+\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1+q^3}+\frac{4^3q^4}{1-q^4}+\ \dots\ \right),$$

and $r_8(n)$ is $16\,\Sigma\,\delta^3$ if n is odd and $8\,\Sigma\,\delta_0^3 - 8\,\Sigma\,\delta_1^3$ if n is even, δ denoting

1918, 10 *Proceedings of the National Academy of Sciences*, 4, 189–93.

a divisor of n, δ_0 a even, and δ_1 an odd divisor. When s exceeds 8 the formulae are less simple, and involve arithmetical functions of a more recondite nature. Liouville gave formulae concerning the cases $s = 10$ and $s = 12$, and Glaisher[1] has worked out systematically all cases up to $s = 18$. More recently important papers on the subject, to which I shall refer later, have been published by Ramanujan[2] and Mordell.[3] In the latter paper the whole subject is exhibited as a corollary of the general theory of modular invariants.

The primary object of my own researches has been *to deduce the formulae for $s = 5$ and $s = 7$ from the theory of elliptic functions*, and so to place the cases in which s is odd and even, so far as may be, on the same footing. The methods which I use have further important applications, but this is the one which I wish to emphasize at the moment. The formulae which I take as my goal are the formulae

$$r_5(n) = \frac{Bn\sqrt{n}}{\pi^2}\sum\left(\frac{n}{m}\right)\frac{1}{m^2}, \tag{1}$$

$$r_7(n) = \frac{Cn^2\sqrt{n}}{\pi^3}\sum\left(\frac{-n}{m}\right)\frac{1}{m^3}, \tag{2}$$

given by Bachmann (pp. 621, 655). Here n as an odd number not divisible by any square (so that there is no distinction between primitive and imprimitive representations); m runs through all odd numbers prime to n; B is 80, 160, 112, or 160, according as n is congruent to 1, 3, 5 or 7 (mod. 8); and C is 448, 560, 448 or 592 in similar circumstances. These formulae are the central formulae of the theory: they involve infinite series, but these series are readily summed in finite terms by the methods of Dirichlet and Cauchy. With them should be associated the formula

$$r_3(n) = \frac{A\sqrt{n}}{\pi}\sum\left(\frac{-n}{m}\right)\frac{1}{m}, \tag{3}$$

where A is 24, 16, 24, or 0: but this formula, as we shall see, stands in some ways on a different footing.

2. My new proof of the formulae (1) and (2) was arrived at incidentally in the course of researches undertaken with a different end, that of finding *asymptotic* formulae (valid for *all* values of s) for $r_s(n)$ and other arithmetical functions which present themselves as coefficients in the expansions of elliptic modular functions. In a paper[4] shortly to appear in the *Proceedings* of the London Mathematical Society, Mr. Ramanujan and I have developed an exceedingly powerful method for the solution of problems of this character, and applied it to the study of $p(n)$, the number of (unrestricted) partitions of n. This method, when applied to our present problem, introduces the function

$$\Theta_s(q) = 1 + \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \sum_{h,k} \left(\frac{S_{h,k}}{k}\right)^s F(qe^{-h\pi i/k}), \tag{4}$$

where

$$S_{h,k} = \sum_{p=0}^{k-1} e^{p^2 h\pi i/k},\ F(q) = \sum_{1}^{\infty} n^{\frac{1}{2}s-1} q^n,$$

and the summation applies to $k = 1, 2, 3, \ldots$, and all positive values of h less than, of opposite parity to, and prime to k ($h = 0$ being associated with $k = 1$ alone). The coefficient of q^n in $\Theta_s(q)$ is

$$\chi_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \sum_{h,k} \left(\frac{S_{h,k}}{k}\right)^s e^{-nh\pi i/k}; \tag{5}$$

and our method leads to the conclusion that

$$r_s(n) = \chi_s(n) + O(n^{\frac{1}{4}s}), \tag{6}$$

at any rate for every value of s exceeding 4.

When s is *even*, $F(q)$ is an elementary function; and $(S_{h,k})^s$ is easily expressible in a form which does not involve the 'Legendre-Jacobi symbol' $\left(\frac{a}{b}\right)$. The function $X_s(n)$ is then susceptible of a variety of elementary transformations and it was shown by Ramanujan, in the second of his two papers quoted above, that $X_s(n)$ is *identical with* $r_s(n)$ when $s = 4$, 6 or 8. In what follows I confine myself to the case in which s is *odd*, merely remarking that my method (which is entirely unlike that used by Ramanujan) leads directly to an alternative proof of his results.

3. When s is odd, $F(q)$ is not an elementary function. But it is not difficult to prove that

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} F(q) = \sum_{-\infty}^{\infty} \frac{1}{[(2n-\tau)i]^{\frac{1}{2}s}}, \tag{7}$$

every term on the right hand side having an argument numerically less than $\frac{1}{4}s\pi$. Further, $S^s_{h,k} = S^{s-1}_{h,k}\, S_{h,k}$; and the first factor can always be expressed in a simple form. Suppose, to fix our ideas, that $s = 5$. Then $S^4_{h,k} = (-1)^h k^2$. Substituting from this equation and from (7) into (4), and effecting some obvious simplications, we obtain

$$\Theta_5(q) = 1 + \sum_{h,k} \frac{(-1)^h S_{h,k}}{\sqrt{k}} \frac{1}{[(h-k\tau)i]^{\frac{5}{2}}} \tag{8}$$

where now h assumes all values of opposite parity to and prime to k. This formula may be simplified further by multiplying each side by

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

We then find

$$\Theta_5(q)=\frac{8}{\pi^2}\sum_{h,k}\frac{(-1)^k S_{h,k}}{\sqrt{k}}\frac{1}{[(h-k\tau)i]^{\frac{5}{2}}},\tag{9}$$

the summation now extending to $k = 0, 1, 2, \ldots$ and *all h* of opposite parity to k. This is our fundamental formula, when $s = 5$. Two steps remain: first, to prove the identity of $\Theta_5(q)$ and ϑ^5; secondly, to deduce the formulae of Smith and Minkowski.

4. The first step presents no very serious difficulty, for it involves nothing beyond an adaptation of the ideas used by Mordell in his paper quoted in §1. We prove first that Θ_5 behaves like ϑ^5 in respect to the linear modular transformations $\tau = T + 2$, $\tau = -1/T$; so that Θ_5/ϑ^5 is an invariant of the modular sub-group called by Klein-Fricke and Mordell Γ_3. Secondly, by studying the transformation $\tau = (T - 1)/T$, we prove that Θ_5/ϑ^5 is *bounded* in the 'fundamental polygon' associated with Γ_3. It then follows that the quotient is a constant which is easily seen to be unity. In all this the only difficulty arises from the use of certain reciprocity-formulae satisfied by Gauss's sums.

We now transform (9) by effecting the summations with respect to h, using certain contour integrals of a type common in the work of Lindelöf and other writers. We thus obtain

$$\vartheta^5=1+\frac{32}{3}\left\{\sum_{1,3,5,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{3}{2}}q^{mk+j}\right.$$
$$\left.-\sum_{2,4,6,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{\frac{3}{2}}q^{mk+j}\right\}\tag{10}$$

a fundamental identity which contains the whole theory of the representation of numbers by sums of 5 squares. The symbols j and μ alone require explanation; j runs through the complete set of least positive residues of $0, 1^2, 2^2, \ldots, (k-1)^2$ to modulus k, each taken as often as it occurs; and μk is the multiple of k deducted in order to arrive at such a residue. And the remainder of the work is purely arithmetical. Picking out the coefficient of q^n, we obtain a series for $r_5(n)$ which is found, after some reduction, to be equivalent to the series given by Bachmann.

4. The formulae which correspond to (10) for $s = 7$ and $s = 3$ are

$$\vartheta^7 = 1 + \frac{256}{15}\left\{\sum_{1,3,5,\ldots} \frac{(-1)^{\frac{1}{2}(k-1)}}{k^3} \sum_j \sum_{m=0}^{\infty} (mk+j)^{\frac{5}{2}} q^{mk+j} \right. \tag{11}$$

$$\left. - \sum_{2,4,6,\ldots} \frac{1}{k^3} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{\frac{5}{2}} q^{mk+j} \right\},$$

$$\vartheta^3 = 1 + 8\left\{\sum_{1,3,5,\ldots} \frac{(-1)^{\frac{1}{2}(k-1)}}{k} \sum_j \sum_{m=0}^{\infty} (mk+j)^{\frac{1}{2}} q^{mk+j} \right. \tag{12}$$

$$\left. + \sum_{2,4,6,\ldots} \frac{1}{k} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{\frac{1}{2}} q^{mk+j} \right\}.$$

The interpretation of j and μ is as before, except that, when k is even, j is a residue of one of the numbers $\frac{1}{2}k$, $\frac{1}{2}k + 1^2$, . . . , $\frac{1}{2}k + (k - 1)^2$. These identities embody the theory for 7 or 3 squares. It should be noted however, that the application of my method becomes very much more difficult when $s = 3$, as the double series used are then not absolutely convergent; and in fact the only proof of (12) which I possess consists in an identification of the results which it gives with those already known.

I conclude by a word concerning the cases in which $s > 8$. Here, when s is odd, we are on untrodden ground. We have the asymptotic formula (6); and we can evaluate $X_s(n)$ as when $s = 5$ or 7, thus obtaining a series of new results. But it is no longer to be expected that our results should be *exact*, and I have verified that, when $s = 9$, they are not exact, even when $n = 1$.

[1] Glaisher, J. W. L., *Proc. London Math. Soc.*, (Ser. 2), **5**, 1907, (479–490).

[2] Ramanujan, S., *Trans. Camb. Phil. Soc.*, **22**, 1916, (159–184); *Ibid.*, (in course of publication).

[3] Mordell, L. J., *Quart. J. Math.*, **48**, 1917, (93–104).

[4] Hardy, G. H., and Ramanujan, S., *Proc. London Math. Soc.*, (Ser. 2), **17**, 1918, (in course of publication).

CORRECTION

p. 192. Formula (9) is incorrect; see the footnote at the end of § 3.22 of the next paper (1920, 10).

ON THE REPRESENTATION OF A NUMBER AS THE SUM OF ANY NUMBER OF SQUARES, AND IN PARTICULAR OF FIVE*

BY

G. H. HARDY

1. Introduction

1. 1. In a short note published recently in the Proceedings of the National Academy of Sciences[1] I sketched the outlines of a new solution of one of the most interesting and difficult problems in the Theory of Numbers, that of determining the number of representations of a given integer as the sum of five or seven squares. The method which I use is one of great power and generality, and has been applied by Mr. J. E. Littlewood, Mr. S. Ramanujan, and myself to the solution of a number of different problems; and it is probable that, in our previous writings on the subject,[2] we have explained sufficiently the general ideas on which it rests. I may therefore confine myself, for the most part, to filling in the details of my previous work. I should observe, however, that the method by which I now sum the "singular series", which plays a dominant rôle in the analysis,

* Presented to the Society, February, 1920.

[1] G. H. Hardy, *On the expression of a number as the sum of any number of squares, and in particular of five or seven,* Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 189–193.

[2] G. H. Hardy and S. Ramanujan: (1) *Une formule asymptotique pour le nombre des partitions de n,* Comptes Rendus, 2 Jan. 1917; (2) *Asymptotic formulae in Combinatory Analysis,* Proceedings of the London Mathematical Society, ser. 2, vol. 17 (1918), pp. 75–115; (3) *On the coefficients in the expansions of certain modular functions,* Proceedings of the Royal Society, (A), vol. 95 (1918), pp. 144–155:

S. Ramanujan, *On certain trigonometrical sums and their applications in the theory of numbers,* Transactions of the Cambridge Philosophical Society, vol. 22 (1918), pp. 259–276:

G. H. Hardy and J. E. Littlewood: (1) *A new solution of Waring's Problem,* Quarterly Journal of Mathematics, vol. 48 (1919), pp. 272–293: (2) *Note on Messrs. Shah and Wilson's paper entitled On an empirical formula connected with Goldbach's Theorem,* Proceedings of the Cambridge Philosophical Society, vol. 19 (1919), pp. 245–254: (3) *Some problems of 'Partitio Numerorum',* (I) *A new solution of Waring's Problem,* Göttinger Nachrichten, 1920: (4) *Some problems of 'Partitio Numerorum',* (II) *Proof that every large number is the sum of at most 21 biquadrates,* Mathematische Zeitschrift, 1920.

The two last papers will be published shortly.

1920, 10 *Transactions of the American Mathematical Society,* **21, 255–84.**

is quite different from that which I sketched in my former note. The new method has important applications to a whole series of problems in Combinatory Analysis, concerning the representation of numbers by sums of squares, cubes, kth powers, or primes. It is in the present problem that it finds its simplest and most elegant application, and it is most instructive to work this application out in detail.

It is well known that the solution of the problem is a good deal simpler when s, the number of squares in question, does not exceed 8. If s is 2, 4, 6, or 8, the number of representations may be expressed in finite form by means of the real divisors of n; if s is 3, 5, or 7, by means of quadratic residues and non-residues. If $s > 8$, other and more recondite arithmetical functions are involved. In this paper I confine myself to the cases in which $s \leqq 8$. Among these, those in which s is odd have always been regarded as notably the more difficult, and one of my principal objects has been to place them all upon the same footing. But I generally suppose $s = 5$ or $s = 8$, cases typical of the odd and even cases respectively.

In Section 2 I construct the *singular series*

$$\frac{\pi^{\frac{1}{2}s}\, n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \sum_{1}^{\infty} A_k,$$

where

$$A_1 = 1, \qquad A_k = \frac{1}{k^s} \sum_{h} (S_{h,k})^s\, e^{-2nh\pi i/k};$$

$S_{h,k}$ denoting the Gaussian sum[3]

$$\sum_{j=1}^{k} e^{2j^2 h\pi i/k},$$

and the summation extending over all positive values of h less than and prime to k. The series may be written in the form

$$\frac{\pi^{\frac{1}{2}s}\, n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \Big[1 + 0 + \frac{2}{3^{\frac{1}{2}s}} \cos\left(\tfrac{2}{3} n\pi - \tfrac{1}{2} s\pi\right) + \frac{2^{\frac{1}{2}s+1}}{4^{\frac{1}{2}s}} \cos\left(\tfrac{1}{2} n\pi - \tfrac{1}{4} s\pi\right)$$
$$+ \frac{2}{5^{\frac{1}{2}s}} \{\cos \tfrac{2}{5} n\pi + \cos\left(\tfrac{4}{5} n\pi - s\pi\right)\} + 0 + \cdots \Big],$$

the zero terms corresponding to $k = 2$ and $k = 6$.

In Section 3 I show that, when $s = 8$ or $s = 5$, the sum of the singular series is in fact $r_s(n)$, the number of representations of n as a sum of s squares. The methods used are equally applicable in the cases of 3, 4, 6, or 7 squares;

[3] In my former note I denoted a typical "rational point" on the unit circle by $e^{h\pi i/k}$, and a typical Gaussian sum by

$$\sum_{0}^{k-1} e^{j^2 h\pi i/k}.$$

In this paper I generally use the forms involving a 2. Each notation has special advantages for particular purposes.

but the case of two squares is abnormal.[4] Throughout this section I am very deeply indebted to a paper by Mr. Mordell, published recently in the Quarterly Journal of Mathematics.[5] My proof of the identity of the functions which I call ϑ^s and Θ_s is in fact based directly on his work. It is true that Mordell considers only the case in which s is even; but his argument is applicable in principle to either case, and was applied by him to the even case only merely because, at the time when his paper was written, he had no method for the construction, when s is odd, of the essential "principal invariant" denoted by him by χ. It is the construction of this invariant by a uniform method in all cases, through the medium of the "singular series", that is my own principal contribution to the subject.

In Section 4 I show how the singular series may be transformed into a product, and give general rules for the calculation of the terms of the product. All the results of this section are independent of the hypothesis $s \leqq 8$. In Section 5 I sum the series when $s = 8$, and obtain Jacobi's well-known results. In Section 6 I consider the case $s = 5$, supposing however that n has no squared factor, so that there is no distinction between primitive and imprimitive representations; and I obtain results equivalent to those enunciated first by Eisenstein and proved later by Smith and Minkowski. In Section 7 I consider the general case, and show that the method leads to the more complete results of Smith. I conclude, in Section 8, by some remarks as to the application of the method when $s > 8$. I do not pursue this subject further because such applications belong more naturally, either to Mr. Littlewood's and my own researches in connection with Waring's problem, or to Mr. Mordell's in connection with the general theory of modular invariants.

It will be noticed that the explicit formulas for the powers of the fundamental theta-function, such as the familiar formula

$$\vartheta^8 = (1 + 2q + 2q^4 + \cdots)^8 = 1 + 16\left(\frac{1^3\,q}{1+q} + \frac{2^3\,q^2}{1-q^2} + \frac{3^3\,q^3}{1+q^3} + \cdots\right),$$

or the new formula[6]

$$\vartheta^5 = 1 + \frac{32}{3}\left\{\sum_{1,3,5,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(mk+j)^{3/2}q^{mk+j} - \sum_{2,4,6,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{3/2}q^{mk+j}\right\},$$

do not appear at all in my present analysis.

[4] See Mr. Ramanujan's paper quoted in footnote 2.

[5] L. J. Mordell, *On the representations of numbers as a sum of 2r squares*, Quarterly Journal of Mathematics, vol. 48 (1917), pp. 93–104. See also a later paper by the same author, *On the representations of a number as a sum of an odd number of squares*, Transactions of the Cambridge Philosophical Society, vol. 22 (1919), no. 17, pp. 361–372.

[6] This is formula (10) of my former note, where the meaning of j and μ is explained. See also p. 360 of Mr. Mordell's second paper cited above.

In the sequel I give references only to isolated results directly required for the objects of my analysis. It is more convenient to collect here some notes concerning the older memoirs dealing with the problem.

Jacobi's classical results concerning 2, 4, 6, or 8 squares are quoted by Smith on p. 307 of his *Report on the Theory of Numbers* (*Collected Papers*, vol. 1). They are contained implicitly in §§ 40–42 of the *Fundamenta Nova* (pp. 103–115).

Liouville gave formulas relating to the cases of 10 and 12 squares in a number of short notes in the second series of the Journal des mathématiques: see in particular vol. 5, p. 143; vol. 6, p. 233; vol. 9, p. 296; vol. 10, p. 1. These notes appeared between 1860 and 1865.

Later Glaisher, in a series of papers published in the Quarterly Journal, worked out systematically all cases in which s is even and between 2 and 18 inclusive. He has given a short summary of his results in a paper *On the numbers of representations of a number as a sum of 2r squares, where 2r does not exceed* 18, published in the Proceedings of the London Mathematical Society, ser. 2, vol. 5 (1907), pp. 479–490. This paper contains full references to his more detailed work.

The results for 5 squares (for numbers which have no square divisors) were stated without proof by Eisenstein on p. 368 of vol. 35 (1847) of Crelle's Journal. They were completed by Smith, who stated the general results at the end of his memoir *On the orders and genera of quadratic forms containing more than three indeterminates* (Proceedings of the Royal Society, vol. 13 (1864), pp. 199–203, and vol. 16 (1867), pp. 197–208; *Collected Papers*, vol. 1, pp. 412–417, 510–523). No detailed proofs, however, appeared before the publication of the prize memoirs of Smith (*Mémoire sur la représentation des nombres par des sommes de cinq carrés*, Mémoires présentés par divers savants à l'Académie, vol. 29, no. 1 (1887), pp. 1–72; *Collected Papers*, vol. 2, pp. 623–680) and Minkowski (*Mémoire sur la théorie des formes quadratiques à coefficients entières*, ibid., no. 2, pp. 1–178; *Gesammelte mathematische Abhandlungen*, vol. 1, pp. 3–144).

The methods for the summation of the series

$$\sum \left(\frac{n}{m}\right)\frac{1}{m^2},$$

which is fundamental in the five square problem, and other series of similar type, are due to Dirichlet (*Recherches sur divers applications de l'analyse infinitésimale à la théorie des nombres*, Crelle's Journal, vol. 19 (1839), pp. 324–369, and vol. 21 (1840), pp. 1–12, 134–155; *Werke*, vol. 1, pp. 411–497) and to Cauchy (*Mémoire sur la théorie des nombres*, Mémoires de

l'Académie des Sçiences, vol. 17 (1840), pp. 249–768; especially Note 12, pp. 665–699).

A systematic account of the whole theory is given by Bachmann in vol. 4 of his *Zahlentheorie*. Bachmann works out the case $s = 7$ also in detail.

2. Formal construction of the singular series

2. 1. I write, as in my former note

$$(2.11)\quad f(q) = 1 + \sum_1^\infty r_s(n) q^n = (1 + 2q + 2q^4 + \cdots)^s$$
$$= \{\vartheta_3(0, \tau)\}^s = \vartheta^s,$$

where $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$; and I consider the behavior of this function when q tends radially to a "rational point" $e^{2h\pi i/k}$ upon the unit circle. We may suppose that $h = 0, k = 1$, or that k is greater than unity and h positive, less than k, and prime to k.

If (2.11)

$$q = \mathbf{q} e^{2h\pi i/k},$$

so that $0 \leqq \mathbf{q} < 1$, $\mathbf{q} \to 1$, we have

$$\vartheta = 1 + 2\sum_1^\infty \mathbf{q}^{n2} e^{2n^2h\pi i/k}$$
$$= 1 + 2\sum_{j=1}^k \sum_{l=0}^\infty \mathbf{q}^{(lk+j)^2} e^{2(lk+j)^2h\pi i/k}$$
$$= 1 + 2\sum_{j=1}^k e^{2j^2h\pi i/k} \sum_{l=0}^\infty \mathbf{q}^{(lk+j)^2}.$$

Now

$$\sum_{k=0}^\infty \mathbf{q}^{(lk+j)^2} \sim \frac{\sqrt{\pi}}{2k}\left(\log\frac{1}{\mathbf{q}}\right)^{-\frac{1}{2}},$$

when $\mathbf{q} \to 1$. It follows that

$$(2.12)\quad \vartheta \sim \sqrt{\pi}\frac{S_{h,k}}{k}\left(\log\frac{1}{\mathbf{q}}\right)^{-\frac{1}{2}},$$

where

$$(2.121)\quad S_{h,k} = \sum_{j=1}^k e^{2j^2h\pi i/k},$$

and

$$(2.13)\quad f(q) \sim \pi^{\frac{1}{2}s}\left(\frac{S_{h,k}}{k}\right)^s\left(\log\frac{1}{\mathbf{q}}\right)^{-\frac{1}{2}s},$$

it being understood that, when $S_{h,k} = 0$ (as is the case if, and only if, k is of the form $4m + 2$), this equation is to be understood as meaning

$$(2.131)\quad f(q) = o\left(\log\frac{1}{\mathbf{q}}\right)^{-\frac{1}{2}s}.$$

2.2. The principle of the method is to write down a power-series

$$f_{h,k}(q) = \sum c_{h,k,n} q^n, \tag{2.21}$$

which (a) is as simple and natural as possible and (b) behaves as much like $f(q)$ as possible when $q \to e^{2h\pi i/k}$; and to endeavor to approximate to the coefficients in $f(q)$ by means of the sums

$$\rho_s(n) = \sum_{h,k} c_{h,k,n}. \tag{2.22}$$

It is plain that, in forming these sums, we may ignore values of k of the form $4m + 2$.

The appropriate auxiliary function (2.21) is

$$f_{h,k}(q) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \left(\frac{S_{h,k}}{k}\right)^s F_s(q), \tag{2.23}$$

where

$$F_s(q) = \sum_1^\infty n^{\frac{1}{2}s-1} q^n. \tag{2.231}$$

It is in fact well known that

$$F_s(x) - \Gamma(\tfrac{1}{2}s)\left(\log\frac{1}{x}\right)^{-\frac{1}{2}s}$$

is regular at $x = 1$.[7] We are thus led to take

$$c_{h,k,n} = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \left(\frac{S_{h,k}}{k}\right)^s e^{-2nh\pi i/k} \tag{2.24}$$

and

$$\rho_s(n) = \sum_{h,k} c_{h,k,n} = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \sum_1^\infty A_k, \tag{2.25}$$

where

$$A_1 = 1, \qquad A_k = k^{-s} \sum_h (S_{h,k})^s e^{-2nh\pi i/k}, \tag{2.251}$$

the summation extending over all positive values of h less than and prime to k. I call the series

$$\rho_s(n) = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \sum A_k = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} S \tag{2.26}$$

the *singular series*. The process by which it has been constructed is of a purely formal character. It remains (1) to investigate more rigorously its bearing on the solution of our problem and (2) to find its sum.

3. Proof that the sum of the singular series, when $s = 8$ or $s = 5$, is the number of representations of n

3.1. *Proof that* $\rho_8(n) = r_8(n)$.

3.11. When s is 3, 4, 5, 6, 7, or 8 (but not 2 or any number greater

[7] See, for example, E. Lindelöf, *Le calcul des résidus*, p. 139.

than 8) the sum of the singular series gives exactly the number of representations of n. In this section I prove this when $s = 8$ and when $s = 5$. These cases are perfectly typical, but formally a little simpler than the others.

Suppose first that $s = 8$. Then

$$(3.111) \quad \Theta_8(q) = 1 + \sum f_{h,k}(q) = 1 + \frac{\pi^4}{6}\sum\left(\frac{S_{h,k}}{k}\right)^8 F_8(qe^{-2h\pi i/k}).$$

Now

$$S_{h,k}^8 = \eta_k k^4,$$

where η_k is 1, 0, or 16 according as k is odd, oddly even, or evenly even. Also, if $x = e^{-y}$, we have

$$F_8(x) = \sum n^3 x^n = \sum n^3 e^{-ny} = \frac{d^2}{dy^2}(\tfrac{1}{4}\operatorname{cosech}^2 \tfrac{1}{2}y) = 6\sum\frac{1}{(y+2n\pi i)^4},$$

where n runs through all integral values. Hence

$$F_8(qe^{-2h\pi i/k}) = \frac{6k^4}{\pi^4}\sum_n \frac{1}{\{2(nk+h)-k\tau\}^4},$$

and

$$(3.112) \qquad \Theta_8(q) = 1 + \sum_{h,k,n}\frac{\eta_k}{\{2(nk+h)-k\tau\}^4},$$

the summation extending over the values of h, k, and n already specified. If $k > 1$, $nk+h$ assumes all values prime to k; if $k = 1$, all values. Thus

$$(3.113) \qquad \Theta_8(q) = 1 + \sum_{*}\frac{\eta_k}{(2h-k\tau)^4};$$

where now $k = 1, 2, 3, \cdots$ and h assumes all values such that (h, k), the highest common factor of h and k, is unity. But this equation may be written

$$(3.114) \quad \Theta_8(q) = 1 + \sum_{k=1,3,\ldots;\,(h,k)=1}\frac{1}{(2h-k\tau)^4} + \sum_{k=4,8,\ldots;\,(h,k)=1}\frac{16}{(2h-k\tau)^4}$$

$$= 1 + \sum_{k=1,3,\ldots;\,(h,k)=1}\frac{1}{(2h-k\tau)^4} + \sum_{k=2,4,\ldots;\,(h,k)=1}\frac{1}{(h-k\tau)^4}$$

$$= 1 + \sum\frac{1}{(h-k\tau)^4},$$

where now $k = 1, 2, 3, \cdots$ and h assumes all values *of opposite parity to and prime to* k.

Multiplying both sides of (3.114) by

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots,$$

we obtain

$$(3.1151) \qquad \frac{\pi^4}{96}\Theta_8(q) = \frac{\pi^4}{96} + \sum\frac{1}{(h-k\tau)^4}$$

or

$$\frac{\pi^4}{96}\Theta_8(q) = \sum \frac{1}{(h-k\tau)^4}. \tag{3.1152}$$

In (3.1151) $k = 1, 2, 3, \cdots$, and in (3.1152) $k = 0, 1, 2, \cdots$; in each equation h assumes *all* values of opposite parity to k.

3.12. We now write

$$\frac{\pi^4}{96}\Theta_8(q) = \chi(\tau), \tag{3.121}$$

and consider the effect on $\chi(\tau)$ of the modular substitutions

(3.1221) $\tau' = \tau \pm 2,$ (3.1222) $\tau' = -1/\tau.$

It is obvious in the first place, from (3.1151) or (3.1152), that

$$\chi(\tau \pm 2) = \chi(\tau). \tag{3.123}$$

Again, we may write (3.1151) in the form

$$\chi(\tau) = \frac{\pi^4}{96} + \frac{1}{2}\sum_k{}' \sum_h \frac{1}{(h-k\tau)^4},$$

where h and k assume all values of opposite parity except that (as is indicated by the dash) the value $k = 0$ is omitted. Thus

$$\chi\left(-\frac{1}{\tau}\right) = \frac{\pi^4}{96} + \frac{1}{2}\tau^4 \sum_k{}' \sum_h \frac{1}{(h\tau+k)^4} = \frac{\pi^4\tau^4}{96} + \frac{1}{2}\tau^4 \sum_k \sum_h{}' \frac{1}{(h\tau+k)^4}.$$

Changing h and k into $-k$ and h, we obtain

$$\chi(-1/\tau) = \tau^4\chi(\tau). \tag{3.124}$$

Now $\{\vartheta_3(0, \tau)\}^8 = \vartheta^8 = \psi(\tau)$ satisfies the equations

$$\psi(\tau \pm 2) = \psi(\tau), \qquad \psi(-1/\tau) = \tau^4\psi(\tau);$$

and so it follows, from (3.123) and (3.124), that the function

$$\eta(\tau) = \chi(\tau)/\vartheta^8 = \chi(\tau)/\psi(\tau) \tag{3.125}$$

is invariant for the substitutions (3.122), and therefore for the modular subgroup which they generate, the group called by Klein-Fricke and Mordell Γ_3.

3.13. The next step in the proof is to show that $\eta(\tau)$ is bounded throughout the "fundamental polygon" G_3 associated with the group Γ_3. This region is defined by

$$\tau = x + iy, \qquad |\tau| \geqq 1, \qquad -1 \leqq x \leqq 1,$$

and has only the points $\tau = \pm 1$ in common with the real axis. It is therefore sufficient to show that $\eta(\tau)$ is bounded when τ approaches one or other

of these points, say $\tau = 1$. For this purpose, following Mordell, I consider the effect of the substitution

$$\tau = 1 - \frac{1}{T}.$$

If we write $T = X + iY$, and suppose that $\tau \to 1$ from inside G_3, then $Y \to \infty$ and $|Q| = |e^{\pi i T}|$ is small. And

$$\chi\left(1 - \frac{1}{T}\right) = \frac{\pi^4}{96} + \frac{1}{2} T^4 \sum_k{}' \sum_h \frac{1}{\{k + (h-k)T\}^4} \tag{3.131}$$
$$= \frac{\pi^4}{96} + \frac{1}{2} T^4 \sum_k{}' \sum_h \frac{1}{(k + hT)^4}$$
$$= \frac{1}{2} T^4 \sum_k \sum_h \frac{1}{(k + hT)^4},$$

where now k assumes all integral values and h all odd values.

Write $hT = a$, $e^{\pi i a} = \zeta$, and sum with respect to k. We have

$$\sum_k \frac{1}{(k+a)^4} = -\frac{2}{3}\pi^4 \operatorname{cosec}^2 a\pi + \pi^4 \operatorname{cosec}^4 a\pi;$$

and this function, when expanded in powers of ζ, begins with the term

$$\tfrac{8}{3}\pi^4 \zeta^2 = \tfrac{8}{3}\pi^4 Q^{2h}.$$

Hence, when $T = X + iY$ and Y is large, we have

$$\chi\left(1 - \frac{1}{T}\right) = \frac{8}{3}\pi^4 T^4 Q^2 + \cdots,$$

$$\Theta_8(q) = 256 T^4 Q^2 + \cdots. \tag{3.132}$$

But we have also

$$\left\{\vartheta_3\left(0, 1 - \frac{1}{T}\right)\right\}^8 = T^4\{\vartheta_2(0, T)\}^8 = 256 T^4 Q^2 + \cdots, \tag{3.133}$$

and so

$$\eta(\tau) = \eta\left(1 - \frac{1}{T}\right) \to 1. \tag{3.134}$$

Thus $\eta(\tau)$ is an invariant of Γ_3 and bounded throughout G_3; and is therefore necessarily a constant, which is plainly unity.

It follows that

$$\vartheta^8 = \Theta_8(q) \tag{3.135}$$

and so that

$$\rho_8(n) = r_8(n). \tag{3.136}$$

3.2. *Proof that* $\rho_5(n) = r_5(n)$.

3.21. When $s = 5$ the proof proceeds on the same lines, but is not quite so simple. We shall require certain well-known identities which I state as lemmas.

LEMMA 3.211.[8] *If h and k are positive integers of opposite parity, then*

$$\sum_{j=1}^{k} e^{j^2 h\pi i/k} = \sqrt{i}\sqrt{\frac{k}{h}}\sum_{j=1}^{h} e^{-j^2 k\pi i/h}; \tag{3.2111}$$

and if h and k are positive integers, and h odd, then

$$\sum_{j=1}^{k}(-1)^j e^{j^2 h\pi i/k} = \sqrt{i}\sqrt{\frac{k}{h}}\sum_{j=1}^{h} e^{-(j-\frac{1}{2})^2 k\pi i/h}. \tag{3.2112}$$

Here $\sqrt{i} = e^{\frac{1}{4}\pi i}$.

LEMMA 3.212. *Suppose that* $0 < \nu < 1$, *and that* σ *and the real part of t are positive. Then*

$$\frac{(2\pi)^\sigma}{\Gamma(\sigma)}\sum_{0}^{\infty}(m+\nu)^{\sigma-1}e^{-2\pi t(m+\nu)} = \sum_{-\infty}^{\infty}\frac{e^{2n\nu\pi i}}{(t+ni)^\sigma}, \tag{3.2121}$$

where

$$(t+ni)^\sigma = exp\{\sigma\log(t+ni)\} = exp(\sigma\log|t+ni| + \sigma\phi i)$$

and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. *The formula still holds for* $\nu = 0$, *if* $\sigma > 1$.

This result is due to Lipschitz.[9] We shall require two special cases.

(i) Suppose that $\sigma = \frac{1}{2}s > 1$, $\nu = 0$, $t = -\frac{1}{2}i\tau$, $x = e^{\pi i\tau}$; so that $\Im(\tau) > 0$ and $|x| < 1$. Then we obtain

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}F_s(x) = \sum_{-\infty}^{\infty}\frac{1}{\{(2n-\tau)i\}^{\frac{1}{2}s}}, \tag{3.2122}$$

where

$$\{(2n-\tau)i\}^{\frac{1}{2}s} = exp\{\tfrac{1}{2}s\log|(2n-\tau)i| + \tfrac{1}{2}s\phi i\} \tag{3.21221}$$

and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$.

(ii) Suppose that $\nu = \frac{1}{2}(1+\theta)$, $\theta = \lambda/K$, where K and λ are integers and $-K < \lambda < K$, and $t = -K\tau i$. Then

$$\sum_{-\infty}^{\infty}\frac{(-1)^n e^{n\theta\pi i}}{\{(n-K\tau)i\}^\sigma} = x^{K+\lambda}P(x), \tag{3.2123}$$

where $P(x)$ is an ascending power series in x.

3.22. Supposing now that $s = 5$, we have

$$\Theta_5(q) = 1 + \frac{4\pi^2}{3}\sum\left(\frac{S_{h,k}}{k}\right)^5 F_5(qe^{-2h\pi i/k}), \tag{3.221}$$

[8] See, for example, G. Landsberg, *Zur Theorie der Gaussschen Summen und der linearen Transformation der Thetafunktionen*, Journal für Mathematik,, vol. 111 (1893), pp. 234–253. Both formulas are included in formula (17*b*), p. 243, of Landsberg's memoir. The first is also proved by Lindelöf, loc. cit., pp. 73–75.

[9] R. Lipschitz, *Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen*, Journal für Mathematik, vol. 105 (1889), pp. 127–156.

and

$$S^4_{h,k} = \eta_k k^2,$$

where now η_k is 1, 0, or -4 according as k is odd, oddly even, or evenly even. Thus

$$\Theta_5(q) = 1 + \frac{4\pi^2}{3}\sum \frac{\eta_k}{k^3} S_{h,k} F_5(qe^{-2h\pi i/k}). \tag{3.222}$$

Substituting from (3.2122), we obtain

$$\Theta_5(q) = 1 + \sum \frac{\eta_k}{\sqrt{k}} \frac{S_{h,k}}{[\{2(nk+h) - k\tau\}i]^{5/2}}, \tag{3.223}$$

or

$$\Theta_5(q) = 1 + \sum \frac{\eta_k}{\sqrt{k}} \frac{S_{h,k}}{\{(2h - k\tau)i\}^{5/2}}, \tag{3.224}$$

the ranges of summation in these equations being the same as in (3.112) and (3.113) respectively. The last equation can be expressed in a more convenient form by introducing the sum

$$T_{h,k} = \sum_{0}^{k-1} e^{j^2 h\pi i/k}. \tag{3.225}$$

In fact (3 224) may be written

$$\Theta_5(q) = 1 + \sum_{k=1,3,\cdots;\,(h,k)=1} \frac{1}{\sqrt{k}} \frac{S_{h,k}}{\{(2h - k\tau)i\}^{5/2}} - \sum_{k=4,8,\cdots;\,(h,k)=1} \frac{4}{\sqrt{k}} \frac{S_{h,k}}{\{(2h - k\tau)i\}^{5/2}}.$$

In the first sum k is odd; $2h = H$ is even and prime to k; and $S_{h,k} = T_{H,k}$. In the second $k = 2K$, where K runs through all even values; h is odd and prime to K; and $S_{h,k} = 2T_{h,K}$. Effecting these substitutions, and then replacing H (or K) by h (or k), we obtain

$$\Theta_5(q) = 1 + \sum \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h - k\tau)i\}^{5/2}},$$

where now $k = 1, 2, 3, \cdots$ and h assumes all values of opposite parity and prime to k.[10]

Multiplying by

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots,$$

and observing that, if λ is odd,

$$(-1)^{\lambda h} = (-1)^h, \qquad \sqrt{\lambda k}\{(\lambda h - \lambda k\tau)i\}^{5/2} = \lambda^3\sqrt{k}\{(h - k\tau)i\}^{5/2},$$

$$T_{\lambda h, \lambda k} = \lambda T_{h,k},$$

[10] This is equation (8) of my former paper.

we obtain

$$\frac{\pi^2}{8}\Theta_5(q) = \frac{\pi^2}{8} + \sum \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h-k\tau)i\}^{5/2}}, \tag{3.226}$$

where h now assumes all values of opposite parity to k.[11]

3.23. The discussion now follows the lines of 3.12 and 3.13. We write

$$\frac{\pi^2}{8}\Theta_5(q) = \chi(\tau), \tag{3.231}$$

and it is obvious at once that

$$\chi(\tau \pm 2) = \chi(\tau). \tag{3.232}$$

The discussion of the transformation $\tau' = -1/\tau$ requires a little more care, owing to the presence of many-valued functions under the sign of summation. It is convenient to begin by including negative values of k.

We write generally

$$z^s = exp\{s \log |z| + i\,\mathrm{am}\, z\}$$

where the particular value of $\mathrm{am}\, z$ to be selected has to be fixed by special convention. Thus in $\{(h-k\tau)i\}^{5/2}$, where $k > 0$, $\mathrm{am}\{(h-k\tau)i\}$ lies (as has already been explained) between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. We now agree that, if k is still positive, $\mathrm{am}(-k) = \pi$, so that $\sqrt{-k} = i\sqrt{k}$, while $\mathrm{am}\{(-h+k\tau)i\}$ lies between $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$. It will easily be verified that

$$\sqrt{-k}\{(-h+k\tau)i\}^{5/2} = \sqrt{k}\{(h-k\tau)i\}^{5/2}. \tag{3.233}$$

Further, we write by definition

$$T_{-h,-k} = T_{h,k}. \tag{3.234}$$

We know from (3.2111) that, when h and k are both positive,

$$T_{h,k} = \sqrt{i}\sqrt{\frac{k}{h}}\,T_{-k,h};$$

and it is easy to verify that, with our conventions, we have generally

$$T_{h,k} = \epsilon\sqrt{i}\sqrt{\frac{k}{h}}\,T_{-k,h}, \tag{3.235}$$

where $\epsilon = 1$ unless $h > 0$, $k < 0$, in which case $\epsilon = -1$.

3.24. We have, from (3.226), (3.233), and (3.234),

$$\chi(\tau) = \frac{\pi^2}{8} + \frac{1}{2}\sum_k{}' \sum_h \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h-k\tau)i\}^{5/2}} \tag{3.241}$$

$$= \frac{\pi^2}{8} + \frac{\pi^2}{8}(-\tau i)^{-5/2} + \frac{1}{2}\sum{}' \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h-k\tau)i\}^{5/2}},$$

[11] This equation takes the place of (9) of my former paper, which is not printed correctly. The first term on the right is omitted, and $k = 0$ is included wrongly under the sign of summation.

where now h and k are any integers other than zero and of opposite parity. Writing $-1/\tau$ for τ in (3.241), using (3.235), and then replacing h and k by K and $-H$, we obtain

$$\text{(3.242)} \quad \chi\left(-\frac{1}{\tau}\right)=\frac{\pi^2}{8}+\frac{\pi^2}{8}\left(\frac{i}{\tau}\right)^{-5/2}+\frac{1}{2}\sqrt{i}\sum{}'\frac{(-1)^K\epsilon}{\sqrt{K}}\frac{T_{H,K}}{\{(K-H/\tau)i\}^{5/2}},$$

where ϵ is 1 unless H and K are both positive, and then -1, and

$$\operatorname{am}\{(K-H/\tau)i\}$$

lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ if $H<0$, between $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$ if $H>0$.

It may be verified without difficulty that

$$\text{(3.243)} \qquad \left\{\left(K-\frac{H}{\tau}\right)i\right\}^{5/2}=\epsilon\left(-\frac{1}{\tau}\right)^{5/2}\{(H-K\tau)i\}^{5/2},$$

where $0<\operatorname{am}(-1/\tau)<\pi$ and the value of $\operatorname{am}\{(H-K\tau)i\}$ is fixed in accordance with our previous conventions. Consider, for example, the case $H>0$, $K<0$. In this case

$$-\tfrac{3}{2}\pi<\alpha=\operatorname{am}\{(K-H/\tau)i\}<-\tfrac{1}{2}\pi,$$

$$0<\beta=\operatorname{am}(-1/\tau)<\pi,$$

and

$$-\tfrac{1}{2}\pi<\gamma=\operatorname{am}\{(H-K\tau)i\}<\tfrac{1}{2}\pi.$$

Thus $\beta+\gamma$ lies between $-\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and, as α differs from $\beta+\gamma$ by a multiple of 2π, we must have $\alpha=\beta+\gamma-2\pi$ and

$$\left\{\left(K-\frac{H}{\tau}\right)i\right\}^{5/2}=e^{-5\pi i}\left(-\frac{1}{\tau}\right)^{5/2}\{(H-K\tau)i\}^{5/2}$$

$$=-\left(-\frac{1}{\tau}\right)^{5/2}\{(H-K\tau)i\}^{5/2},$$

in agreement with (3.243). The other possible cases may be treated similarly.

Thus

$$\text{(3.244)} \quad \chi\left(-\frac{1}{\tau}\right)=\frac{\pi^2}{8}+\frac{\pi^2}{8}\left(\frac{i}{\tau}\right)^{-5/2}$$

$$+\tfrac{1}{2}\sqrt{i}\left(-\frac{1}{\tau}\right)^{-5/2}\sum{}'\frac{(-1)^K}{\sqrt{K}}\frac{T_{H,K}}{\{(H-K\tau)i\}^{5/2}},$$

where $-\frac{1}{2}\pi<\operatorname{am}(i/\tau)<\frac{1}{2}\pi$, $0<\operatorname{am}(-1/\tau)<\pi$. And this equation

leads to

$$(3.245)\quad \chi\left(-\frac{1}{\tau}\right)=\frac{\pi^2}{8}\left(\frac{i}{\tau}\right)^{-5/2}+\tfrac{1}{2}\sqrt{i}\left(-\frac{1}{\tau}\right)^{-5/2}\sum_K{}'\sum_H\frac{(-1)^K}{\sqrt{K}}\frac{T_{H,K}}{\{(H-K\tau)i\}^{5/2}}$$

$$=\left(\frac{\tau}{i}\right)^{5/2}\left[\frac{\pi^2}{8}+\frac{1}{2}\sum_K{}'\sum_H\frac{(-1)^H}{\sqrt{K}}\frac{T_{H,K}}{\{(H-K\tau)i\}^{5/2}}\right]$$

$$=\left(\frac{\tau}{i}\right)^{5/2}\chi(\tau).^{12}$$

This is the same functional equation as is satisfied by ϑ^5. Hence

$$(3.246)\qquad \eta(\tau)=\chi(\tau)/\vartheta^5$$

is an invariant for each of the substitutions (3.122), and so for Γ_3.

3.25. It remains to verify that $\eta(\tau)$ is bounded in G_3. As in 3.13, it is only necessary to consider the neighborhood of $\tau=1$. Putting $\tau=1-1/T$, as in 3.13, in (3.241), and then writing $h=H-K$, $k=H$, we obtain

$$(3.251)\quad \frac{\pi^2}{8}\chi\left(1-\frac{1}{T}\right)=\frac{\pi^2}{8}+\frac{1}{2}\sum_H{}'\sum_K\frac{(-1)^H}{\sqrt{H}}\frac{T_{H-K,H}}{\{(-K+H/\tau)i\}^{5/2}},$$

where H assumes all values save 0 and K all odd values, and

$$\operatorname{am}\{(-K+H/\tau)i\}$$

lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, or between $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$, according as $H>0$ or $H<0$.

Now

$$(3.252)\qquad T_{H-K,H}=\sum_0^{H-1}(-1)^j e^{-j^2K\pi i/H}=U_{K,H},$$

say. It follows from (3.2112) that, if h and k are both positive, and h is odd, and

$$(3.253)\qquad U_{h,k}=\sum_0^{k-1}(-1)^j e^{-j^2h\pi i/k},\qquad W_{h,k}=\sum_1^k e^{(j-\frac{1}{2})^2h\pi i/k},$$

then

$$U_{h,k}=\sqrt{-i}\sqrt{\frac{k}{h}}\,W_{k,h};$$

and it is easily verified that, if we adopt the same conventions as in 3.23 concerning the meanings of $\sqrt{-h}$, $\sqrt{-k}$, $U_{-h,-k}$, and $W_{-h,-k}$, we have

[12] We first restore the terms for which $H=0$, and then observe that

$$(-1)^H=-(-1)^K.$$

We have to verify that, with our conventions,

$$-\sqrt{i}(-1/\tau)^{-\frac{5}{2}}(-\tau i)^{-\frac{5}{2}}=1$$

and

$$-\sqrt{i}(-1/\tau)^{-\frac{5}{2}}=-(i/\tau)^{-\frac{5}{2}}:$$

these verifications present no difficulty.

generally

$$U_{h,\,k} = \epsilon\,\sqrt{-\,i}\sqrt{\frac{k}{h}}\,W_{k,\,h}, \tag{3.254}$$

where $\epsilon = 1$ unless $h < 0$, $k > 0$, in which case $\epsilon = -1$.

Using this equation in (3.251), we obtain

$$\begin{aligned} \frac{\pi^2}{8}\chi\left(1 - \frac{1}{T}\right) &= \frac{\pi^2}{8} - \tfrac{1}{2}\sqrt{-\,i}\sum_H{}'\sum_K \frac{(-1)^H\epsilon}{\sqrt{K}}\frac{W_{H,K}}{\{(-K + H/T)i\}^{5/2}} \\ &= -\tfrac{1}{2}\sqrt{-\,i}\sum_H\sum_K \frac{(-1)^H\epsilon}{\sqrt{K}}\frac{W_{H,K}}{\{(-K + H/T)i\}^{5/2}}. \end{aligned} \tag{3.255}$$

It is now easy to verify, by arguments similar to those of 3.24, that

$$\{(-K + H/T)i\}^{5/2} = \epsilon T^{-5/2}\{(H - KT)i\}^{5/2},$$

where $\operatorname{am} T$ lies between 0 and π, while $\operatorname{am}\{(H - KT)i\}$ obeys our previous conventions. We thus obtain

$$\frac{\pi^2}{8}\chi\left(1 - \frac{1}{T}\right) = -\tfrac{1}{2}\sqrt{-\,i}\,T^{5/2}\sum_H\sum_K\frac{(-1)^H}{\sqrt{K}}\frac{W_{H,K}}{\{(H - KT)i\}^{5/2}},$$

the summation being now limited only by the fact that K is odd. In virtue of (3.233), this equation may be written

$$\frac{\pi^2}{8}\chi\left(1 - \frac{1}{T}\right) = -\sqrt{-\,i}\,T^{5/2}\sum_{K=1,3,5,\ldots}\frac{1}{\sqrt{K}}\sum_H\frac{(-1)^H W_{H,K}}{\{(H - KT)i\}^{5/2}}. \tag{3.256}$$

3.26. The series in (3.256) may be expressed in the form

$$\sum_{K=1,3,5,\ldots}\frac{1}{\sqrt{K}}\sum_{j=1}^{K}\sum_H\frac{(-1)^H e^{-(j-\frac{1}{2})^2 H\pi i/K}}{\{(H - KT)i\}^{5/2}}.$$

Suppose that

$$(j - \tfrac{1}{2})^2 \equiv \lambda_j \pmod{2K} \qquad (-K < \lambda_i < K),$$

and $\theta_j = \lambda_j/K$. Then

$$\sum_H\frac{(-1)^H e^{-(j-\frac{1}{2})^2 H\pi i/K}}{\{(H - KT)i\}^{5/2}} = \sum_H\frac{(-1)^H e^{-H\theta_j\pi i}}{\{(H - KT)i\}^{5/2}}$$

may, by (3.2123), be expanded as a power series in $Q = e^{\pi iT}$, in which the lowest power of Q is

$$Q^{K+\lambda_j}.$$

The smallest possible values of $K + \lambda_j$ are $\frac{1}{4}, \frac{5}{4}, \cdots$; and $K + \lambda_j = \frac{1}{4}$ involves

$$j^2 - j + \tfrac{1}{4} \equiv -K + \tfrac{1}{4} \pmod{2K}$$

or

$$j^2 - j = (2p - 1)K,$$

where p is an integer, i.e. an equation whose left-hand side is even and whose right-hand side is odd. Thus $K + \lambda_j \geqq \frac{5}{4}$; and the left-hand side of (3.256) is the product of $T^{5/2} Q^{5/4}$ by an ascending power-series in Q. But

$$\vartheta_3\left(0, 1 - \frac{1}{T}\right) = \frac{\sqrt{T}}{\sqrt{i}} \vartheta_2(0, T)$$

is the product of $T^{1/2} Q^{1/4}$ by a power-series in Q. It now follows, just as in 3.13, that $\eta(\tau)$ is bounded, and so is a constant, which is plainly unity.

We have thus established the identity of Θ_s and ϑ^s, and so of $\rho_s(n)$ and $r_s(n)$, when $s = 8$ and $s = 5$. The same method may be used for any value of s from 5 to 8 inclusive.[13] In order to complete the solution of our problem, we have to sum the singular series (2.26).

4. General rules for the summation of the singular series

4.1. *The value of* $S_{h,k}$.

The known results concerning the value of the Gaussian sum $S_{h,k}$ are as follows.[14] We assume that $(h, k) = 1$.

If k and k' are prime to one another

$$S_{h,kk'} = S_{hk',k} S_{hk,k'}. \tag{4.11}$$

We need therefore consider only the cases in which $k = 2^\lambda$ or $K = p^\lambda$, p being an odd prime.

If $k = 2$

$$S_{h,k} = 0. \tag{4.121}$$

If $k = 2^\lambda = 2^{2\mu+1}$, and $\mu > 0$,

$$S_{h,k} = 2^{\mu+1} e^{\frac{1}{4}h\pi i}. \tag{4.122}$$

If $k = 2^\lambda = 2^{2\mu}$, and $\mu > 0$,

$$S_{h,k} = 2^\mu (1 + i^h) = 2^{\mu+1} \cos \tfrac{1}{4}h\pi\, e^{\frac{1}{4}h\pi i}. \tag{4.123}$$

If $k = p$,

$$S_{h,k} = \left(\frac{h}{p}\right) i^{\frac{1}{4}(p-1)^2} \sqrt{p}, \tag{4.131}$$

where (h/p) is the well known symbol of Legendre and Jacobi.

[13] When $s = 2$ or $s > 8$, the conclusion is false. The cases $s = 3$ and $s = 4$ are exceptional. The conclusion is true, but new difficulties arise in the proof because the series used are not all absolutely convergent. These difficulties are easily surmounted when $s = 4$, but are more serious when $s = 3$.

[14] For proofs of these assertions see the chapter on Gauss's sums in the second volume of Bachmann's *Zahlentheorie*. A less complete account is given in Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie*, ed. 4, 1894, pp. 287 et seq.

If $k = p^\lambda = p^{2\mu+1}$, and $\mu > 0$,

$$S_{h,k} = p^\mu S_{h,p}. \tag{4.132}$$

Finally, if $k = p^\lambda = p^{2\mu}$, and $\mu > 0$,

$$S_{h,k} = p^\mu. \tag{4.133}$$

These formulas enable us to write down the value of $S_{h,k}$ for all co-prime pairs of values of h and k.

The multiplication rule for A_k.

4.2. The first step is to prove that

$$A_{kk'} = A_k A_{k'} \tag{4.21}$$

whenever k and k' are prime to one another.

In the formula which defines A_k, viz.

$$k^s A_k = \sum (S_{h,k})^s e^{-2nh\pi i/k},$$

h assumes all positive values less than and prime to k. Let us call this set of values, or any set congruent to this set to modulus k, a *k-set*. It is easy to see that if h runs through a k-set, and h' through a k'-set, then

$$\mathbf{h} = hk' + h'k$$

runs through a kk'-set. For the number of values of $\mathbf{h}$ is

$$\phi(k)\phi(k') = \phi(kk'),$$

and it is obvious that all are prime to kk' and incongruent to modulus kk'.

Thus

$$(kk')^s A_{kk'} = \sum_{\mathbf{h}} (S_{\mathbf{h},kk'})^s e^{-2n\mathbf{h}\pi i/kk'}$$
$$= \sum_{\mathbf{h}} (S_{\mathbf{h}k',k})^s (S_{\mathbf{h}k,k'})^s e^{-2n\mathbf{h}\pi i/kk'},$$

by (4.11). But

$$S_{\mathbf{h}k',k} = \sum_{j=1}^{k} e^{2j^2\mathbf{h}k'\pi i/k} = \sum_{j=1}^{k} e^{2(jk')^2h\pi i/k} = S_{h,k},$$

since jk' runs through a complete system of residues to modulus k; and similarly $S_{\mathbf{h}k,k'} = S_{h',k'}$. Thus

$$(kk')^s A_{kk'} = \sum_{h,h'} (S_{h,k})^s (S_{h',k'})^s e^{-2n\mathbf{h}\pi i/kk'}$$
$$= \sum_{h} (S_{h,k})^s e^{-2nh\pi i/k} \sum_{h'} (S_{h',k'})^s e^{-2nh'\pi i/k'}$$
$$= (kk')^s A_k A_{k'};$$

which proves (4.21).

It follows[15] that

(4.22) $$S = A_1 + A_2 + A_3 + \cdots = 1 + A_2 + A_3 + \cdots = \prod \chi_p$$

where

(4.23) $$\chi_p = 1 + A_p + A_{p^2} + A_{p^3} + \cdots$$

and p runs through all prime values.

Calculation of A_{2^λ}.

4.3. Suppose first that $p = 2$. Then the value of A_{2^λ} is given by the following system of rules.

4.31. $$A_2 = 0.$$

4.32. *If λ is odd and greater than* 1,

(4.321) $$A_{2^\lambda} = 0,$$

unless

(4.3221) $$n \equiv 0 \pmod{2^{\lambda-3}}$$

and

(4.3222) $$\nu - s = 2^{-(\lambda-3)}\, n - s \equiv 0 \pmod{4},$$

in which case

(4.3223) $$A_{2^\lambda} = 2^{-(\frac{1}{2}s-1)(\lambda-1)}\, e^{-\frac{1}{4}(\nu-s)\pi i}.$$

If $\lambda = 3$, (4.3221) *is satisfied automatically.*

Let $\lambda = 2\mu + 1\ (\mu > 0)$. Then

$$S_{h,\,2^\lambda} = 2^{\mu+1} e^{\frac{1}{4}h\pi i} = 2^{\mu-1} S_{h,\,8},$$

by (4.122). We write

$$h = 8z + h' \qquad (z = 0, 1, \cdots, 2^{\lambda-3} - 1;\ h' = 1, 3, 5, 7).$$

Then

$$A_{2^\lambda} = 2^{-s(\mu+2)} \sum_h (S_{h,\,8})^s\, e^{-2nh\pi i/2^{2\mu+1}}$$

$$= 2^{-s(\mu+2)} \sum_{h'} (S_{h',\,8})^s\, e^{-2nh'\pi i/2^{2\mu+1}} \sum_z e^{-2nz\pi i/2^{2\mu-2}},$$

which vanishes (in virtue of the summation with respect to z) unless $n \equiv 0 \pmod{2^{2\mu-2}}$.

If $n = 2^{2\mu-2}\,\nu$, we have

$$A_{2^\lambda} = 2^{2\mu-2-s(\mu+2)} \sum_{h'} (S_{h',\,8})^s\, e^{-\frac{1}{4}\nu h'\pi i}.$$

The sum with respect to h' is

$$2^{2s}\{e^{-\frac{1}{4}(\nu-s)\pi i} + e^{-\frac{3}{4}(\nu-s)\pi i} + e^{-\frac{5}{4}(\nu-s)\pi i} + e^{-\frac{7}{4}(\nu-s)\pi i}\},$$

[15] We assume that the series and product are absolutely convergent. This is obviously the case if $s > 4$, as $S_{h,k} = O(\sqrt{k})$, $A_k = O(k^{1-\frac{1}{2}s})$, and $1 - \frac{1}{2}s < -1$.

which is 0 or $2^{2s+2}e^{-\frac14(\nu-s)\pi i}$ according as (4.3222) is not or is satisfied. This completes the proof of 4.32.

4.33. *If λ is even and greater than* 1,

(4.331) $$A_{2^\lambda} = 0$$

unless

(4.3321) $$n \equiv 0 \pmod{2^{\lambda-2}},$$

in which case

(4.3322) $$A_{2^\lambda} = 2^{-(\frac12 s-1)(\lambda-1)}\cos\left(\tfrac12\nu\pi - \tfrac14 s\pi\right),$$

where $n = 2^{\lambda-2}\nu$. If $\lambda = 2$, the last formula holds in any case.

If $\lambda = 2\mu$, we have

$$S_{h,2^\lambda} = 2^{\mu+1}\cos\tfrac14 h\pi\, e^{\frac14 h\pi i} = 2^{\mu-1}S_{h,4},$$

by (4.123). We write

$$h = 4z + h' \qquad (z = 0, 1, \cdots, 2^{\lambda-2}-1;\ h' = 1, 3).$$

Then

$$\begin{aligned} A_{2^\lambda} &= 2^{-s(\mu+1)}\sum_h (S_{h,4})^s e^{-2nh\pi i/2^{2\mu}} \\ &= 2^{-s(\mu+1)}\sum_{h'} (S_{h',4})^s e^{-2nh'\pi i/2^{2\mu}}\sum_z e^{-2nz\mu i/2^{2\mu-2}}, \end{aligned}$$

which vanishes (in virtue of the summation with respect to z) unless $n \equiv 0 \pmod{2^{2\mu-2}}$. But if $n = 2^{2\mu-2}\nu$, we have

$$A_{2^\lambda} = 2^{2\mu-2-s(\mu+1)}\sum_{h'}(S_{h',4})^s e^{-\frac12\nu h'\pi i},$$

and the sum here is

$$2^{2s}\{(\cos\tfrac14\pi)^s e^{\frac14 s\pi i-\frac12\nu\pi i} + (\cos\tfrac34\pi)^s e^{\frac34 s\pi i-\frac32\nu\pi i}\} = 2^{\frac32 s+1}\cos\left(\tfrac12\nu\pi - \tfrac14 s\pi\right).$$

This completes the proof of 4.33. If $\lambda = 2$, z disappears from the argument and h and h' are identical.

Calculation of A_{p^λ} when p is odd.

4.4. The corresponding results when p is odd are as follows.

4.41. *If $n \not\equiv 0 \pmod p$ then*

(4.411) $$A_p = -p^{-\frac12 s} \qquad (s \equiv 0),$$

(4.412) $$A_p = \left(\frac{n}{p}\right)p^{-\frac12(s-1)} \qquad (s \equiv 1),$$

(4.413) $$A_p = -(-1)^{\frac12(p-1)}p^{-\frac12 s} \qquad (s \equiv 2),$$

(4.414) $$A_p = (-1)^{\frac12(p-1)}\left(\frac{n}{p}\right)p^{-\frac12(s-1)} \qquad (s \equiv 3);$$

the congruences for s referring to modulus 4. *But if $n \equiv 0 \pmod p$ then*

(4.415) $$A_p = (p-1)p^{-\frac12 s} \qquad (s \equiv 0),$$

(4.416) $$A_p = 0 \qquad (s \equiv 1, 3),$$

(4.417) $$A_p = (-1)^{\frac{1}{2}(p-1)}(p-1)p^{-\frac{1}{2}s} \qquad (s \equiv 2).$$

We have

$$A_p = p^{-s}\sum_h (S_{h,\,p})^s e^{-2nh\pi i/p} = i^{\frac{1}{2}s(p-1)^2} p^{-\frac{1}{2}s} \sum_h \left(\frac{h}{p}\right)^s e^{-2nh\pi i/p}.$$

If s is even, this is

$$i^{\frac{1}{2}s(p-1)^2} p^{-\frac{1}{2}s} \sum_h e^{-2nh\pi i/p},$$

and the sum is equal to -1 or to $p-1$ according as n is not or is a multiple of p. This leads at once to the results stated for even values of s.

If on the other hand s is odd, we have

$$A_p = i^{\frac{1}{2}s(p-1)^2} p^{-\frac{1}{2}s} \sum_h \left(\frac{h}{p}\right) e^{-2nh\pi i/p},$$

which is equal to 0 if n is a multiple of p, and to

$$i^{\frac{1}{2}(s-1)(p-1)^2}\left(\frac{n}{p}\right) p^{-\frac{1}{2}(s-1)}$$

otherwise. We thus obtain the results stated for odd values of s.

4.42. *If λ is odd and greater than 1, then*

(4.421) $$A_{p^\lambda} = 0$$

if $n \not\equiv 0 \pmod{p^{\lambda-1}}$;

(4.422) $$A_{p^\lambda} = p^{-(\frac{1}{2}s-1)(\lambda-1)} A_p(\nu)$$

if $n = p^{\lambda-1}\nu$ *and* $\nu \not\equiv 0 \pmod p$; *and*

(4.4231) $$A_{p^\lambda} = (p-1)p^{\lambda-1-\frac{1}{2}s\lambda} \qquad (s \equiv 0),$$

(4.4232) $$A_{p^\lambda} = 0 \qquad (s \equiv 1, 3),$$

(4.4233) $$A_{p^\lambda} = (-1)^{\frac{1}{2}(p-1)}(p-1)p^{\lambda-1-\frac{1}{2}s\lambda} \qquad (s \equiv 2),$$

if $n \equiv 0 \pmod{p^\lambda}$.

If $\lambda = 2\mu+1$, $\mu > 0$, we have $S_{h,\,p^\lambda} = p^\mu S_{h,\,p}$, by (4.132). We write

$$h = pz + h' \qquad (z = 0, 1, \cdots, p^{\lambda-1}-1;\ h' = 1, 2, \cdots, p-1).$$

Then

$$A_{p^\lambda} = p^{-s(\mu+1)} \sum_h (S_{h,\,p})^s e^{-2nh\pi i/p^{2\mu+1}}$$

$$= p^{-s(\mu+1)} \sum_{h'} (S_{h',\,p})^s e^{-2nh'\pi i/p^{2\mu+1}} \sum_z e^{-2nz\pi i/p^{2\mu}}.$$

If $n \not\equiv 0 \pmod{p^{2\mu}}$, the sum with respect to z vanishes, and we obtain (4.421).

If $n = p^{2\mu}\nu$, we obtain

$$A_{p^\lambda} = p^{2\mu - s(\mu+1)} \sum_{h'} (S_{h',\,p})^s e^{-2\nu h'\pi i/p}.$$

If $\nu \not\equiv 0 \pmod{p}$, this is $p^{-(s-2)\mu} A_p(\nu)$, and we obtain (4.422). But if $\nu \equiv 0 \pmod{p}$ we have

$$\sum_{h'} (S_{h',\,p})^s = i^{\frac{1}{4}s(p-1)^2} p^{\frac{1}{2}s} \sum_{h'} \left(\frac{h'}{p}\right)^s,$$

and we obtain the equations (4.423).

4.43. *If λ is even and greater than* 1, *then*

(4.431) $$A_{p^\lambda} = 0$$

if $n \not\equiv 0 \pmod{p^{\lambda-1}}$;

(4.432) $$A_{p^\lambda} = -\,p^{\lambda-1-\frac{1}{2}s\lambda}$$

if $n = p^{\lambda-1}\nu$ *and* $\nu \not\equiv 0 \pmod{p}$; *and*

(4.433) $$A_{p^\lambda} = (p-1)\,p^{\lambda-1-\frac{1}{2}s\lambda}$$

if $n \equiv 0 \pmod{p^\lambda}$.

If $\lambda = 2\mu$, we have $S_{h,\,p^\lambda} = p^\mu$, by (4.133). Hence

$$A_{p^\lambda} = p^{-s\mu} \sum_{h'} e^{-2nh'\pi i/p^{2\mu}} \sum_{z} e^{-2nz\pi i/p^{2\mu-1}},$$

which is zero if $n \not\equiv 0 \pmod{p^{2\mu-1}}$, and

$$p^{2\mu-1-s\mu} \sum_{h'} e^{-2\nu h'\pi i/p}$$

if $n = p^{2\mu-1}\nu$; and the sum here is equal to -1 or $p-1$ according as ν is not or is divisible by p.

5. Summation of the singular series when $s = 8$

5. 1. The formulas of Section 4 enable us to sum the singular series whatever the value of s. I take as typical the cases $s = 8$ and $s = 5$. I suppose first that $s = 8$ and that n has no squared factor. We have to determine the factors χ_p of (4.22).

In the first place, let $p = 2$. Then, as n is not divisible by 4, we have $A_{16} = A_{32} = \cdots = 0$, by (4.321) and (4.331); and also $A_8 = 0$, by (4.321), since $\nu = 1$ and $\nu - s$ is not a multiple of 4. If n is odd, $A_4 = 0$, by (4.331); but if n is even,

$$A_4 = 2^{-3}\cos\tfrac{1}{2}\nu\pi = -\tfrac{1}{8},$$

by (4.3322). Finally $A_2 = 0$ in any case, by 4.31. Thus

(5.11) $$\chi_2 = 1 \ (n \text{ odd}), \qquad \chi_2 = \tfrac{7}{8} \ (n \text{ even}).$$

Next, suppose p odd and $p \nmid n$.[16] Then $A_{p^2} = A_{p^3} = \cdots = 0$, by (4.421) and (4.431), and $A_p = -p^{-4}$, by (4.411). Thus

$$\chi_p = 1 - p^{-4} \qquad (p \nmid n). \tag{5.12}$$

Finally suppose p odd and $p \mid n$. Then $A_{p^3} = A_{p^4} = \cdots = 0$, by (4.421) and (4.431); $A_{p^2} = p^{-7}$, by (4.432); and

$$A_p = (p-1)p^{-4},$$

by (4.415). Thus

$$\chi_p = 1 + (p-1)p^{-4} - p^{-7} = (1+p^{-3})(1-p^{-4}). \tag{5.13}$$

We have therefore

$$S = \chi_2 \prod_{p \nmid n} (1-p^{-4}) \prod_{p \mid n} \{(1+p^{-3})(1-p^{-4})\}$$
$$= \chi_2 \prod (1-p^{-4}) \prod_{p \mid n} (1+p^{-3}) = \frac{96}{\pi^4}\chi_2 \prod_{p \mid n} (1+p^{-3}),$$

since

$$\prod (1-p^{-4}) = \frac{16}{15\zeta(4)} = \frac{96}{\pi^4}.$$

If n is odd,

$$\rho_8(n) = \frac{\pi^4 n^3}{\Gamma(4)} \frac{96}{\pi^4} \prod_{p \mid n} (1+p^{-3}) = 16\sigma_3(n), \tag{5.14}$$

where $\sigma_3(n)$ is the sum of the cubes of the divisors of n. If n is even,

$$\rho_8(n) = 16n^3(1-2^{-3}) \prod_{p \mid n} (1+p^{-3}) = 16\{\sigma_3'(n) - \sigma_3''(n)\}, \tag{5.15}$$

where $\sigma_3'(n)$ and $\sigma_3''(n)$ are the sums of the cubes of the even and odd divisors respectively. These are Jacobi's well-known results, proved at present, however, only when n is not divisible by any square.

5.2. Proceeding to the general case, suppose that

$$n = 2^\alpha \omega^a \omega'^{a'} \cdots \qquad (\alpha \geqq 0;\ a, a', \cdots > 0) \tag{5.21}$$

and consider first A_{2^λ}.

If $\lambda = 1$, $A_{2^\lambda} = 0$, by 4.31. If λ is odd and greater than 1, $A_{2^\lambda} = 0$, by (4.321), unless $\nu = n/2^{\lambda-3} \equiv 0 \pmod 4$, i.e. unless $n \equiv 0 \pmod{2^{\lambda-1}}$, or unless $\lambda \leqq \alpha + 1$. If this condition is satisfied, and $n = 2^\alpha N$, so that N is odd, we have

$$A_{2^\lambda} = 2^{-3(\lambda-1)} e^{-2^{\alpha+1-\lambda} N\pi i},$$

by (4.3223); and so

$$A_{2^\lambda} = 2^{-3(\lambda-1)} \quad (\lambda < \alpha+1), \qquad A_{2^\lambda} = -2^{-3(\lambda-1)} \quad (\lambda = \alpha+1). \tag{5.22}$$

On the other hand, if λ is even, $A_{2^\lambda} = 0$, by (4.331), unless $n \equiv 0 \pmod{2^{\lambda-2}}$,

[16] Following Landau, I write $p \mid n$ for 'p is a divisor of n' and $p \nmid n$ for 'p is not a divisor of n'.

i.e. unless $\lambda \leqq \alpha + 2$. If this condition is satisfied we have, by (4.3322),

$$A_{2^\lambda} = 2^{-3(\lambda-1)} \cos(2^{\alpha+1-\lambda} N\pi).$$

The cosine is 1 if $\lambda < \alpha + 1$, -1 if $\lambda = \alpha + 1$, and 0 if $\lambda = \alpha + 2$. Thus the equations (5.22) still hold for even values of λ. We have therefore

$$\chi_2 = 1 + 0 + 2^{-3} + 2^{-6} + \cdots + 2^{-3(\alpha-1)} - 2^{-3\alpha}, \tag{5.23}$$

the zero term corresponding to $\lambda = 1$.

Next suppose that p is odd. If p is not an ω, $\chi_p = 1 - p^{-4}$, as before. If $p = \omega$ and $\lambda < a + 1$, $n \equiv 0 \pmod{\omega^\lambda}$, and

$$A_{\omega^\lambda} = (\omega - 1)\omega^{-3\lambda-1},$$

by (4.415), (4.4231), or (4.433). If $\lambda = a + 1$,

$$A_{\omega^\lambda} = -\omega^{-3\lambda-1} = -\omega^{-3a-4},$$

by (4.422), (4.411), and (4.432). And if $\lambda > a + 1$,

$$A_{\omega^\lambda} = 0,$$

by (4.421) and (4.431). Thus

$$\begin{aligned} \chi_\omega &= 1 + (\omega-1)\omega^{-4} + (\omega-1)\omega^{-7} + \cdots + (\omega-1)\omega^{-3a-1} - \omega^{-3a-4} \\ &= (1 - \omega^{-4})(1 + \omega^{-3} + \omega^{-6} + \cdots + \omega^{-3a}). \end{aligned} \tag{5.24}$$

From (5.23) and (5.24) it follows, as at the end of 5.1, that

$$\rho_8(n) = 16n^3\{1 + 2^{-3} + \cdots + 2^{-3(\alpha-1)} - 2^{-3\alpha}\} \prod_\omega (1 + \omega^{-3} + \cdots + \omega^{-3a}),$$

it being understood that the factor in curly brackets is to be replaced by unity when $\alpha = 0$; and it is easily verified that the formulas (5.14) and (5.15) are still correct.

6. Summation of the singular series when $s = 5$ and n has no squared factors

6.1. Suppose next that $s = 5$ and that n is not divisible by any square, and first that $p = 2$. Then $A_{16} = A_{32} = \cdots = 0$, by (4.321) and (4.331). And $A_8 = 0$, by (4.321), unless $n \equiv 1 \pmod 4$, in which case, by (4.3223),

$$A_8 = 2^{-3} e^{-\frac{1}{4}(n-5)\pi i}.$$

Thus $A_8 = -\frac{1}{8}$ if $n \equiv 1 \pmod 8$, $A_8 = \frac{1}{8}$ if $n \equiv 5 \pmod 8$, and otherwise $A_8 = 0$.

Next,

$$A_4 = 2^{-3/2} \cos(\tfrac{1}{2} n\pi - \tfrac{5}{4}\pi),$$

by (4.3322), so that $A_4 = -\frac{1}{4}$ if $n \equiv 1 \pmod 4$ and $A_4 = \frac{1}{4}$ otherwise. Finally $A_2 = 0$ in all cases, by 4.31.

Collecting these results we find that

(6.11) $\chi_2 = \frac{5}{8}(n \equiv 1), \qquad \chi_2 = \frac{5}{4}(n \equiv 2, 3, 6, 7), \qquad \chi_2 = \frac{7}{8}(n \equiv 5),$

the congruences being to modulus 8.

If p is odd and $p \nmid n$, we have

$$A_p = \left(\frac{n}{p}\right)\frac{1}{p^2}, \qquad A_{p^2} = A_{p^3} = \cdots = 0,$$

by (4.412), (4.421), and (4.431). If p is odd and $p|n$, we have

$$A_p = 0, \qquad A_{p^2} = -p^{-4}, \qquad A_{p^3} = A_{p^4} = \cdots = 0,$$

by (4.416), (4.432), (4.421), and (4.431). Thus

$$\chi_p = 1 + \left(\frac{n}{p}\right)\frac{1}{p^2} \quad (p \nmid n), \qquad \chi_p = 1 - \frac{1}{p^4} \quad (p|n). \tag{6.12}$$

If $n \not\equiv 1 \pmod 4$, we have

$$\rho_5(n) = \frac{\pi^{5/2}\, n^{3/2}}{\Gamma(5/2)}\frac{5}{4}\prod_{p \nmid n}\left\{1 + \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}\prod_{p|n}\left(1 - \frac{1}{p^4}\right)$$
$$= \frac{5}{3}\pi^2 n^{3/2}\prod\left(1 - \frac{1}{p^4}\right)\prod_{p \nmid n}\left\{1 - \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}^{-1}.$$

Also

$$\prod\left(1 - \frac{1}{p^4}\right) = \frac{16}{15\,\zeta(4)} = \frac{96}{\pi^4}, \qquad \prod_{p \nmid n}\left\{1 - \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}^{-1} = \sum\left(\frac{n}{m}\right)\frac{1}{m^2},$$

where m runs through all odd values prime to n. Hence finally

$$\rho_5(n) = \frac{160}{\pi^2}\, n^{3/2}\sum\left(\frac{n}{m}\right)\frac{1}{m^2}. \tag{6.13}$$

If $n \equiv 1 \pmod 8$ the value of χ_2 is $\frac{5}{8}$ instead of $\frac{5}{4}$; and if $n \equiv 5 \pmod 8$ it is $\frac{7}{8}$. In these cases the numerical factor 160 must be replaced by 80 and by 112 respectively.

These are the results of Eisenstein, subsequently proved by Smith and Minkowski by means of the arithmetical theory of quadratic forms. The series (6.13) is easily summed in a finite form, by methods due to Dirichlet and to Cauchy. I have nothing to add to this part of the discussion.

7. The general case when $s = 5$

7.1. So far it has not been necessary to distinguish between one type of representation and another. At this stage the distinction between "primitive" and "imprimitive" representations becomes of importance.

A representation

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$

is said to be *imprimitive* if x_1, x_2, x_3, x_4, x_5 possess a common factor, and *primitive* in the contrary case. It is plain that imprimitive representations can exist only when n is divisible by a square. When $s = 8$ (and the remark applies equally when s is $2, 4$, or 6) the distinction is, for our purposes, irrelevant, even when n is divisible by a square: the formulas (5.14) and (5.15) are valid in any case. But when $s = 5$ the distinction is important. It will be remembered in fact, by anyone familiar with the work of Minkowski and Smith, that the right-hand side of (6.13) represents, in general, not the total number of representations but the number of primitive representations. Our series (2.26), on the other hand, gives the total number of representations; and its relation to the Smith-Minkowski series must therefore generally be more intricate than in the simplest case treated in 6.1.

The theorem which I shall prove is as follows:

The sum of the series

$$\frac{C}{\pi^2} n^{3/2} \sum \left(\frac{n}{m}\right) \frac{1}{m^2}, \tag{7.11}$$

where m runs through all odd numbers prime to n, and

$$C = 80\,(n \equiv 0, 1, 4), \quad C = 160\,(n \equiv 2, 3, 6, 7), \quad C = 112\,(n \equiv 5),$$

the congruences being to modulus 8, is $r_5(n)$, the number of primitive representations of n.

We shall require the following

Lemma. *If*[17]

$$r(n) = \sum \phi\left(\frac{n}{q^2}\right), \tag{7.12}$$

where q^2 runs through all squared divisors of n, then

$$\phi(n) = \bar{r}(n). \tag{7.13}$$

To prove this, suppose first that n is divisible by p^2, but by no other square. Then

$$\phi(n) = r(n) - \phi\left(\frac{n}{p^2}\right) = r(n) - r\left(\frac{n}{p^2}\right) = \bar{r}(n).$$

Next, if n is divisible by p^2, p'^2, and $(pp')^2$, where $p' \neq p$, but by no other square, we have

$$\begin{aligned}\phi(n) &= r(n) - \phi\left(\frac{n}{p^2}\right) - \phi\left(\frac{n}{p'^2}\right) - \phi\left(\frac{n}{p^2 p'^2}\right)\\ &= r(n) - \bar{r}\left(\frac{n}{p^2}\right) - \bar{r}\left(\frac{n}{p'^2}\right) - r\left(\frac{n}{p^2 p'^2}\right) = \bar{r}(n).\end{aligned}$$

[17] In what follows I omit the suffix 5 in $r_5(n)$, etc.

A similar proof applies if $p = p'$; and it is plain that a repetition of the argument leads to a general proof of the lemma.

7.2. Suppose first that n is congruent to 2, 3, 6, or 7, so that n is not divisible by 4 and $\chi_2 = \frac{5}{4}$, by (6.11). If we write

$$n = 2^{\alpha} N = 2^{\alpha} \omega^{a} \omega'^{\,a'} \cdots \qquad (\alpha = 0, 1), \tag{7.21}$$

the value of χ_p requires reconsideration when p is an ω and $a > 1$. Using the formulas of 4.4, we obtain the following results.

If $a = 2b + 1$ then

$$A_{\omega} = 0, \quad A_{\omega^2} = (\omega - 1)\omega^{-4}, \quad A_{\omega^3} = 0, \quad A_{\omega^4} = (\omega - 1)\omega^{-7},$$
$$\cdots, \quad A_{\omega^{2b-1}} = 0, \quad A_{\omega^{2b}} = (\omega - 1)\omega^{-3b-1},$$
$$A_{\omega^{2b+1}} = 0, \quad A_{\omega^{2b+2}} = -\omega^{-3b-4}, \quad A_{\omega^{2b+3}} = A_{\omega^{2b+4}} = \cdots = 0.$$

If $a = 2b$, the values of the A's, up to $A_{\omega^{2b}}$, are as above, but

$$A_{\omega^{2b+1}} = \left(\frac{\nu}{\omega}\right)\omega^{-3b-2}, \quad A_{\omega^{2b+2}} = A_{\omega^{2b+3}} = \cdots = 0,$$

where $\nu = \omega^{-a} n$. We thus find that

$$\chi_{\omega} = \frac{(1 - \omega^{-4})(1 - \omega^{-3b-3})}{1 - \omega^{-3}} \tag{7.22}$$

if a is odd, and

$$\chi_{\omega} = \frac{(1 - \omega^{-4})(1 - \omega^{-3b})}{1 - \omega^{-3}} + \left\{1 + \left(\frac{\nu}{\omega}\right)\frac{1}{\omega^2}\right\}\omega^{-3b} \tag{7.23}$$

if a is even.

Suppose now that $n = 2^{\alpha} \omega^{a} \omega'^{\,a'} \cdots = \omega^2 d$, where d has no squared factor. Then the odd primes p fall into four classes characterized as follows.

(i) $p = \boldsymbol{p}, \boldsymbol{p} \nmid n$. In this case

$$\chi_{\boldsymbol{p}} = 1 + \left(\frac{n}{\boldsymbol{p}}\right)\boldsymbol{p}^{-2}. \tag{7.241}$$

(ii) $p = \omega_1, \omega_1 \nmid \omega, \omega_1 | d$. In this case

$$\chi_{\omega_1} = 1 - \omega_1^{-4}. \tag{7.242}$$

(iii) $p = \omega_2, \omega_2 | \omega, \omega_2 \nmid d$. In this case a is even, say equal to $2b$, and

$$\chi_{\omega_2} = (1 - \omega_2^{-4})\left\{1 + \omega_2^{-3} + \cdots + \omega_2^{-3b+3} + \frac{\omega_2^{-3b}}{1 - \left(\frac{\nu}{\omega_2}\right)\frac{1}{\omega_2^2}}\right\}, \tag{7.243}$$

by (7.23).

(iv) $p = \omega_3, \omega_3 | \omega, \omega_3 | d$. In this case a is odd, say equal to $2b + 1$, and

$$\chi_{\omega_3} = (1 - \omega_3^{-4})(1 + \omega_3^{-3} + \cdots + \omega_3^{-3b}), \tag{7.244}$$

by (7.22). And we have

(7.25) $$S = \tfrac{5}{4}\prod \chi_p = \tfrac{5}{4}\prod \chi_p \prod \chi_{\omega_1} \prod \chi_{\omega_2} \prod \chi_{\omega_3}.$$

7.3. We now multiply out the product (7.25), treating the second factors of χ_{ω_2} and χ_{ω_3} each as a sum of $b+1$ separate terms. We thus obtain

(7.31) $$S = \tfrac{5}{4}\prod_p (1 - p^{-4}) \prod_p \gamma_p(n) \sum_{\lambda,\mu} \Big(\prod_{\omega_2} \theta \omega_2^{-3\lambda} \prod_{\omega_3} \omega_3^{-3\mu}\Big),$$

where

(7.311) $$\gamma_p(n) = \left\{1 - \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}^{-1};$$

$\lambda \leqq b_2$, if $\omega_2^{a_2} = \omega_2^{2b_2}$ is the highest power of ω_2 which divides n; $\mu \leqq b_3$, if $\omega_3^{a_3} = \omega_3^{2b_3+1}$ is the highest power of ω_3 which divides n; and θ is an additional factor which is equal to 1 unless $\lambda = b_2$, and then to

$$\gamma_{\omega_2}(\nu) = \gamma_{\omega_2}(\omega_2^{-a_2} n).$$

If we denote the product which appears under the sign of summation in (7.31) by $\sigma_{\lambda,\mu}$, we have

(7.32) $$\rho(n) = \frac{4\pi^2}{3} n^{3/2} S = \frac{160}{\pi^2} n^{3/2} \prod_p \gamma_p(n) \sum \sigma_{\lambda,\mu} = \sum \rho_{\lambda,\mu},$$

say.

Suppose first that λ does not, for any ω_2, assume its maximum value b_2, so that all the θ's in $\sigma_{\lambda,\mu}$ are equal to unity; and write

(7.34) $$\psi(n) = \frac{160}{\pi^2} n^{3/2} \sum \left(\frac{n}{m}\right)\frac{1}{m^2},$$

so that

$$\psi(n) = \frac{160}{\pi^2} n^{3/2} \prod_p \gamma_p(n),$$

the product extending over all odd primes which do not divide n. Then

$$\rho_{\lambda,\mu} = \frac{160}{\pi^2} n^{3/2} \Big(\prod_{\omega_2} \omega_2^{-2\lambda} \prod_{\omega_3} \omega_3^{-2\mu}\Big)^{3/2} \prod_p \gamma_p(n) = \psi\left(\frac{n}{q^2}\right),$$

where

$$q^2 = \prod_{\omega_2} \omega_2^{-2\lambda} \prod_{\omega_3} \omega_3^{-2\mu}$$

is a typical square divisor of n, division by which does not eliminate completely any prime factor of n.

This transformation would not, as it stands, be valid if $\lambda = b_2$ for some ω_2, since there are then certain primes ω_2 which divide n and not n/q^2. But with each of these primes ω_2 there is associated an additional factor $\theta = \gamma_{\omega_2}(\nu)$ in $\sigma_{\lambda,\mu}$, and these factors provide exactly the corrective required. We have

therefore in any case $\rho_{\lambda,\mu} = \psi(n/q^2)$ and

$$r(n) = \rho(n) = \sum \psi\left(\frac{n}{q^2}\right),$$

the summation extending over all square divisors of n. And therefore, by (7.13), $\psi(n) = \bar{r}(n)$, the result required.

Our theorem is thus proved when n is congruent to $2, 3, 6$, or 7 to modulus 8. In order to prove it when n is congruent to 1 or 5, we have only to write $\frac{5}{8}$ or $\frac{7}{8}$ instead of $\frac{5}{4}$ throughout our argument. It is only when n is divisible by 4 that further discussion is required.

7. 4. We have now

$$n = 2^{\alpha} N = 2^{\alpha} \omega^{a} \omega'^{a'} \cdots \qquad (\alpha \geqq 2).$$

The value of χ_p, when p is odd, is the same as before. The value of χ_2 may be calculated by means of the results of 4.3; and we find that

$$\chi_2 = 1 - \frac{1}{4} - \frac{1}{4.8} - \frac{1}{4.8^2} - \cdots - \frac{1}{4.8^{\beta-1}} + \frac{1}{4.8^{\beta}}, \tag{7.41}$$

where α is odd and equal to $2\beta + 1$ or even and equal to 2β.

Let us denote by $r^*(n)$ the number of representations of n which are *primitive so far as* 2 *is concerned*, that is to say in which x_1, x_2, x_3, x_4, and x_5 are not all even. It is plain that

$$r(n) = r^*(n) + r^*\left(\frac{n}{4}\right) + \cdots + r^*\left(\frac{n}{4^{\beta}}\right), \tag{7.42}$$

and that

$$r^*(n) = \sum \bar{r}\left(\frac{n}{q^2}\right), \tag{7.43}$$

where now the summation applies to all *odd* square divisors of n. Further, as in 7.1, we can show that *if*

$$r^*(n) = \sum \phi\left(\frac{n}{q^2}\right), \tag{7.44}$$

where the summation applies to all odd square divisors of n, *then*

$$\phi(n) = \bar{r}(n). \tag{7.45}$$

Bearing these remarks in mind, we can complete the proof of the theorem as follows. Since $\rho(n)$, $\rho(\frac{1}{4}n)$, $\cdots$ differ only in the factor χ_2 and the outside power of n, $\frac{1}{4}n$, $\cdots$, we have, by (7.41),

$$\begin{aligned}\rho(n) - \rho\left(\frac{n}{4}\right) = \tfrac{4}{5}\rho\left(\frac{n}{4^{\beta}}\right)\Bigg\{ & 8^{\beta}\left(1 - \frac{1}{4} - \cdots - \frac{1}{4.8^{\beta-1}} + \frac{1}{4.8^{\beta}}\right) \\ & - 8^{\beta-1}\left(1 - \frac{1}{4} - \cdots - \frac{1}{4.8^{\beta-2}} + \frac{1}{4.8^{\beta-1}}\right)\Bigg\} \\ = {} & 4.8^{\beta-1}\rho\left(\frac{n}{4^{\beta}}\right).\end{aligned}$$

But, by (7.42), we have

$$r^*(n) = r(n) - r\left(\frac{n}{4}\right) = \rho(n) - \rho\left(\frac{n}{4}\right) = 4.8^{\beta-1}\rho\left(\frac{n}{4^\beta}\right) = 4.8^{\beta-1}r\left(\frac{n}{4^\beta}\right);$$

and therefore, by our previous results,

$$r^*(n) = 4.8^{\beta-1}\sum \bar{r}\left(\frac{n}{4^\beta q^2}\right) = 4.8^{\beta-1}\sum \psi\left(\frac{n}{4^\beta q^2}\right) = \tfrac{1}{2}\sum \psi\left(\frac{n}{q^2}\right),$$

the summation applying to all square divisors of $n/4^\beta$, or, what is the same thing, to all odd square divisors of n. Hence, by (7.45),

$$\psi(n) = 2\bar{r}(n).$$

This completes the proof of the theorem. It is easily verified that the results are in complete agreement with those of Smith.[18]

8. Concluding remarks

8.1. I have assumed, throughout this paper, that $s \leqq 8$; and it is well known that the analogous results when $s > 8$ are false.

The analysis of the paper breaks down, when $s > 8$, in one section only, namely Section 3. We can still form the singular series, and sum it by methods differing only in detail from those of Sections 4–7. We obtain a simple function of the divisors of n when s is even, a series of the Smith-Minkowski type when s is odd; and this series can still be summed in terms of the quadratic residues and non-residues of n. We can still prove, moreover, that the sum of the singular series behaves, in respect to the fundamental transformations of the modular subgroup Γ_3, exactly like the appropriate power of the theta-function ϑ, and that the function corresponding to $\eta(\tau)$ is an invariant of the group. What we cannot prove is that $\eta(\tau)$ is bounded; and the conclusion which would follow from this, namely that $\eta(\tau)$ is constant, is in fact false.

We have still, however, all the materials for a complete solution of the problem. But it is necessary to replace the analysis of Section 3 by a more complex discussion in which we deal not with a single invariant but with a linear combination of invariants, among which that represented by the sum of the singular series is the first and most important. And our conclusion will be that the number of representations of n is the sum of a function of the types considered in this paper and of a number of other arithmetical functions defined in a more recondite manner. Some of these functions have already appeared in the work of Liouville, Glaisher, and Mordell. If I do not pursue

[18] See in particular pp. 673 et seq. of the second volume of his *Collected Papers*.

this subject further, it is because such developments seem to be a part of Mr. Mordell's researches rather than of mine.

There is another question which arises more naturally out of my own researches. The singular series or principal invariant yields in any case an *asymptotic* formula for $r_s(n)$, valid without restriction on s. But, with the entry of asymptotic formulas, the peculiar interest of squares as such departs, and the problem becomes merely a somewhat trivial case of the much larger problem usually described as Waring's problem, and so of the investigations which Mr. Littlewood and I are publishing elsewhere.

(*b*) Waring's Problem

INTRODUCTION TO PAPERS ON WARING'S PROBLEM

Hardy's papers on Waring's problem were all written in collaboration with J. E. Littlewood. In a sense, they followed naturally from his earlier papers on the representation of a number as a sum of squares, but the new difficulties which had to be overcome were severe.

The starting-point of the method is the same. For each integer $k \geqslant 3$ the generating function is defined as

$$f(x) = 1+2\sum_{n=1}^{\infty} x^{n^k},$$

so that

$$f(x)^s = \sum_{n=0}^{\infty} r_{k,s}(n)x^n,$$

where $r_{k,s}(n)$ is the number of representations of n as the sum of s absolute values of integral kth powers. For even k this is, of course, the same as the number of representations by kth powers. Then by Cauchy's Theorem the number $r(n)$ is given by

$$r(n) = r_{k,s}(n) = \frac{1}{2\pi i}\int (f(x))^s x^{-n-1}\,dx,$$

where the integral is taken along the circle $|x| = e^{-1/n}$. This circle, $x = e^{-n^{-1}+2\pi i\theta}$, is divided into 'Farey arcs' by first marking all rational points $\theta = p/q$ with

$$1 \leqslant q \leqslant n^{1-k^{-1}}, \qquad (p,q) = 1, \qquad 0 \leqslant p < q.$$

Two neighbouring rationals $\theta = p/q$, $\theta = p'/q'$ are separated by their mediant $\theta = (p+p')/(q+q')$, so that to each rational p/q there belongs a zone of influence, bounded by its two neighbouring mediants. These zones are the Farey arcs $M_{p,q}$. In the treatment of Waring's Problem they fall into two classes, *major arcs* if $q \leqslant n^{1/k}$, and *minor arcs* if $n^{1/k} < q \leqslant n^{1-1/k}$.

On each major arc Hardy and Littlewood introduced an approximating function

$$F_{p,q}(x) = C\Gamma(s/k)(q^{-1}S_{p,q})^s\{\log(e^{2\pi ip/q}x^{-1})\}^{-s/k},$$

where

$$C = \{2\Gamma(1+1/k)\}^s\{\Gamma(s/k)\}^{-1},$$

$$S_{p,q} = \sum_{h=0}^{q-1} e^{2\pi ih^kp/q}.$$

They were then able to show that on each major arc $M_{p,q}$ the function $f^s(x)$ can be approximated by $F_{p,q}(x)$ with a sufficiently good error term not to upset the final result. The difficulties they had to overcome were formidable and their first paper on the subject (1920, 2) makes impressive reading even today. In one respect they were perhaps fortunate. H. Weyl's celebrated paper (*Math. Annalen*, 77 (1916), 313–52) on exponential sums had just appeared and the methods developed in it made the work more manageable.

The second difficulty arose on the minor arcs, which do not occur at all in the case of squares. Whereas Weyl's inequality was a convenience on the major arcs, it was and to a large extent still is a necessity on the minor arcs, because the error in the approximation to $f^s(x)$ by $F_{k,s}(x)$ becomes too large; indeed it is larger than the principal term.

The third difficulty was to deal with the *singular series*. Having replaced $f^s(x)$ by $F_{p,q}(x)$ on all major arcs, Hardy and Littlewood obtained for $s > k2^{k-1}$ the asymptotic formula

$$r(n) \sim Cn^{-1+s/k}S,$$

where

$$S = \sum_{q=1}^{\infty} q^{-s} \sum_{\substack{p=1 \\ (p,q)=1}}^{q} (S_{p,q})^s e^{-2\pi i np/q}$$

is the singular series.

In their first paper (1920, 2) they did no more than prove that the series is absolutely convergent and greater than a positive constant for $k = 4$, $s = 33$ and all n, though they suggested that there would be no serious difficulties in extending this result to general $k \geqslant 3$ if $s > k2^{k-1}$. This programme was carried through in (1920, 5),* the first of their famous papers on 'partitio numerorum'.

In P.N. II (1921, 1), the asymptotic formula is proved for $k = 4$, $s = 21$ and it is shown that the singular series is uniformly positive. Apart from a detailed study of the singular series the paper introduces an improvement in minor arc technique, namely the use of the inequality ($\mathfrak{m}$ denoting the union of all the minor arcs)

$$\int\limits_{\mathfrak{m}} |f(re^{2\pi i\theta})|^{2s}\, d\theta \leqslant \max_{\theta \text{ in } \mathfrak{m}} |f(re^{2\pi i\theta})|^{2(s-2)} \int\limits_0^1 |f(re^{2\pi i\theta})|^4\, d\theta.$$

For the first factor on the right Weyl's estimate is available, and the second factor can easily be estimated by elementary means.

In P.N. IV (1922, 4) the authors prove their asymptotic formula for $r(n)$ for all $s > (k-2)2^{k-1}+5$. But the real interest of the paper lies in the study of the singular series. It is now commonplace to realize that the inequality $S > 0$ implies, subject to absolute convergence, that the congruence

$$x_1^k + \ldots + x_s^k \equiv n \pmod m$$

is soluble for every modulus m; conversely if the congruence is soluble for all m, then $S > 0$. In fact we can be more precise. If in the summation formula for S we restrict q to the powers of a fixed prime ϖ, the series turns out to be finite for $n \neq 0$ and represents $\varpi^{-t(s-1)}$ times the number of solutions of the above congruence mod ϖ^t, for sufficiently large t. Moreover, S equals the product of the restricted sums extended over all primes ϖ.

This idea, which is fundamental not only for Waring's Problem, but for all applications of the Hardy–Littlewood method, appears first in P.N. IV. It is historically interesting that the important relation between the singular series and the congruence was formulated by the authors as a lemma only (Lemma 2).

* There was a preliminary announcement in *Proc. London Math. Soc.* (2), 18 (1920), vii–viii.

In P.N. VI (1925, 1)† the authors break new ground. They prove first that almost all positive integers are the sum of fifteen fourth powers, and more generally the sum of

$$(\tfrac{1}{2}k-1)2^{k-1}+3$$

non-negative kth powers for $k = 3$ and $k \geqslant 5$. Secondly they prove that all large integers are the sum of

$$(\tfrac{1}{2}k-1)2^{k-1}+k+5+\left[\frac{(k-2)\log 2-\log k+\log(k-2)}{\log k-\log(k-1)}\right]$$

non-negative kth powers. The idea of the proof is to introduce a sequence of integers which have a relatively large density and are representable as the sum of few kth powers. It was the first step in the combination of analytical with elementary methods, serving as a signpost to the future, and in particular to the work of Vinogradov.

Finally they introduced in P.N. VI the Hypothesis K, which asserts that the number of solutions of

$$n = x_1^k+\ldots+x_k^k$$

is $O(n^\epsilon)$ for each $\epsilon > 0$. Under the assumption of Hypothesis K they proved that their asymptotic formula for $r(n)$ holds for $s \geqslant 2k+1$, in particular that each large integer is a sum of $2k+1$ non-negative kth powers, if k is not a power of 2. Unfortunately, K. Mahler proved (*Journal London Math. Soc.* 11 (1936), 136–8) that Hypothesis K does not hold for $k = 3$ by means of the simple identity

$$(9x^4)^3+(3xy^3-9x^4)^3+(y^4-9x^3y)^3 = y^{12}.$$

Whether the hypothesis is true for $k \geqslant 4$ is still an unanswered question. For the application to Waring's Problem it would suffice if the hypothesis held in a weaker 'mean square' form.

P.N. VIII (1928, 4) differs from the preceding papers in posing a purely arithmetical problem. Let $\Gamma(k)$ be the smallest integer s such that for all n and all primes p, and all $m > 0$, the congruence

$$x_1^k+\ldots+x_s^k \equiv n \pmod{p^m}$$

has a solution in which not all the x_σ are $\equiv 0 \pmod{p}$.

The authors determine five distinct classes of k for which $\Gamma(k) > k$ and calculate $\Gamma(k)$ explicitly. They also prove that $\Gamma(k) = k$ if $2k+1$ is a prime. For all other cases they prove the inequality $\Gamma(k) \leqslant k$.

As a conjecture they mention the innocent-looking relation

$$\varliminf_{k\to\infty} \Gamma(k) \geqslant 4,$$

which has remained unproved. It would be sufficient to find for all large k a prime π such that

$$\pi \equiv 1 \pmod{k}, \qquad \pi < k^{4/3}.$$

Finally, a footnote should be quoted:

'We may add that "P.N. 7", which is still unpublished, contains an application of our methods to the problem of the order of magnitude of the difference between

† An abstract appeared in *Proc. London Math. Soc.* (2), 23 (1925), xx–xxi.

consecutive primes. We prove (subject to our generalized form of the Riemann hypothesis) that

$$\varliminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log n} \leqslant \tfrac{2}{3}.\text{'}$$

The paper was never published, but R. A. Rankin (*Proc. Camb. Phil. Soc.* 36 (1940), 255–66) proved that the lower limit in question does not exceed 3/5 under the generalized Riemann Hypothesis, whereas P. Erdős (*Duke Math. J.* 6 (1940), 438–41) proved without any assumptions that it is less than 1. Later R. A. Rankin (*Journal London Math. Soc.* 22 (1947), 226–30) proved that the lower limit is less than 57/59.

The subsequent major improvements upon the Hardy–Littlewood results in Waring's Problem are due to I. M. Vinogradov. An account of his results is given in his book, *The Method of Trigonometrical Sums in the Theory of Numbers*, English translation by K. F. Roth and A. Davenport, 1954.

He proved first (*Annals of Math.* 36 (1935), 395–405) that

$$G(k) \leqslant 6k\log k+(4+\log 216)k,$$

where $G(k)$ is defined as the smallest s such that $\varliminf_{n\to\infty} r_{k,s}(n) > 0$. A simplified version was given by H. Heilbronn (*Acta Arithmetica*, 1 (1935), 212–21). Subsequently I. M. Vinogradov improved this result progressively in various papers, and in his book he proves

$$G(k) < k(3\log k+11).$$

He also in a series of papers considered the validity of the asymptotic formula for $r_{k,s}(n)$. The best result given in his book is that the formula holds for

$$s > 10k^2\log k-1.$$

L. K. Hua had previously (*Quarterly Journal*, 9 (1938), 199–202) proved the asymptotic formula for $s \geqslant 2^k+1$.

The inequality $G(3) \leqslant 7$ was first proved by U. V. Linnik (*Recueil Math.* 12 (1943), 218–24) and subsequently a simpler proof was given by G. L. Watson (*Journal London Math. Soc.* 26 (1951), 153–6).

H. Davenport proved that almost all numbers are sums of four positive cubes (*Acta Math.* 71 (1939), 123–43); that all large integers not $\equiv 0$ or $15 \pmod{16}$ are sums of fourteen fourth powers (*Annals of Math.* 40 (1939), 731–47); and that $G(5) \leqslant 23$ and $G(6) \leqslant 36$ (*American J. of Math.* 64 (1942), 199–207).

Finally, the problem raised in P.N. VIII, that of improving the upper bound for $\Gamma(k)$, has not received much attention. It was, however, proved by I. Chowla (*Proc. Nat. Acad. Sci. India*, sect. A 13 (1943), 195–220) that $\Gamma(k) < k^c$, for some c with $0 < c < 1$, in all cases in which Hardy and Littlewood had not shown that $\Gamma(k) \geqslant k$.

In recent years there has been renewed interest in the Hardy–Littlewood method, which has been successfully applied to prove the existence of integral solutions of homogeneous algebraic equations with integral coefficients in a sufficiently large number of variables (subject to certain conditions if the degree is even). The proofs are very sophisticated, requiring a good deal of 'old-fashioned' algebraic geometry.

For some of these results, and for references, the reader may consult H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Ann Arbor, Michigan, 1963.

H. H.

A NEW SOLUTION OF WARING'S PROBLEM.

By G. H. HARDY and J. E. LITTLEWOOD.

Introduction.

1. IT was asserted by Waring* in 1782 that every number is the sum of at most four squares, nine positive cubes, nineteen fourth powers, and, in general, $g(k)$ positive k-th powers, where $g(k)$ is a number depending upon k alone. Waring advanced no argument of any kind in support of his assertion; and there is no reason to suppose that it rested on any basis more substantial than the examination of a number of particular cases.

If the proposition 'every number is the sum of at most m positive k-th powers' is true for any particular value of m, it is true *a fortiori* for any larger value. The number $g(k)$ is, by definition, the smallest value of m for which the proposition is true. The problem suggested by Waring then falls naturally into two parts. The first is the proof of the existence of $g(k)$, the second the determination of its actual value as a function of k. It is the first of these two problems that has generally been described as *Waring's Problem*. This problem was solved by Hilbert† in 1909, the the existence of $g(k)$ having been proved before only when k is 2, 3, 4, 5, 6, 7, 8, or 10. The second problem is still unsolved, except when k is 2 or 3.

Hilbert based his proof, in the first instance, on considerations drawn from the integral calculus. The proof falls into two parts. In the first he considers the properties of a volume integral in space of five dimensions‡, from which he deduces the existence of certain algebraical identities leading to an induction from k to $2k$. As the theorem is known to be true when $k=2$, it follows that it is true when k is any power of 2. This completes the first part of the proof: the second, in which the conclusion is extended to an arbitrary k, is purely algebraical.

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* E. Waring, *Meditationes Algebraicae*, ed. 3, 1782, pp. 349–350.

† D. Hilbert, 'Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl $n$ter Potenzen (Waringsche Problem)', *Göttinger Nachrichten*, 1909, pp. 17–36, and *Math. Annalen*, vol. lxvii., 1909, pp. 281–300.

‡ Twenty-five in the first version of his memoir.

1920, 2 (with J. E. Littlewood) *Quarterly Journal of Mathematics*, 48, 272–93.

Hilbert's proof was reconsidered and simplified by Hausdorff*, Stridsberg†, and Remak‡, who succeeded ultimately in eliminating all reference to the integral calculus, and indeed all reasoning of a transcendental character. The proof, as left by Remak, is by no means easy; but it is purely algebraical and, in the technical sense, entirely elementary§.

2. In this note we propose to give a short account of another solution of Waring's Problem which we have discovered recently and which proceeds on entirely different lines. This solution is not, in any sense of the word, elementary. It is based throughout on Cauchy's Theorem and the ordinary machinery of the theory of analytic functions, and has, from beginning to end, no point of contact with Hilbert's solution. It might seem that a highly transcendental proof of a theorem which has already been proved, and that in an entirely elementary manner, is unnecessary. This view, we think, would rest upon a misapprehension. It seems to us most desirable and important that Waring's Problem, and all similar problems of Combinatory Analysis, should be brought into relation with the transcendental side of the Analytic Theory of Numbers. Further, the method which we follow, and which we describe shortly as *the method of Farey dissection*, is in many ways the most natural and, in spite of its considerable technical difficulties, the most straightforward method for the discussion of any problem of this character; and it is a method of great power and wide scope, applicable to almost any problem concerning the decomposition of integers into parts of a particular kind‖, and to many against which it is difficult to suggest any other obvious method of attack.

* F. Hausdorff, 'Zur Hilbertschen Lösung des Waringschen Problems', *Math. Annalen*, vol, lxvii., 1909, pp. 301–305.

† E. Stridsberg, 'Sur la démonstration de M. Hilbert du théorème de Waring', *Math. Annalen*, vol. lxxii., 1912, pp. 145–152. This paper gives references to two earlier notes by the author in the *Arkiv för Matematik*, vol. vi., 1910 and 1911.

‡ R. Remak, 'Bemerkung zu Herrn Stridsbergs Beweis des Waringschen Theorems', *Math. Annalen*, vol. lxxii., 1912, pp. 153–156.

§ That is to say, it does not depend upon arguments which involve limiting processes.

‖ See G. H. Hardy and S. Ramanujan, 'Une formule asymptotique pour le nombre des partitions de *n*', *Comptes Rendus*, 2 Jan. 1917; 'Asymptotic formulæ in Combinatory Analysis', *Proc. London Math. Soc.*, ser. 2, vol. xvii, 1918, pp. 75–115; 'On the coefficients in the expansions of certain modular functions', *Proc. Royal Soc.* (A), vol. xcv., 1918, pp. 144–155: S. Ramanujan, 'On certain trigonometrical sums and their applications in the analytic theory of numbers', *Trans. Camb. Phil. Soc.*, vol. xxii., 1918, pp. 259–276: G. H. Hardy, 'On the expression of a number as the sum of any number of squares, and in particular of five or seven', *Proc. National Acad. of Sciences* (Washington), vol. iv., 1918, pp. 189–193: G. H. Hardy and J. E. Littlewood, 'Note on Messrs. Shah and Wilson's paper entitled *An empirical formula connected with Goldbach's Theorem*', *Proc. Camb. Phil. Soc.*, vol. xix. (1919), pp. 245–254.

Moreover, when our method is applied to this particular problem, it yields a great deal more than can be obtained by more elementary methods. In particular it enables us to assign a definite upper bound, in the form of a function of k, not indeed for $g(k)$ but for another number $G(k)$ which seems in some ways more fundamental. This number $G(k)$ is the least number m of which it is true that every number *from a certain point onwards* is the sum of at most m positive k-th powers. It is obvious that $G(k) \leq g(k)$, and it would seem that in general $G(k) < g(k)$; and that $G(k)$ is more fundamental than $g(k)$ because its value is less likely to be determined, in any particular case, by a number of arithmetical coincidences. The value of $G(k)$ is not known for any value of k save 2; nor has any general upper bound for $G(k)$ yet been determined. Our method enables us to prove that

$$(2.1) \qquad G(k) \leq 2^{k-1}k + 1.$$

Finally our method yields not only a proof that representations of n in the form required exist, but also asymptotic formulæ for the number of representations.

We propose to give here a short account of our analysis, not complete in detail, but full enough to shew clearly the main ideas on which it is founded. The complete investigation will be published later.

The function $f_{k,s}(x)$.

3. Suppose first that k is even, and let us denote by

$$r_{k,s}(n)$$

the number of representations of n in the form

$$n = a_1^{\,k} + a_2^{\,k} + \ldots + a_s^{\,k},$$

where the a's are integers, positive, negative, or zero, and representations which differ only in the order of the a's are reckoned as distinct. Then $r_{k,s}(n)$ is the coefficient of x^n in the expansion of the function

$$(3.1) \quad f_{k,s}(x) = \{f_k(x)\}^s = (1 + 2\sum_1^\infty x^{n^k})^s = \sum_0^\infty r_{k,s}(n)\, x^n.$$

We shall sometimes omit the suffix k to f or r, when no ambiguity results.

It is convenient to conserve the same notation when k is odd. The arithmetical interpretation of $r_{k,s}(n)$ is then not

quite so simple, but the function is equally well adapted for our purpose. In particular, whether k is odd or even, Hilbert's Theorem may be stated as follows:

HILBERT'S THEOREM. *To every k corresponds a $g(k)$ such that*

$$r_{k,s}(n) > 0$$

for $s \geq g(k)$ and every value of n.

It should be observed that, as zero values of the a's are admitted,

$$r_{k,s+1}(n) > r_{k,s}(n)$$

for all values of the variables.

The singular series.

4. The unit circle is (apart from the trivial case in which $k=1$) a line of essential singularities of the function $f_s(x)$. There are however certain points on the circle which stand out as being, in a sense, both the *simplest* and the *heaviest* singularities. These are the points

$$x = e^{2p\pi i/q},$$

where p and q are positive* coprime integers and $p<q$. We shall call these points *the rational points* p/q or $x_{p,q}$ of the circle.

We write $$x = X/x_{p,q} = Xe^{-2p\pi i/q},$$

so that X is positive when x lies on a radius vector to the point p/q. It is easily shewn that, when $X \to 1$ through positive values†

$$(4.1)\quad f_k(x) = 1 + 2\sum^{\infty} x^{n^k} \sim 2\Gamma\left(1+\frac{1}{k}\right)\frac{S_{p,q}}{q}\left(\log\frac{1}{X}\right)^{-1/k},$$

where

$$(4.2)\qquad S_{p,q} = \sum_{h=0}^{q-1} e^{2h^k p\pi i/q}$$

is one of the Gaussian sums associated with the theory of the

* We admit the value 0 for p when $q=1$ only.

† These asymptotic formulæ are valid when $x \to x_{p,q}$ along any regular path which does not touch the unit circle. The behaviour of $f_k(x)$ when x tends to an 'irrational' point of the circle is far more complex: for a detailed study of the case $k=2$, in which $f_k(x)$ is a Theta-function, see G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine Approximation', *Acta Mathematica*, vol. xxxvii., 1914, pp. 193–238.

division of the circle. It may happen that $S_{p,q}=0$: in this case the equation (4.1) is to be understood as meaning

$$f_k(x)=o\left(\log\frac{1}{X}\right)^{-1/k}.$$

It follows that

$$(4.3)\quad f_{k,s}(x)\sim\left\{2\Gamma\left(1+\frac{1}{k}\right)\right\}^s\left(\frac{S_{p,q}}{q}\right)^s\left(\log\frac{1}{X}\right)^{-s/k}.$$

5. Our fundamental idea is that of approximating to $f_{k,s}(x)$, and so to its coefficients, by means of a sum of 'simple elements' each associated with a particular point p/q. The functions indicated, as appropriate to this purpose, by the formula (4.3), are of the type

$$A\left(\frac{S_{p,q}}{q}\right)^s\left(\log\frac{1}{X}\right)^{-s/k},$$

where A is a constant. These functions are not power-series in x. It is however well known that

$$F_a(x)=\sum_1^\infty \nu^{a-1}x^\nu=\Gamma(a)\left(\log\frac{1}{x}\right)^{-a}+G(x),$$

where $G(x)$ is regular at $x=1$.* This suggests that we should adopt, as the formal basis of our analysis, the formula

$$f_{k,s}(x)\sim C\left(\frac{S_{p,q}}{q}\right)^s F_{s/k}(X),$$

where
$$C=\left\{2\Gamma\left(1+\frac{1}{k}\right)\right\}^s\Big/\Gamma\left(1+\frac{s}{k}\right),$$

instead of (4.3). We are thus led to replace $f_{k,s}(x)$ by

$$(5.1)\qquad \phi_{k,s}(x)=1+C\Sigma\left(\frac{S_{p,q}}{q}\right)^s F_{s/k}(xe^{-2p\pi i/q}),$$

and $r_{k,s}(n)$ by the coefficient of x^n in $\phi_{k,s}(x)$, viz. by

$$(5.2)\qquad \rho_{k,s}(n)=Cn^{(s/k)-1}\Sigma\left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q}.$$

The summations apply to all positive values of q, and all positive values of p less than and prime to q, $p=0$ being

* See, for example, Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions*, 1905, pp. 138–141.

associated with $q=1$ alone; and the constant term in (5.1) is added merely for the sake of formal correspondence. And it is of course to be understood that our whole argument, up to this point, is of a purely formal character. It proves nothing; it merely suggests that $\rho_{k,s}(n)$ may not unreasonably be expected to furnish some sort of approximation for $r_{k,s}(n)$, a hope which more rigorous analysis proves to be well-founded.

We shall call the series

$$(5.3) \qquad \Sigma \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q}$$

the *singular series.*

The order of magnitude of a Gaussian sum.

6. It will evidently be essential to prove that the singular series (5.3) is convergent for a given k and all sufficiently large values of s. The most obvious way of effecting this is to shew that

$$S_{p,q}=O(q^{\alpha}),$$

uniformly in p, α being a function of k only and $0<\alpha<1$. It is well known that $S_{p,q}=O(\sqrt{q})$ when $k=2$; it is not difficult to prove, by the use of known results in the theory of the division of the circle, that

$$S_{p,q}=O(q^{\frac{2}{3}+\epsilon})$$

when $k=3$ and $\qquad S_{p,q}=O(q^{\frac{3}{4}+\epsilon})$

when $k=4$, ϵ being any positive number; and it appears that we may suppose, in general, that α is any number greater than $(k-1)/k$.

We shall however find it necessary to consider more general sums than $S_{p,q}$, viz.

$$(6.1) \qquad S_{p,q,m}=\sum_{h=0}^{q-1} e^{2h^k p\pi i/q}\cos\frac{2mh\pi}{q},$$

where m is an arbitrary integer; and we are unable to prove, for these more general sums, results as accurate as those stated above for the special sums $S_{p,q}$, which seem to be substantially the best possible of their kind. We have not at present proved any result, concerning the order of $S_{p,q,m}$, which seems likely to embody the best that can be stated truly.* For some

* We have proved that, if q is prime,

$$S_{p,q,m}=O(q^{\frac{3}{4}})$$

uniformly in p and m.

purposes, in particular that of solving the second problem of § 1, it is of the utmost importance that the best possible result of this kind should be discovered. But for our first purpose, that of proving Hilbert's Theorem, not so much is necessary. All that is required is *some* result of the form

$$S_{p,q,m} = O(q^{\alpha}) \quad (0 < \alpha < 1),$$

and such a result is fortunately not difficult to find. We can in fact prove, by the use of certain very elegant transformations due to Weyl,* that

$$S_{p,q,m} = O(q^{\kappa+\epsilon}) \tag{6.2}$$

uniformly in p and m, ϵ being any positive number and

$$\kappa = (K-1)/K, \quad K = 2^{k-1}. \tag{6.3}$$

Further discussion of the singular series.

7. It follows at once that the singular series is convergent for sufficiently large values of s. Further, its absolute value may be made, by choice of a sufficiently large value of s, as near to unity as we please. The first term is unity: let us write the series, then, in the form

$$S = 1 + S'.$$

We have
$$|S_{p,q}/q| < 1$$
for $q \geq 2$; and
$$|S_{p,q}/q| < Hq^{-2\mu},$$
where $\mu = 1/4K$ and H depends on k only, for all values of q. Choose a value of ν such that

$$\nu > H^{4K},$$

and let
$$\delta = \operatorname*{Max}_{2 \leq q \leq \nu} \left| \frac{S_{p,q}}{q} \right|.$$

Then†
$$|S'| < \delta^s \sum_{2}^{\nu} q + \sum_{\nu+1}^{\infty} q^{1-s\mu}$$
$$< \nu^2 \delta^s + \frac{\nu^{2-s\mu}}{s\mu - 2},$$

and can therefore be made as small as we please by choice of s.

* H. Weyl, 'Über die Gleichverteilung von Zahlen mod. Eins', *Math. Annalen*, vol. lxxvii., 1916, pp. 313–352. See in particular p. 330, equation (11).

† The number of values of p associated with each value of q is $\phi(q)$, the number of numbers less than and prime to q.

It follows from (5.2) that

(7.1) $$|\rho_{k,s}(n)| > \tfrac{1}{2} C n^{(s/k)-1}$$

for all sufficiently large values of s and all values of n.

The Farey dissection.

8. We have now to consider whether $\rho_{k,s}(n)$ is a genuine approximation to $r_{k,s}(n)$. We have

(8.1) $$r_{k,s}(n) = \frac{1}{2\pi i}\int \frac{f_{k,s}(x)}{x^{n+1}}\,dx,$$

the contour of integration being a circle Γ whose centre is the origin and whose radius R is less than 1. We take

(8.2) $$R = 1 - \frac{1}{n}.$$

In order to study the integral (8.1), we divide the circle Γ into a large number of small pieces by what we call a *Farey dissection.* It is the use of this method of dissection which is the most characteristic feature of our analysis.

The *Farey's series of order N* is the aggregate of irreducible rational fractions p/q, where $0 \le p \le q \le N$, arranged in ascending order of magnitude; it is plain that $p > 0$ except when $p = 0$, $q = 1$, and $p < q$ except when $p = 1$, $q = 1$. Thus

$$\tfrac{0}{1},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{1}{1}$$

is the Farey's series of order 5. It is known that, if p/q and p'/q' are two successive terms in a Farey's series, then

$$p'q - pq' = 1.$$

Further, if p/q is a term in the Farey's series of order N, and p''/q'' and p'/q' are the adjacent terms on the left and the right, and $j_{p,q}$ denotes the interval

$$\frac{p}{q} - \frac{1}{q(q+q'')},\quad \frac{p}{q} + \frac{1}{q(q+q')},^{*}$$

then the intervals $j_{p,k}$ exactly fill up the continuum (0, 1), and the length of the two parts into which $j_{p,q}$ is divided by p/q is greater than $1/2qN$ and less than $1/qN$.* The essential point is that each of the parts of $j_{p,q}$ is of order $1/qN$.†

* When p/q is 0/1 or 1/1, $j_{p,q}$ consists of a single part only, one or other of p''/q'' and p'/q' being non-existent.

† For proofs of all these assertions see the second paper of Hardy and Ramanujan quoted on p. 273, §§ 4.21, 4.22.

If now we suppose that Γ is defined by

$$x = Re^{2\pi i\theta} \quad (0 \leq \theta \leq 1),$$

that the range of variation of θ is divided up by taking as points of division all points belonging to the Farey's series of order N, and that the two extreme intervals, ending at $\theta = 0$ and $\theta = 1$ respectively, are regarded as defining a single arc of the circle, we obtain the dissection of the circle which we describe as the Farey dissection of order N. The arc associated with p/q we call the *Farey arc* $\xi_{p,q}$.

We shall find it convenient to write

$$(8.3) \qquad \frac{1}{k} = a, \quad \frac{k-1}{k} = 1 - a, \quad \frac{1}{k-1} = b, \quad \frac{k-2}{k-1} = 1 - b,$$

so that $a = b/(1+b)$ and $b = a/(1-a)$. We take $N = [n^{1-a}]$; and we shall find that, for the purposes of this problem, the Farey arcs fall into two classes, arcs for which $q \leq n^a$, which we call *major* arcs, and arcs for which $n^a < q \leq n^{1-a}$, which we call *minor* arcs. The behaviour of $f_s(x)$ has to be studied by quite different methods, according as x lies on an arc of one or the other of these two categories. When $k = 2$, $a = 1 - a$, and the minor arcs disappear: thus one of the characteristic difficulties of the problem is absent in this particular case.

The behaviour of $f_k(x)$ on a major arc.

9. We begin by considering the major arcs. The characteristic of a major arc is that $f_s(x)$ is to a certain extent dominated by the approximative function (4.3), viz.

$$A \left(\frac{S_{p,q}}{q}\right)^s \left(\log \frac{1}{X}\right)^{-sa}.$$

We can prove in fact that, on a major arc

$$(9.1) \qquad f(x) = 1 + 2\Sigma x^{n^k} = \phi_{p,q} + \Phi_{p,q},$$

where

$$(9.2) \qquad \phi_{p,q} = 2\Gamma(1+a) \frac{S_{p,q}}{q} \left(\log \frac{1}{X}\right)^{-a}$$

and

$$(9.3) \qquad \Phi_{p,q} = O(n^{a\kappa+\epsilon}),$$

for every positive value of ϵ. Let us suppose, to fix our ideas, that q is not large and that X is real. Then $\phi_{p,q}$ is, in general, of order n^a, and therefore, since $\kappa < 1$, of higher order than

$\Phi_{p,q}$. It is consequently a genuine approximation to $f(x)$. We shall find that, on a minor arc, this is in general untrue.

The fundamental transformation.

10. In order to study the behaviour of $f(x)$ on a major arc, we employ a transformation which is in essence a generalisation of one of the fundamental formulæ of the theory of the linear transformation of the Theta-functions. We write

(10.1) $$x = Xe^{2p\pi i/q}, \quad X = e^{-\eta}, \quad Y = q^k \eta,$$

so that

(10.2) $$\eta = \log \frac{1}{X} = \log \frac{1}{R} - i\theta = \nu - i\theta,$$

where

(10.3) $$\nu = -\log\left(1 - \frac{1}{n}\right) \sim \frac{1}{n}$$

and θ varies over an interval extending, on each side of 0, a distance greater than $1/2qN$ and less than $1/qN$. The transformation in question is then

(10.4) $$f_k(x) = 2S_{p,q}\phi(0) + 4\sum_1^\infty S_{p,q,m}\phi(N)$$
$$= 2\Gamma(1+a)\,S_{p,q}Y^{-a} + 4\sum_1^\infty S_{p,q,m}\phi(N),$$

where

(10.5) $$\phi(m) = \int_0^\infty e^{-Yu^k} \cos 2m\pi u \, du.$$

When $k = 2$, $\phi(m)$ can be evaluated in terms of elementary functions, and we are led back to a familiar formula in the theory of the Theta-functions. The general formula (10.4) may be proved without difficulty by the methods of the calculus of residues, as expounded by Lindelöf in his work quoted in § 5.

The function $\phi(m)$.

11. It is now necessary to study the integral (10.5) in some detail; and a variety of methods are available for this discussion. We observe first that θ or am X rises, towards the ends of a major arc, to the order of magnitude $1/Nq$. Since $q < n^a$ and $N \leq n^{1-a}$, this is never of lower order than $1/n$, and in general of greater. Thus in general am Y is nearly equal to $-\frac{1}{2}\pi$ when x is at the upper end of a major

arc, and to $\frac{1}{2}\pi$ when x is at the lower end, though on some major arcs the range of variation of am Y is somewhat more restricted.

(i) Whatever method is followed, the analysis is a little simpler when k is even*. Let us suppose first, to fix our ideas, that $k=4$. We have

$$\phi(m) = Y^{-\frac{1}{4}} \int_0^\infty e^{-w^4} \cos \lambda w \, dw = \tfrac{1}{4} Y^{-\frac{1}{4}} \chi(\lambda),$$

where $$\lambda = 2m\pi Y^{-\frac{1}{4}}$$

and $$\chi(\lambda) = \sum_0^\infty \frac{(-1)^p \Gamma(\frac{1}{2}p + \frac{1}{4})}{2p!} \lambda^{2p},$$

the powers of Y having their principal values. This function is substantially one of the linear combinations of generalised hypergeometric functions studied by Barnes†. In fact we have, in Barnes' notation,

$$\chi(\lambda) = \Gamma(\tfrac{1}{4})\, {}_0F_2\{\tfrac{1}{2}, \tfrac{3}{4};\ (\tfrac{1}{4}\lambda)^4\} - \tfrac{1}{2}\Gamma(\tfrac{3}{4})\, \lambda^2\, {}_0F_2\{\tfrac{5}{4}, \tfrac{3}{2};\ (\tfrac{1}{4}\lambda)^4\}.$$

Using Barnes' results, we are led to the conclusion that, for the range of values of Y under consideration, we have

$$(11.1) \quad \phi(m) = O\,|\,m^{-\frac{1}{3}} Y^{-\frac{1}{6}} \exp(-He^{\pm\frac{1}{3}\pi i} m^{\frac{4}{3}} Y^{-\frac{1}{3}})\,|\,;$$

where H is positive and, together with the constant implied by the O, independent both of Y and of m, and the ambiguous sign is to be interpreted in the most unfavourable manner; chosen, that is to say, so that, when x is near the end of the major arc in question,

$$\pm\tfrac{1}{3}\pi - \tfrac{1}{3}\operatorname{am} Y$$

is as nearly as possible equal to $\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$. It is clear that we must take the plus sign when θ is positive and am Y negative, and the minus sign in the opposite case*.

It was by the analysis sketched above that we first obtained the formula (11.1)†, on which the most critical part of our proof (when $k=4$) depends. It would no doubt be possible to treat the general case on similar lines. The formal complications of the necessary analysis would however be considerable,

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* So far as the mere proof of Hilbert's Theorem is concerned, we might without prejudice suppose k even throughout; for if (*e.g.*) a number is always expressible as the sum of a fixed number of sixth powers, it is *a fortiori* always expressible as the sum of a fixed number of positive cubes. For other purposes it would be very undesirable that k should be subject to any restriction.

† E. W. Barnes, 'The asymptotic expansion of integral functions defined by hypergeometric series', *Proc. London Math. Soc.*, ser. 2, vol. v., pp. 59–116.

since the actual functions with which we deal are very special cases of those studied by Barnes, which involve a large number of arbitrary parameters. It would moreover be necessary to consider in detail the validity of Barnes' results in certain limiting cases which are more important for our purposes than for those of his memoir. A more direct method is therefore desirable; and such a method is available in what has been described as the 'Methode der Sattelpunkte', 'méthode du col', or 'method of steepest descents', which has been applied by a number of recent writers, and notably by Brillouin, Debye, Perron, and Watson‡, to the solution of problems concerning the asymptotic expansion of functions defined by definite integrals.

(ii) When k is even

$$\phi(m) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-Yu^k+2m\pi ui}\,du.$$

The path of integration may be deformed into a suitable path through the 'col' given by

$$u = \left(\frac{2m\pi i}{kY}\right)^b \S\,;$$

and we find without serious difficulty that

$$\begin{aligned}(11.2)\quad \phi(m) &= O\,|m^{-\frac{1}{2}(1-b)}Y^{-\frac{1}{2}b}\exp\{-He^{\pm\frac{1}{2}(1-b)\pi i}m^{1+b}Y^{-b}\}|\\ &= O\,|m^{-\frac{1}{2}(1-b)}Y^{-\frac{1}{2}b}E|\\ &= O(\psi),\end{aligned}$$

say. The interpretations of H, and of the ambiguous sign, are the same as in (11.1), to which (11.2) reduces when $k=4$ and $b=\frac{1}{3}$.

When k is odd,

$$\phi(m) = \frac{1}{2}\int_0^{\infty} e^{-Yu^k+2m\pi ui}\,du + \frac{1}{2}\int_0^{\infty} e^{-Yu^k-2m\pi ui}\,du.$$

Each of the integrals on the right may be analysed by the method of steepest descents, but there will be two terms in

---

* When x is at the centre of the arc the sign is indifferent, since Y is real and $\cos\frac{1}{3}\pi = \cos(-\frac{1}{3}\pi) = \frac{1}{2}$.

† It would be possible, by using the full force of Barnes' results, to replace (11.1) by a far more accurate formula, but such refinements would be irrelevant to our purpose.

‡ See, for example, G. N. Watson, 'The harmonic functions associated with the parabolic cylinder', *Proc. London Math. Soc.*, ser. 2, vol. xvii. (1918), pp. 116–148, where many references to the literature of the subject are given.

§ There are $k-1$ 'cols': the one with which we are concerned is that which is real when Y is purely imaginary.

the final result, one arising from a 'col' and one from the fixed end of the path of integration. It will be found that

$$(11.3) \qquad \phi(m) = O(\psi) + O(m^{-k-1}|Y|).$$

The second term is trivial: there is nothing of importance in in our analysis which depends upon the distinction between odd and even values of k.

Behaviour of $f_k(x)$ on a major arc (continued.)

12. We have, from (10.4),

$$f_k(x) = \phi_{p,q} + \Phi_{p,q};$$

where

$$\phi_{p,q} = 2\Gamma(1+a) S_{p,q} Y^{-a} = 2\Gamma(1+a)\frac{S_{p,q}}{q}\left(\log\frac{1}{X}\right)^{-a},$$

and

$$(12.1) \qquad \Phi_{p,q} = 4\sum_1^\infty S_{p,q,m}\phi(m) = O\left(q^{\kappa+\epsilon}\sum_1^\infty|\phi(m)|\right).$$

If k is odd, our upper bound (11.3) for $|\phi(m)|$ contains two terms. The second of these, when substituted into (12.1), will contribute

$$O(q^{\kappa+\epsilon}|Y|\Sigma m^{-k-1}) = O(q^{\kappa+\epsilon}|Y|).$$

This contribution is altogether trivial in comparison with other error terms which we shall be compelled to retain. We may therefore proceed with the argument as though k were even.

Suppose now that $\theta > 0$ and

$$Y = q^k(\nu - i\theta) = q^k\rho e^{-i\psi},$$

so that

$$\rho = \nu\sec\psi \sim 1/(n\cos\psi).$$

Then

$$\mathbf{R}\{-He^{\frac{1}{2}(1-b)\pi i}m^{1+b}Y^{-b}\} = -Hm^{1+b}q^{-1-b}\nu^{b}(\cos\psi)^b\cos\Psi,$$

where

$$\Psi = \tfrac{1}{2}(1-b)\pi + b\psi.$$

It is easy to verify that the ratio $\cos\Psi/\cos\psi$ remains between positive bounds for $0 \leq \psi \leq \frac{1}{2}\pi$. Thus

$$|E| = O[\exp\{-Jm^{1+b}q^{-1-b}n^b(\cos\psi)^{1+b}\}],$$

where J, like H, is independent of n, q, or m. Finally, from (11.2) and (12.1), we deduce

$$(12.2) \qquad \Phi_{p,q} = O\Big[q^{\kappa+\epsilon}|Y|^{-\frac{1}{2}b} \times \sum_1^\infty m^{-\frac{1}{2}(1-b)}\exp\{-Jm^{1+b}q^{-1-b}n^b(\cos\psi)^{1+b}\}\Big].$$

13. In using (12.2) we must distinguish two cases.

(i) Suppose first that $n\theta \leq 1$.* Then θ/ν is less than, and $\cos\psi$ greater than, a positive constant. Also

$$n^b q^{-1-b} = (nq^{-k})^b \geq 1$$

and
$$Jm^{1+b} q^{-1-b} n^b (\cos\psi)^{1+b} > Lm^{1+b},$$

where L is a constant like H and J. Finally, $|Y|$ is greater than a constant multiple of q^k/n. Hence

$$(13.1)\quad \Phi_{p,q} = O\left\{q^{\kappa+\epsilon}\left(\frac{q^k}{n}\right)^{-\frac{1}{2}b}\sum_1^\infty \exp(-Lm^{1+b})\right\}$$
$$= O(n^{\frac{1}{2}b} q^{\kappa+\epsilon-\frac{1}{2}kb})$$
$$= O(n^{a\kappa+\epsilon}),$$

since $q^k \leq n$.†

(ii) Suppose that $n\theta > 1$. Then $\cos\psi$ is greater than a constant multiple of $1/n\theta$; and

$$q^{-1-b} n^b (\cos\psi)^{1+b}$$

is greater than a constant multiple of

$$q^{-1-b} n^{-1} \theta^{-1-b},$$

and therefore, since $\theta < 1/qn^{1-a}$, than a constant multiple of

$$n^{-1+(1-a)(1+b)} = 1.$$

Also $|Y|$ is greater than a constant multiple of $q^k\theta$. Hence we obtain an equation of the form

$$\Phi_{p,q} = O\{q^{\kappa+\epsilon}(q^k\theta)^{-\frac{1}{2}b}\sum_1^\infty \exp(-Lm^{1+b})\}.$$

Finally, $\theta > 1/n$, so that we are again led to (13.1).‡ We have thus proved equations (9.1)—(9.3) of § 9.

The behaviour of $f_k(x)$ on the minor arcs.

14. On a minor arc $q^k > n$; and $|Y| > q^k/n$, so that Y is in general *large*. Thus the argument of the preceding sections, which was based on the assumption that Y is generally small, fails in principle.

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* Any other positive constant might be used instead of 1.

† We assume here that $k > 2$. In this case $\kappa \geq \frac{1}{2}kb$ (the sign of equality occurring when $k=3$, $\kappa = \frac{3}{4}$, $b=\frac{1}{2}$). The argument requires modification when $k=2$.

‡ The argument again requires modification when $k=2$.

It is easy to see, moreover, that the formulæ (9.1)–(9.3), if true, will no longer express a genuine approximation to $f(x)$. For

$$S_{p,q}Y^{-a}=O\{q^{\kappa+\epsilon}(q^k/n)^{-a}\}$$
$$=O(n^a q^{\kappa-1+\epsilon})$$
$$=O(n^{a\kappa+\epsilon}),$$

since $\kappa<1$, so that the term $\phi_{p,q}$ will be irrelevant, and we shall have simply

(14.1) $$f(x)=O(n^{a\kappa+\epsilon}).$$

Now there is nothing to suggest any abnormal rise in the order of magnitude of $f(x)$, near particular points of the circle of convergence, except in so far as such a rise may be accounted for by the influence of a term $\phi_{p,q}$. It is such a term which dominates $f(x)$ on a major arc: on a minor arc, where there is no such dominating influence, there is no reason to expect that $f(x)$ will exceed the order of the error terms which affect all our analysis. In short, it is reasonable to expect (14.1) to hold over any minor arc; and this expectation is in fact fulfilled, though an entirely different proof is required.

15. Our proof rests upon the following lemma in the theory of 'Diophantine Approximation'.

If $n^{a-\epsilon}<q<n^{1-a+\epsilon}$ *and* $\mu<n^{a+\epsilon}$, *then*

(15.1) $$S_\mu=\sum_0^\mu e^{2h^k p\pi i/q}=O(n^{a\kappa+\epsilon}),$$

uniformly in p.

A word of explanation is required as to the use of ϵ. The symbol means, as usual, 'an arbitrary (small) positive number'; and the lemma might be stated more formally as follows. *If* $\eta>0$, $\zeta>0$ *and* $n^{a-\eta}<q<n^{1-a+\eta}$, $\mu<n^{a+\zeta}$, *then* $S_\mu=O(n^{ak+\epsilon})$, *where* $\epsilon=\epsilon(\eta,\zeta)$ *is a positive function of* η *and* ζ *which tends to zero when* η *and* ζ *tend to zero.* The proof of the lemma depends upon the same transformations of Weyl that were quoted in § 6. The result is sufficient for our purpose but (like the result of § 6) is probably not final, and for this reason we reserve the details of the proof.

Our lemma leads at once to a proof of (14.1). Suppose, for simplicity, that X is real (so that we are considering the central point of a minor arc). Then

$$f(x)=1+2\sum_{1\le\nu<\mu}x^{\nu^k}+2\sum_{\mu\le\nu}x^{\nu^k}$$
$$=g(x)+O(1),$$

where $\mu = n^{a+\epsilon}$ and

$$g(x) = 2 \sum_{1 \leq \nu < \mu} X^{\nu^k} e^{2\nu^k p\pi i/q}.$$

From (15.1) it follows at once, by partial summation, that

$$|g(x)| \leq 2 \operatorname*{Max}_{1 \leq \nu < \mu} |S_\nu| = O(n^{a\kappa+\epsilon}).$$

The argument which precedes applies, as it stands, only to the central point of a minor arc, but there is no difficulty in extending the conclusion to the whole of the arc.*

To sum up the conclusions of §§ 9–15 : *we have*

$$f_k(x) = \phi_{p,q} + O(n^{a\kappa+\epsilon}) \tag{15.2}$$

on a major arc, and

$$f_k(x) = O(n^{a\kappa+\epsilon}) \tag{15.3}$$

on a minor arc; where

$$\phi_{p,q} = 2\Gamma(1+a)\frac{S_{p,q}}{q}\left(\log\frac{1}{X}\right)^{-a}, \tag{15.4}$$

$$X = x/x_{p,q} = xe^{-2p\pi i/q}, \quad a = 1/k, \quad \kappa = (K-1)/K = 1 - 2^{-k+1},$$

and ϵ is any positive number.

Proof of Hilbert's Theorem.

16. Returning to the contour integral (8.1), we have

$$r_s(n) = \frac{1}{2\pi i}\int \frac{f_s(x)}{x^{n+1}}\,dx = \sum \frac{1}{2\pi i}\int_{\xi_{p,q}} \frac{f_s(x)}{x^{n+1}}\,dx, \tag{16.1}$$

the summation extending over all arcs $\xi_{p,q}$ of the Farey dissection of order $m = [n^{1-a}]$. We write this equation in the form

$$r_s(n) = S_1 + S_2, \tag{16.2}$$

where S_1 comprises the terms contributed by major arcs and S_2 those contributed by minor arcs. It follows at once from (15.3) that

$$S_2 = O(n^{sa\kappa+\epsilon}); \tag{16.3}$$

but S_1 requires considerably more discussion.

* The value of $|\theta|$ at the end of a minor arc is less than $1/qn^{1-a} < 1/n$, since $q > n^a$. Hence the lines from $x_{p,q}$ to the ends of its associated arc make with the radius vector an angle less than $\frac{1}{4}\pi$, so that the whole arc lies within a region of the type employed (*e.g.* by Stolz) in the ordinary extensions of Abel's theorem. See for example Bromwich, *Infinite Series*, pp. 210–212. On major arcs the lines referred to are generally nearly parallel to the tangent at $x_{p,q}$.

On a major arc, we have

$$f = \phi_{p,q} + \Phi_{p,q} = \phi + \Phi, \quad f_s = (\phi + \Phi)^s,$$

or

$$f_s = \phi^s + \Psi,$$

where

$$\Psi = O\{\mathrm{Max}(|\phi^{s-1}\Phi|, |\Phi^s|)\}.$$

Hence

$$(16.4) \quad \frac{1}{2\pi i}\int_{\xi_{p,q}} \frac{f_s}{x^{n+1}}\,dx = \frac{1}{2\pi i}\int_{\xi_{p,q}} \frac{\phi^s}{x^{n+1}}\,dx + \mathrm{R}_{p,q} = i_{p,q} + \mathrm{R}_{p,q},$$

where

$$(16.5) \quad \mathrm{R}_{p,q} = O\left(\int_{\xi_{p,q}} \frac{|\phi^{s-1}\Phi|}{|x|^{n+1}}\,|dx|\right) + O\left(\int_{\xi_{p,q}} \frac{|\Phi^s|}{|x|^{n+1}}\,|dx|\right)$$

$$= \rho_{p,q} + \sigma_{p,q},$$

say. It is plain, in the first place, that

$$(16.6) \quad \Sigma\sigma_{p,q} = O(n^{sa\kappa+\epsilon}).$$

To estimate $\Sigma\rho_{p,q}$, we observe that

$$\phi = O\{q^{\kappa+\epsilon}(q^k|\nu - i\theta|)^{-a}\} = O\{q^{-\lambda+\epsilon}(\nu^2+\theta^2)^{-\frac{1}{2}a}\},$$

where $\lambda = 1 - \kappa = 1/K$, and

$$\Phi = O(n^{a\kappa+\epsilon}).$$

Thus

$$\rho_{p,q} = O\left\{n^{a\kappa+\epsilon}q^{-(s-1)\lambda+\epsilon}\int_{-\infty}^{\infty}(\nu^2+\theta^2)^{-\frac{1}{2}a(s-1)}\,d\theta\right\}$$

$$= O\{n^{sa-a\lambda-1+\epsilon}q^{-(s-1)\lambda+\epsilon}\}.$$

Now the series

$$\Sigma q^{-(s-1)\lambda+\epsilon}$$

will be convergent if $(s-1)\lambda > 2$*, *i.e.* if

$$(16.7) \quad s > 2K + 1;$$

a condition we shall henceforth suppose to be satisfied. And so

$$\Sigma\rho_{p,q} = O\{n^{sa-a\lambda-1+\epsilon}\},$$

$$\Sigma\mathrm{R}_{p,q} = O(n^{sa\kappa+\epsilon}) + O(n^{sa-a\lambda-1+\epsilon}),$$

$$(16.8) \quad r_s(n) = S_1 + S_2 = \Sigma i_{p,q} + O(n^{sa\kappa+\epsilon}) + O(n^{sa-a\lambda-1+\epsilon}).$$

17. We next perform two transformations on $i_{p,q}$.

(i) We first replace ϕ^s, in $i_{p,q}$, by a function of the type considered in § 5. The functions

$$\phi^s = \{2\Gamma(1+a)\}^s\left(\frac{S_{p,q}}{q}\right)^s\left(\log\frac{1}{X}\right)^{-sa}$$

* It must be remembered that Σ implies summation with respect to p as well as q.

and $$F_{p,q} = \frac{\{2\Gamma(1+a)\}^s}{\Gamma(1+as)} \left(\frac{S_{p,q}}{q}\right)^s F_{sa}(X),$$

where F is defined as in § 5, differ by a function regular for $X=1$; and it is easy to see that, in substituting one for the other, we introduce only an error term trivial in comparison with those already present in (16.8).

(ii) We next replace the arc of integration $\xi_{p,q}$ by the complete circle of which it is a part. This process also is easily shown to introduce only an error term of less importance than those present in (16.8).

We have thus replaced $\Sigma i'_{p,q}$, in (16.8), by

$$\frac{C}{2\pi i} \Sigma \left(\frac{S_{p,q}}{q}\right)^s \int \frac{F_{sa}(X)}{x^{n+1}}\, dx,$$

where C is defined as in § 5 and the path of integration is the circle Γ; and this is equal to

$$(17.1) \qquad Cn^{sa-1} \Sigma \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q}.$$

18. The summation in (17.1) extends over all values of q which do not exceed n^a, and all values of p less than and prime to q. The series remains convergent, when q ranges to infinity, if $s\lambda > 2$ or $s > 2K$, a condition included in (16.7). Also

$$n^{sa-1} \sum_{q>n^a} \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q} = O\left(n^{sa-1} \sum_{q>n^a} q^{1-s\lambda}\right)$$
$$= O\left(n^{sa\kappa+2a-1+\epsilon}\right) = O\left(n^{sa\kappa+\epsilon}\right).$$

We have therefore

(18.1)
$$r_s(n) = Cn^{sa-1} \Sigma \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q} + O\left(n^{sa\kappa+\epsilon}\right) + O\left(n^{sa-a\lambda-1+\epsilon}\right)$$
$$= \rho(n) + O\left(n^{sa\kappa+\epsilon}\right) + O\left(n^{sa-a\lambda-1+\epsilon}\right),$$

in the notation of § 5.

The equation (18.1) contains the proof of Hilbert's Theorem. In fact, we have seen in § 7 that the coefficient of

n^{sa-1} is positive (and as near to unity as we please) for every n, if only s is large enough. Also

$$sa - a\lambda - 1 < sa - 1,$$

and $$sa\kappa < sa - 1$$

if $sa\lambda > 1$ or $$s > kK = 2^{k-1}k.$$

It is to be observed that this last condition includes the condition $s > 2K+1$, previously imposed upon s, in all cases except that in which $k=2$.

It follows that $r_s(n) > 0$ for $s \geq G(k)$, where $G(k)$ is a number which depends only on k, and for all sufficiently large values of n. In other words *every large number is the sum of at most $G(k)$ positive k^{th} powers:* and from this it obviously follows that *every number is the sum of at most $g(k)$ positive k^{th} powers*, which is Hilbert's Theorem.

It is plain that we have in reality proved much more than Hilbert's Theorem; in particular that *to every k corresponds a $G_1(k)$ such that*

$$r_{k,s}(n) \sim \frac{[2\Gamma\{1+(1/k)\}]^s}{\Gamma\{1+(s/k)\}} n^{(s/k)-1} \Sigma \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q}$$

for $s \geq G_1(k)$. Here $G_1(k)$, like $g(k)$ and $G(k)$, is a number which depends only on k.

On the values of $g(k)$ and $G(k)$.

19. The numbers of $g(k)$ and $G(k)$ are the least numbers such that (1) every number is the sum of at most $g(k)$ positive k^{th} powers and (2) every *large* number is the sum of at most $G(k)$ positive k^{th} powers. It is obvious that

$$G(k) \leq g(k).$$

The preceding analysis establishes the existence of these numbers; but does not, as it stands, lead to a definite value of either, though it suggests very forcibly that

$$G(k) \leq 2^{k-1}k + 1. \tag{19.1}$$

The known results, for the first few values of k, are as follows.*

* See the dissertations of A. J. Kempner, *Über das Waringsche Problem und einige Verallgemeinerungen*, Göttingen 1912, and W. S. Baer, *Beiträge zum Waringschen Problem*, Göttingen 1913.

In the first place, it is known that $g(k)$ does not exceed the numbers given in the following table:

$$k = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$$

$$g(k) \leq 4,\ 9,\ 37,\ 58,\ 478,\ 3806,\ 31353.^{*}$$

These numbers furnish, *a fortiori*, upper bounds for $G(k)$. It has also been proved by Landau that $G(3) \leq 8$.

The corresponding lower bounds for $g(k)$ are

$$g(k) \geq 4,\ 9,\ 19,\ 37,\ 73,\ 143,\ 279.$$

Thus $g(2) = 4$ and $g(3) = 9$, while there is a wide range of uncertainty for all higher values of k.

As regards $G(k)$, there is even greater uncertainty. It is known that $G(k) \geq k+1$ for all values of k. Thus $4 \leq G(3) \leq 8$, but the actual value of $G(3)$ is unknown: empirical data point to 5 or 6. Similarly, nothing is known about $G(5)$ save that it lies between 6 and 58. When k is *a power of* 2, a little more is known, viz. that

$$G(k) \geq 4k:$$

in particular $G(4) \geq 16$, so that $G(4)$ lies between 16 and 37.

20. If $s > 2^{k-1}k$, the error terms in (18.1) are of lower order than the power of n which multiplies the singular series. Our analysis therefore makes it appear very probable that the inequality (19.1) is true. The values of the right-hand side of the inequality are 5, 13, 33, 81, 193, 449, 1025; so that it would give new values for $G(k)$ except when k is 2, 3, or 5. In order to prove it generally, it is necessary to study the singular series much more closely.

The singular series may be written in the form

$$S = 1 + A_2 + A_3 + \ldots + A_q + \ldots .$$

For particular values of k and s, the values of the most important terms of the series may be calculated without excessive labour, and upper bounds assigned to the remainder. The series may thus be expressed as a sum of terms which oscillate according to simple laws, and show in a most illuminating manner the source of the irregular variations of $r_{k,s}(n)$. Suppose, for example, that $k = 4$, $s = 33$. It is plain that the most im-

* The numbers towards the end of the table need not be taken too seriously, as they have been obtained by writers anxious only to establish *some* numerical result. But a good deal of work has been devoted to the earlier cases, and in particular to the cases $k=3$ and $k=4$.

portant terms in the series are those for which $|S_{p,q}|$ is most nearly equal to unity: these correspond to $q=16$, $q=8$, $q=5$ and certain associated values of p. When s is as large as 33, all other terms of the series are quite small; and we find that

$$r_{4,33}(n) = Cn^{\frac{29}{4}}\{1 + 1{\cdot}054 \cos(\tfrac{1}{8}n\pi - \tfrac{1}{16}\pi) + {\cdot}147 \cos(\tfrac{1}{4}n\pi - \tfrac{1}{8}\pi) + ({\cdot}038)\} + O(n^{\psi+\epsilon}),$$

where (·038) denotes a number whose modulus is less than ·038, and

$$\psi = \tfrac{7}{8} \times \tfrac{33}{4} = \tfrac{231}{32} < \tfrac{232}{32} = \tfrac{29}{4}.$$

It is easily verified that

$$1 + 1{\cdot}054 \cos(\tfrac{1}{8}n\pi - \tfrac{1}{16}\pi) + {\cdot}147 \cos(\tfrac{1}{4}n\pi - \tfrac{1}{8}\pi) > {\cdot}102,$$

so that $r_{4,33}(n) > 0$ for all sufficiently large values of n. It follows that $G(4) \leq 33$, or in other words that *every large number is the sum of at most* 33 *fourth powers.*

It may be shown similarly that

$$G(6) \leq 193.$$

In this case it will be found that *all* the terms in the singular series, except the first, are quite small: a fact which in itself suggests very forcibly that the result, although a considerable improvement on what is known, is a long way from the ultimate truth.

21. Our general proof of (19.1) does not involve any difficulty of principle, or novelty of idea, other than those which we have explained in §§ 3–17. But the completion of the proof demands a considerable number of simple algebraical calculations of which it would be difficult to give a short and intelligible account. We must therefore confine ourselves for the moment to a few general remarks.

We suppose that $s = 2^{k-1}k + 1$; and what we have to prove is that there is a number δ, independent of n, such that

$$|S| > \delta.$$

It would no doubt be possible to prove this, for any particular value of k, by an argument proceeding on the lines of § 19. The general proof, however, is very much simplified by a preliminary transformation of our series into an infinite product.

It is easy to prove that

$$A_q = \Sigma \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q}$$

satisfies the equation $A_{qq'} = A_q A_{q'}$,

whenever q and q' are prime to one another. From this it follows that

$$S = \Pi \chi_{\varpi}$$

where ϖ runs through the primes 2, 3, 5, ... and

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + A_{\varpi^3} + \dots .$$

The problem of finding a lower bound for S is thus reduced to that of finding lower bounds for each of the factors χ_{ϖ}.

The discussion of χ_{ϖ} depends upon the arithmetical relations between ϖ and k, the most troublesome factors being those for which ϖ is a divisor of k or is of the form $mk+1$. Suppose, to fix our ideas, that k is *prime*. Then the factors in question correspond to $\varpi = k$ and $\varpi = mk+1$. The other factors are easily shown to be unimportant.

If ϖ is of the form $mk+1$, and n is not divisible by ϖ, A_{ϖ^2}, A_{ϖ^3}, ... are found to be zero, and χ_{ϖ} reduces to the simple form

$$\chi_{\varpi} = 1 + A_{\varpi}.$$

All that is necessary, then, is to find an appropriate upper bound for $|A_{\varpi}|$, and this can be done if we know such a bound for $|S_{p,\varpi}|$. Now it is easy to prove, first that

$$|S_{p,\varpi}| \leq (k-1)\sqrt{\varpi},$$

and secondly that

$$|S_{p,\varpi}| \leq \varpi - 2(\varpi - 1)\sin^2\frac{\pi}{\varpi}.$$

Using one or other of these two inequalities, the first if $\varpi > k^2$ and the second if $\varpi < k^2$, we obtain in either case an inequality for $|A_{\varpi}|$ which proves sufficient for our purpose. If n is divisible by ϖ, the argument is slightly more complicated, but in principle the same. The factor χ_k, corresponding to $\varpi = k$, may be dealt with in a similar manner.

The consideration of composite k's does not introduce any fresh difficulties of a serious nature; but it is hardly possible, in such a sketch as this, to give a coherent account of the various complications which arise; and we must pass them by until we are able to publish our proof in full. Our object in this paper has been merely to give a general account of our method, full enough to make clear to the reader the fundamental ideas on which it rests, and the nature of the difficulties which arise and the analysis which is necessary to overcome them.

CORRECTIONS

There is a correction to (10.4) in 1920, 5 (footnote on p. 45).

In the formula for C, above (5.1) on p. 276, the denominator should read $\Gamma(s/k)$. Similarly at the top of p. 289 and in the asymptotic formula for $r_{k,s}(n)$ on p. 290.

Some problems of 'Partitio Numerorum'; I: A new solution of Waring's Problem.

Von

G. H. Hardy in Oxford und **J. E. Littlewood** in Cambridge.

Vorgelegt von E. Landau in der Sitzung vom 30. Januar 1920.

Introduction.

1. The present memoir is the first of a series, in which we propose to develop in detail the new analytic method which we have found for the discussion of Waring's Problem and a number of allied problems in 'additiver Zahlentheorie'. The general lines of our method, in so far as it concerns Waring's Problem in particular, were explained in a recent paper[1]) in the *Quarterly Journal of Mathematics.* This paper contains also full references to the literature connected with the problem. Our object here is to give full details of the proofs, up to the point at which Hilbert's famous theorem, first proved in this journal[2]) in 1909, emerges as a corollary from our analysis.

Notation and terminology.

2. 1. The following notation will be adhered to throughout the memoir: —

k, s, n, p, q, and m are positive integers or zero, $k > 2$, $s > 0$, $0 \leqq p < q$, $(p, q) = 1$, $p > 0$ if $q > 1$;

1) G. H. Hardy and J. E. Littlewood, *A new solution of Waring's Problem,* Quarterly Journal, Vol. 48 (1919), pp. 272—293.

2) D. Hilbert, *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^{ter} Potenzen (Waringsches Problem),* Göttinger Nachrichten, 1909, pp. 17—36.

1920, 5 (with J. E. Littlewood) *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen,* Math.-phys. Klasse, 1920, 33–54.

$$K = 2^{k-1}, \quad \varkappa = 1 - \frac{1}{K}, \quad a = \frac{1}{k}, \quad b = \frac{1}{k-1},$$

$$C = \frac{(2\Gamma(1+a))^s}{\Gamma(sa)};$$

$$e(x) = e^{2\pi i x}, \quad e_q(x) = e\left(\frac{x}{q}\right),$$

$$S_{p,q} = \sum_{h=0}^{q-1} e_q(h^k p), \quad S_{p,q,m} = \sum_{h=0}^{q-1} e_q(h^k p) \cos \frac{2mh\pi}{q},$$

$$S'_{p,q,m} = \sum_{h=0}^{q-1} e_q(h^k p) \sin \frac{2mh\pi}{q};$$

$$f(x) = 1 + 2x^{1^k} + 2x^{2^k} + \cdots \qquad (|x| < 1),$$

$$F_\sigma(x) = \sum_{\nu=1}^{\infty} \nu^{\sigma-1} x^\nu \quad (\sigma > 0, \ |x| < 1).$$

We denote by $r_{k,s}(n)$ the coefficient of x^n in $(f(x))^s$, so that

$$(1 + 2x^{1^k} + 2x^{2^k} + \cdots)^s = 1 + \sum_{n=1}^{\infty} r_{k,s}(n) x^n.$$

It is evident that $r_{k,s}(n) \geqq r_{k,s'}(n)$ if $s > s'$. If k is *even*, $r_{k,s}(n)$ is the number of representations of n in the form

$$n = n_1^k + n_2^k + \cdots + n_s^k,$$

where n_j is an integer, positive, negative, or zero, and representations which differ in the sign or order of the numbers n_j are reckoned as distinct. If k is *odd*, the arithmetical interpretation of $r_{k,s}(n)$ is not quite so simple. We have in fact, if $n > 0$,

$$r_{k,s}(n) = \sum_{\mu=1}^{s} \binom{s}{\mu} 2^\mu \bar{r}_{k,\mu}(n),$$

where $\bar{r}_{k,\mu}(n)$ is the number of representations of n by exactly μ positive k-th powers.

2. 2. It is plain that, if $n > 1$,

$$r_{k,s}(n) = \frac{1}{2\pi i} \int \frac{(f(x))^s}{x^{n+1}} dx, \tag{2.21}$$

the contour of integration being the circle Γ whose centre is the origin and whose radius is $R = 1 - \frac{1}{n}$.

We divide Γ into arcs $\xi_{p,q}$, which we call *Farey arcs*, as follows. We form the Farey's series of order $N = [n^{1-a}]$, the first

and last terms being $\frac{0}{1}$ and $\frac{1}{1}$. We suppose that $\frac{p}{q}$, where $q > 1$, is a term of the series, and $\frac{p'}{q'}$ and $\frac{p''}{q''}$ the adjacent terms on the right and left; and we denote by $j_{p,q}$ the interval

$$\frac{p}{q} - \frac{1}{q(q+q'')}, \quad \frac{p}{q} + \frac{1}{q(q+q')}.$$

The intervals $j_{0,1}$ and $j_{1,1}$ are defined to be $\left(0, \frac{1}{N+1}\right)$ and $\left(1 - \frac{1}{N+1}, 1\right)$ respectively. The intervals $j_{p,q}$ just fill up the interval (0, 1), and the length of each of the parts into which $j_{p,q}$ is divided by $\frac{p}{q}$ is not less than $\frac{1}{2qN}$ and less than $\frac{1}{qN}$ [3]). If now the intervals $j_{p,q}$ are considered as intervals of variation of $\frac{\theta}{2\pi}$, where $\theta = \arg x$, and the two extreme intervals amalgamated into one, we obtain the required dissection of Γ into arcs $\xi_{p,q}$.

We call $\xi_{p,q}$ a *major arc* if $q \leqq n^a$; a *minor arc* if $n^a < q \leqq n^{1-a}$. The arc of Γ complementary to $\xi_{p,q}$ we denote by $\eta_{p,q}$.

In considering any particular Farey arc $\xi_{p,q}$, we shall write

$$x = Xx_{p,q} = Xe^{\frac{2p\pi i}{q}} = Xe_q(p),$$

so that X is positive when x lies on the radius vector to the point $x_{p,q}$. And on $\xi_{p,q}$ we have

$$X = |X|e^{i\theta} = \left(1 - \frac{1}{n}\right)e^{i\theta}$$

where

$$|\theta| < \frac{2\pi}{qN}.$$

We shall in general use ε to denote a fixed, but arbitrary, positive number, and A to denote a positive number depending upon k only. Different ε's or A's will, when necessary, but not always, be distinguished by suffixes. The symbols O and o will be used in the manner which is now classical. It is always to be understood, however, that the inequalities or asymptotic relations implied

3) For formal proofs of these well known propositions see G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc., ser. 2, vol. 17 (1918), pp. 75—115 (§§ 4. 21—4. 22).

in the use of these symbols are, unless the contrary is stated explicitly, satisfied uniformly in all parameters other than k and (when it occurs) ε. Thus, if ψ is defined for $n = 2, 3, 4, \ldots$,

$$\psi = O\left(n^{a\varkappa+\varepsilon}\right)$$

means that to every positive ε corresponds an $H = H(k, \varepsilon)$ such that $|\psi| < Hn^{a\varkappa+\varepsilon}$ for $n = 2, 3, 4, \ldots$.

Statement of the main theorem.

3. Hilbert's theorem may be stated as follows.

Theorem A. *To every k corresponds a $g = g(k)$ such that $r_{k,s}(n) > 0$ for $s \geqq g$ and $n \geqq 1$.*

We shall prove

Theorem B. *To every k corresponds a $G_1 = G_1(k)$ such that*[4]

$$r_{k,s}(n) \sim Cn^{\frac{s}{k}-1} \sum_{p,q} \left(\frac{S_{p,q}}{q}\right)^s e_q(-np)$$

for $s \geqq G_1$.

It will be shown that this series is absolutely convergent for $s \geqq s_1 = s_1(k)$, and that its sum — which is real, since the terms (p, q) and $(q-p, q)$ are conjugate — is positive.

From this theorem we shall deduce, first

Theorem C. *To every k corresponds a $G = G(k)$ and an $n_0 = n_0(k)$ such that $r_{k,s}(n) > 0$ for $s \geqq G$, $n \geqq n_0$;*

and then Hilbert's theorem.

We shall use g, G_1, G to denote the *least* numbers for which these assertions are true.

The singular series.

4. We call the series

$$S = \sum_{p,q} \left(\frac{S_{p,q}}{q}\right)^s e_q(-np)$$

the *singular series*. The use of this series, which is the dominating idea of our work, is suggested by the following considerations.

If we write $x = Xx_{p,q}$, we have

$$f(x) = -1 + 2\sum_{\nu=0}^{\infty} X^{\nu^k} e_q(\nu^k p) = -1 + 2\sum_{h=0}^{q-1} e_q(h^k p) \sum_{l=0}^{\infty} X^{(lq+h)^k}.$$

4) $f(n) \sim g(n)$ means that $\dfrac{f}{g} \to 1$ when $n \to \infty$.

If now $X \to 1$ by positive values, so that x tends to $x_{p,q}$ along a radius, then

$$\sum_{l=0}^{\infty} X^{(lq+h)^k} = O(1) + \int_{-\frac{h}{q}}^{\infty} X^{(lq+h)^k} dl = O(1) + \frac{1}{q}\int_0^{\infty} X^{t^k} dt$$
$$= O(1) + \frac{\Gamma(1+a)}{q}\left(\log\frac{1}{X}\right)^{-a},$$

and

$$f(x) \sim 2\Gamma(1+a)\frac{S_{p,q}}{q}\left(\log\frac{1}{X}\right)^{-a};$$

it being understood that, when $S_{p,q} = 0$, this formula means

$$f(x) = o\left(\log\frac{1}{X}\right)^{-a}.$$

We have also

$$F_\sigma(X) = \Gamma(\sigma)\left(\log\frac{1}{X}\right)^{-\sigma} + \bar{F}(X),$$

where $\bar{F}(X)$ is regular for $X = 1$ [5]). Thus

$$(f(x))^s \sim C\left(\frac{S_{p,q}}{q}\right)^s F_{\frac{s}{k}}(X) = F_{p,q},$$

say.

Our fundamental idea is that of approximating to the coefficients of f^s by a sum formed from those of the various approximating functions $F_{p,q}$. We therefore replace X by $xe_q(-p)$, $F_{\frac{s}{k}}(X)$ by its expansion as a power series in X, pick out the

---

5) This well known proposition may be proved as follows. The function

$$F(y) = \int_0^{\infty} t^{-\sigma}\left(\frac{1}{e^{t+y}-1} - \frac{1}{t+y}\right) dt \qquad (0 < \sigma < 1)$$

is regular for $y = 0$ (the integral being uniformly convergent in the neighbourhood of $y = 0$). And if $y > 0$ we have

$$\int_0^{\infty} \frac{t^{-\sigma}\,dt}{e^{t+y}-1} = \sum_{n=1}^{\infty} e^{-ny}\int_0^{\infty} e^{-nt}\,t^{-\sigma}\,dt = \Gamma(1-\sigma)\sum_{n=1}^{\infty} n^{\sigma-1}e^{-ny},$$
$$\int_0^{\infty} \frac{t^{-\sigma}\,dt}{t+y} = \Gamma(1-\sigma)\,\Gamma(\sigma)\,y^{-\sigma}.$$

These equations establish the truth of the proposition for $0 < \sigma < 1$. It is trivial for $\sigma = 1$; and it may at once be extended to other values of $\sigma > 0$ by differentiation. Incidentally we see (from the integral expression of $\sum n^{\sigma-1}e^{-ny} = F_\sigma(e^{-y})$) that $F_\sigma(x)$ is regular at all points of the unit circle except $x = 1$.

coefficient of x^n, and sum with respect to p and q. We thus obtain

$$\varrho_{k,s}(n) = Cn^{\frac{s}{k}-1} \sum_{p,q} \left(\frac{S_{p,q}}{q}\right)^s e_q(-np) = Cn^{\frac{s}{k}-1} S$$

as the function by means of which we are hoping to approximate to $r_{k,s}(n)$.

Properties of the singular series.

5. 1. **Lemma 1.** *If* $\beta, \gamma, \ldots$ *are integers then*

$$T_k = \sum_{h=0}^{q-1} e_q(p h^k + \beta h^{k-1} + \gamma h^{k-2} + \cdots) = O\left(q^{\varkappa+\varepsilon}\right).$$

In accordance with our use of the symbol O, this equation holds uniformly in $\beta, \gamma, \ldots$.

Our proof of this fundamental lemma is based on certain transformations due in substance to Weyl[6]). For the sake of formal simplicity we consider first the case $k = 4$. We have

$$|T_4|^2 = \sum_{h,l} e_q(p(h^4 - l^4) + \beta(h^3 - l^3) + \gamma(h^2 - l^2) + \delta(h - l)).$$

the summations with respect to h and l extending over a complete set of residues to modulus q. Writing $h = l + h_1$ we obtain

$$|T_4|^2 = \sum_{l, h_1} e_q\left(4 p h_1 l^3 + B_{h_1} l^2 + C_{h_1} l + D_{h_1}\right),$$

where $B_{h_1}, C_{h_1}, \ldots$ are quadratic, cubic, ... polynomials in h_1, with integral coefficients, whose precise form is irrelevant to the argument. Writing h again in place of l, we have

$$|T_4|^2 = \sum_{h_1} (1) \sum_h e_q\left(4 p h_1 h^3 + B_{h_1} h^2 + C_{h_1} h\right),$$

where (1) denotes a number whose modulus does not exceed unity.

Applying the familiar inequality of Cauchy and Schwarz, we obtain

$$\begin{aligned} |T_4|^4 &\leqq q \sum_{h_1} \left| \sum_h e_q\left(4 p h_1 h^3 + B_{h_1} h^2 + C_{h_1} h\right) \right|^2 \\ &= q \sum_{h_1} \sum_{h,l} e_q\left(4 p h_1 (h^3 - l^3) + B_{h_1}(h^2 - l^2) + C_{h_1}(h - l)\right). \end{aligned}$$

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6) H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Mathem. Annalen, vol. 77 (1916), pp. 313—352.

Writing $h = l + h_2$, and repeating the argument,

$$|T_4|^4 \leqq q \sum_{h_1, h_2} (1) \sum_l e_q\left(12 p h_1 h_2 l^2 + B_{h_1, h_2} l\right),$$

where B_{h_1, h_2} is a polynomial in h_1 and h_2 with integral coefficients. Applying the Cauchy-Schwarz inequality again, and replacing l once more by h, we have

$$|T_4|^8 \leqq q^2 \cdot q^2 \sum_{h_1, h_2} \left| \sum_h e_q\left(12 p h_1 h_2 h^2 + B_{h_1, h_2} h\right) \right|^2$$

$$= q^4 \sum_{h_1, h_2} \sum_{h, l} e_q\left(12 p h_1 h_2 (h^2 - l^2) + B_{h_1, h_2}(h - l)\right).$$

Finally, writing $h = l + h_3$, we have

$$|T_4|^8 \leqq q^4 \sum_{h_1, h_2, h_3} \left| \sum_l e_q(24 p h_1 h_2 h_3 l) \right|.$$

It is plain that the argument used here is perfectly general; and the resulting inequality in the general case is

$$|T_k|^K \leqq q^{K-k} \sum_{h_1, h_2, \ldots, h_{k-1}} \left| \sum_l e_q(k!\, p H l) \right| \text{[7]},$$

where

$$H = h_1 h_2 \ldots h_{k-1}.$$

5. 2. If $(k!, q) = \delta$, we write $k! = \delta k_0$, $q = \delta Q$, so that $(k_0 p, Q) = 1$.

The summation with respect to $h_1, h_2, \ldots, h_{k-1}$ is defined by $0 \leqq h_j < q \ (1 \leqq j \leqq k-1)$, so that $H < q^{k-1}$. And evidently

$$(5.21) \qquad |T_k|^K \leqq q^{K-k} \sum{}' \left| \sum_l \right| + q^{K-k} \sum{}'' \left| \sum_l \right| = T' + T'',$$

where $\sum'$ is defined by

$$H = 0, \quad 0 \leqq h_j < q,$$

and $\sum''$ by

$$0 < H < q^{k-1}.$$

The number of terms in $\sum'$ is $O(q^{k-2})$, and in each of them $\left|\sum_l\right| = q$. Thus

$$(5.22) \qquad T' = O(q^{K-1}).$$

In $\sum''$, $\sum_l$ is zero unless $H \equiv 0 \pmod{Q}$, when it is q. The con-

7) The index of the power of q may be at once verified by induction: for

$$2(2^{k-1} - k) + k - 1 = 2^k - k - 1.$$

gruence is satisfied for at most $\frac{q^{k-1}}{Q} = O(q^{k-2})$ values of H; and therefore[8]) for at most $O\left(q^{k-2+\varepsilon}\right)$ sets $(h_1, h_2, \ldots, h_{k-1})$. Thus

$$T'' = O\left(q^{K-1+\varepsilon}\right). \tag{5.23}$$

From (5.21), (5.22), and (5.23) the lemma follows immediately.

As obvious corollaries we have

$$S_{p,q} = O\left(q^{\varkappa+\varepsilon}\right), \quad S_{p,q,m} = O\left(q^{\varkappa+\varepsilon}\right), \quad S'_{p,q,m} = O\left(q^{\varkappa+\varepsilon}\right).$$

5. 3. **Lemma 2.** *For all sufficiently large values of s and for all values of n, the singular series S is absolutely convergent, and $S > \frac{1}{2}$.*

The term for which $p = 0$, $q = 1$, is 1; write then $S = 1 + S'$. We have

$$\frac{S_{p,q}}{q} = O\left(q^{-(1-\varkappa)+\varepsilon}\right) = O\left(q^{-\frac{1}{K}+\varepsilon}\right),$$

and therefore

$$\left|\frac{S_{p,q}}{q}\right| < A q^{-2\mu},$$

where $\mu = \frac{1}{4K}$. Also

$$\left|\frac{S_{p,q}}{q}\right| < 1 \qquad (q > 1).$$

Choose $\nu > A^{4K}$, and let

$$\delta = \operatorname*{Max}_{2 \leqq q \leqq \nu} \left|\frac{S_{p,q}}{q}\right|.$$

Thus

$$|S'| < \delta^s \sum_{q=2}^{\nu} q + \sum_{q=\nu+1}^{\infty} A^s q^{1-2\mu s} < \nu^2 \delta^s + \sum_{q=\nu+1}^{\infty} q^{1-\mu s}$$
$$< \nu^2 \delta^s + \frac{\nu^{2-\mu s}}{\mu s - 2} < \frac{1}{2},$$
$$S = 1 + S' > \frac{1}{2},$$

if only s is sufficiently large.

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8) See E. Landau, *Über die Anzahl der Gitterpunkte in gewissen Bereichen,* Göttinger Nachrichten, 1912, pp. 687—771 (p. 717).

Behaviour of $f(x)$ on a minor arc.

6. 1. Lemma 3. *On a minor arc*

$$f(x) = O\left(n^{a\varkappa + \varepsilon}\right).$$

We deduce this from

Lemma 4. *If η and ζ are arbitrary fixed positive numbers, and μ an integer, and*

$$n^{a-\eta} < q < n^{1-a+\eta}, \quad 0 < \mu < n^{a+\zeta},$$

then

$$U_\mu = \sum_{h=0}^{\mu} e_q(ph^k) = O\left(n^{a\varkappa + \varepsilon}\right),$$

where $\varepsilon = \varepsilon(\eta, \zeta)$ is a function of η and ζ which tends to zero with η and ζ.

We denote generally by $\varepsilon_1, \varepsilon_2, \ldots$, functions of η and ζ which have the property stated.

We begin by performing on U_μ a series of transformations analogous to those of § 5. 1. In this case, however, we may at once quote the final result, as it is given explicitly by Weyl[9]). We have

$$|U_\mu|^K \leqq A\mu^{K-k} \sum_{h_1, h_2, \ldots, h_{k-1}} \left|\sum_l e_q(k!\,pHl)\right|,$$

where $H = h_1 h_2 \ldots h_{k-1}$, and the summation with respect to $h_1, h_2, \ldots, h_{k-1}$ is defined by

$$|h_1| + |h_2| + \cdots + |h_{k-1}| \leqq \mu,$$

and that with respect to l is over an unbroken sequence of not more than $\mu + 1$ values.

We replace this inequality by

$$(6.11) \quad |U_\mu|^K \leqq A\mu^{K-k} \sum{}' \left|\sum_l\right| + A\mu^{K-k} \sum{}'' \left|\sum_l\right| = U' + U'',$$

$\sum'$ and $\sum''$ being defined as in § 5. 2. And we have, in the first place,

$$(6.12) \quad U' = O(\mu^{K-k}\mu^{k-2}\mu) = O(\mu^{K-1}) = O\left(n^{(K-1)a + \varepsilon_1}\right).$$

6. 2. In discussing U'' we observe that

$$\left|\sum_l\right| \leqq \mu + 1 \quad (k!\,pH \equiv 0 \ (\text{mod.}\ q)),$$

9) Weyl, *loc. cit.*, p. 330.

$$\left|\sum_{l}\right| \leqq \left|\operatorname{cosec}\frac{k!\,pH\pi}{q}\right| \quad (k!\,pH \not\equiv 0 \pmod{q}).$$

Suppose now that $k!\,pH \equiv \lambda \pmod{q}$, where $0 \leqq \lambda < q$, and write $U'' = U_1'' + U_2''$, where $\lambda = 0$ in U_1'' and $\lambda > 0$ in U_2''. If $\lambda = 0$ we must have (in the notation of § 5. 2) $H \equiv 0 \pmod{Q}$; and this can happen for at most $2\frac{\mu^{k-1}}{Q} = O\left(\frac{\mu^{k-1}}{q}\right)$ values of H (for none if $\mu^{k-1} < Q$). To each of these correspond at most $O\left(\mu^{(k-1)\varepsilon_2}\right)$ or $O(\mu^{\varepsilon_3})$ sets $(h_1, h_2, \ldots, h_{k-1})$. Hence

$$U_1'' = O\left(\mu^{K-k}\frac{\mu^{k-1}}{q}\mu^{\varepsilon_3}\mu\right) = O\left(\mu^{K-1+\varepsilon_3}\frac{\mu}{q}\right).$$

Since $\mu < n^{a+\zeta}$, $q > n^{a-\eta}$, we have $\frac{\mu}{q} = O(n^{\varepsilon_4})$ and so

$$U_1'' = O\left(\mu^{K-1+\varepsilon_3}n^{\varepsilon_4}\right) = O\left(n^{(K-1)a+\varepsilon_5}\right).$$

Next, suppose that $\lambda > 0$, and that

$$k!\,pH \equiv k!\,pH' \equiv \lambda \pmod{q},$$

where $H' = h_1' h_2' \ldots h_{k-1}'$. Then $H' - H \equiv 0 \pmod{Q}$, or $H' = mQ + H$, where m is an integer. Since $|H|$ and $|H'|$ are each less than μ^{k-1}, this can happen for at most

$$2\frac{\mu^{k-1}}{Q} + 1$$

different values of H', and so for at most

$$O\left(\operatorname{Max}\left(\frac{\mu^{k-1+\varepsilon_6}}{q}, \mu^{\varepsilon_7}\right)\right)$$

sets $(h_1', h_2', \ldots, h_{k-1}')$. Hence

$$U_2'' = O\left(\mu^{K-1+\varepsilon_6}\frac{\sigma}{q}\right) + O\left(\mu^{K-k+\varepsilon_7}\sigma\right),$$

where

$$\sigma = \sum_{\lambda=1}^{q-1}\operatorname{cosec}\frac{\lambda\pi}{q} = O(q\log q) = O\left(q^{1+\varepsilon_8}\right).$$

The first term gives

$$O\left(n^{(K-1)a+\varepsilon_9}\right).$$

In discussing the second term we may suppose $\mu^{k-1} < Q$; and we obtain

$$O\left(n^{a(K-k)+1-a+\varepsilon_{10}}\right) = O\left(n^{(K-1)a+\varepsilon_{11}}\right),$$

since $q < n^{1-a+\eta}$. Thus U_1'' and U_2'', and therefore U'', are of the form $O\left(n^{(K-1)a+\varepsilon}\right)$. So, by (6. 12), is U'; and Lemma 4 now follows from (6. 11).

6. 3. In order to deduce Lemma 3, we write

$$f(x) = -1 + 2 \sum_{0 \leqq \nu \leqq \omega} x^{\nu^k} + 2 \sum_{\nu > \omega} x^{\nu^k} = -1 + f_1(x) + f_2(x),$$

where $\omega = \left[n^{a+\zeta}\right]$ and ζ is an arbitary positive number. Then

$$\left|x^{\nu^k}\right| = \left(1 - \frac{1}{n}\right)^{\nu^k} < e^{-\frac{\nu^k}{n}},$$

and

$$f_2(x) = O\left(\sum_{\nu > \omega} e^{-\frac{\nu^k}{n}}\right) = O\left(\int_{\omega}^{\infty} e^{-\frac{x^k}{n}} dx\right)$$

$$= O\left(\int_{2^{-k} n^{1+\zeta k}}^{\infty} e^{-\frac{w}{n}} w^{a-1} dw\right) = O\left(n^a \int_{2^{-k} n^{\zeta k}}^{\infty} e^{-u} u^{a-1} du\right) = o(1).$$

Hence we need only consider

$$f_1(x) = 2 \sum_{0 \leqq \nu \leqq \omega} X^{\nu^k} e_q(p\nu^k) = 2 \sum_{0 \leqq m \leqq \omega^k} a_m X^m,$$

say. Writing $a_0 + a_1 + \cdots + a_m = s_m$, we have

$$f_1(x) = 2 \sum_{0 \leqq m \leqq \omega^k} s_m (X^m - X^{m+1}) + 2 s_{\omega^k} X^{\omega^k + 1},$$

$$|f_1(x)| \leqq 2 \sum_{0 \leqq m \leqq \omega^k} |s_m| |X^m| |1 - X| + 2 \left|s_{\omega^k}\right| \left|X^{\omega^k + 1}\right|.$$

Now, on a minor arc, $X = |X| e^{i\theta} = \left(1 - \frac{1}{n}\right) e^{i\theta}$, where

$$|\theta| < \frac{2\pi}{q\left[n^{1-a}\right]} < \frac{A_1}{q n^{1-a}} < \frac{A_2}{n},$$

and

$$|1 - X| = \sqrt{\left(1 - 2\left(1 - \frac{1}{n}\right)\cos\theta + \left(1 - \frac{1}{n}\right)^2\right)}$$

$$= \sqrt{\left(\frac{1}{n^2} + 4\left(1 - \frac{1}{n}\right)\sin^2 \frac{1}{2}\theta\right)} < A_3 \sqrt{\left(\frac{1}{n^2} + \theta^2\right)} < \frac{A_4}{n}$$

$$= A_4(1 - |X|).$$

Hence

$$|f_1(x)| = O\left(\operatorname*{Max}_{0 \leqq m \leqq \omega^k} |s_m|\right) = O\left(\operatorname*{Max}_{0 \leqq \mu \leqq \omega} |U_\mu|\right) = O\left(n^{a\varkappa + \varepsilon}\right),$$

by Lemma 4.

Behaviour of $f(x)$ on a major arc.

7. 1. We now suppose that x lies on a major arc. We write

$$x = X x_{p,q}, \quad X = e^{-\eta}, \quad Y = q^k \eta,$$

so that

$$\eta = \log \frac{1}{X} = \log \frac{1}{R} - i\theta = \nu - i\theta$$

where

$$\nu = -\log\left(1 - \frac{1}{n}\right) \sim \frac{1}{n}.$$

It should be observed that, on a major arc, $|Y|$ cannot exceed a fixed upper bound (and is in general small); for $\frac{q^k}{n} \leqq 1$ and

$$|q^k \theta| = O\left(\frac{q^k}{q n^{1-a}}\right) = O\left(\frac{q^k}{n}\right)^{1-a} = O(1).$$

Lemma 5. *If*

$$\varphi(m) = \int_0^\infty e^{-Yu^k} \cos 2m\pi u \, du,$$

$$\chi(m) = \int_0^\infty e^{-Yu^k} \sin 2m\pi u \, du,$$

then

$$(7.11) \quad f(x) = 2 S_{p,q}\, \varphi(0) + 4 \sum_{m=1}^{\infty} S_{p,q,m}\, \varphi(m) + 4 \sum_{m=1}^{\infty} S'_{p,q,m}\, \chi(m).$$

We have [10])

$$(7.12) \quad \sum_{l=0}^{\infty} \varepsilon_{l+j}\, e^{-Y(l+j)^k} = 2 \sum_{m=0}^{\infty} \varepsilon_m \int_0^\infty e^{-Yu^k} \cos 2m\pi(u-j)\, du,$$

---

10) This formula may be proved, by classical methods, in the following way. The function

$$f(j) = \sum_{l=-\infty}^{\infty} \varepsilon_{l+j}\, e^{-Y(l+j)^k},$$

where $\varepsilon_x = 0$ for $x < 0$, has the period 1; and it may be verified at once that it has a derivative $f'(j)$ bounded for $0 < j < 1$ and that $f(0) = \frac{1}{2}(f(-0) + f(+0))$. Hence it is expressible in a Fourier's series $\sum_{m=0}^{\infty} 2\varepsilon_m \gamma_m$, where

$$\gamma_m = \int_0^1 \cos 2m\pi(u-j) f(u)\, du = \sum_{l=-\infty}^{\infty} \int_0^1 \cos 2m\pi(l+u-j)\, \varepsilon_{l+u}\, e^{-Y(l+u)^k} du$$

$$= \int_{-\infty}^{\infty} \varepsilon_u\, e^{-Yu^k} \cos 2m\pi(u-j)\, du = \int_0^\infty e^{-Yu^k} \cos 2m\pi(u-j)\, du.$$

where $\Re(Y) > 0$, $1 > j \geqq 0$, and

$$\varepsilon_x = 1 \quad (x > 0), \qquad \varepsilon_0 = \tfrac{1}{2}.$$

Also

$$(7.13) \quad f(x) = 1 + 2\sum_{n=1}^{\infty} x^{n^k} = 2\sum_{h=0}^{q-1} e_q(ph^k) \sum_{l=0}^{\infty} \varepsilon_{lq+h} X^{(lq+h)^k}$$

$$= 2\sum_{h=0}^{q-1} e_q(ph^k) f_h(X),$$

say. But

$$(7.14) \qquad f_h(X) = \sum_{l=0}^{\infty} \varepsilon_{l+j}\, e^{-Y(l+j)^k} \qquad \left(j = \frac{h}{q}\right).$$

Substituting from (7.12) and (7.14) into (7.13), and changing the order of summation, we obtain the result of the lemma. It may be written [11])

$$(7.15) \quad f(x) = 2\Gamma(1+a) S_{p,q} Y^{-a} + 4\sum_{m=1}^{\infty} S_{p,q,m}\,\varphi(m) + 4\sum_{m=1}^{\infty} S'_{p,q,m}\,\chi(m),$$

when Y^{-a} has its principal value, viz. $e^{-a\log Y}$, the imaginary part of $\log Y$ lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

Asymptotic formulae for $\varphi(m)$ and $\chi(m)$.

7. 2. **Lemma 6.** *If x lies on a major arc, so that $|Y|$ is not large,*

$$\int_0^{\infty} \exp\left(-(Yu^k \mp 2m\pi i u)\right) du = \pm \frac{i}{2m\pi} + O(|Y| m^{-k-1})$$
$$+ O\left(m^{-\frac{1}{2}(1-b)} |Y|^{-\frac{1}{2}b} E\right),$$

where

$$E = \exp(-A m^{1+b} |Y|^{-b} \cos\psi), \qquad \psi = \arg Y.$$

It is enough to consider the integral in which the ambiguous sign is negative; the other assertion follows by changing ψ into $-\psi$. Let

$$u = \frac{kZ}{2m\pi} v, \qquad Z = \left(\frac{2m\pi}{k}\right)^{1+b} Y^{-b},$$

so that

$$-(Yu^k - 2m\pi i u) = -Z(v^k - kiv).$$

11) In our paper in the Quarterly Journal the term in $S'_{p,q,m}$ was omitted in error in the case when k is odd: $S'_{p,q,m} = 0$ when k is even. The omission affects none of the results of the paper.

When Y is real (and positive) we have

$$(7.21) \qquad \int_0^\infty \exp\left(-(Yu^k - 2m\pi i u)\right) du = \frac{kZ}{2m\pi}\int_0^\infty \exp\left(-Z(v^k - kiv)\right) dv.$$

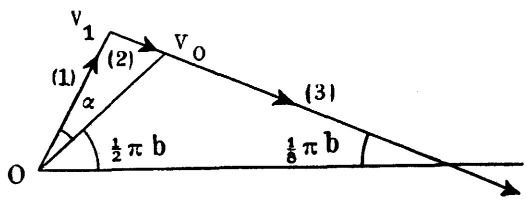
The right hand side is an analytic function of Z when $\Re(Z) > 0$; the equation (7.21) therefore holds for $\Re(Y) > 0$.

Let

$$v_0 = e^{\frac{1}{2}\pi i b}, \quad v_1 = v_0 - c e^{-\frac{1}{8}\pi i b} = \xi v_0 e^{i\alpha},$$

where c and ξ are positive. We can choose A_1 so that $|v_1| \leqq 1$ and $0 < \alpha \leqq \frac{\pi}{4k}$ when $c < A_1$. We shall ultimately choose c to be a number depending only on k and satisfying this condition. It may be observed, in order to avoid any possible misunderstanding, that α, and all geometrical elements of the figure adjoined, are functions of k only.

Since $|\arg Z| = |-b\psi| < \frac{1}{2}\pi b$, the path of integration may be changed in the right hand side of (7.21) from the real axis to



(1) + (2) + (3), when (1) is the straight line from 0 to v_1, (2) the straight line from v_1 to v_0, and (3) the continuation of this last line to infinity [12]. We write then

$$(7.22) \qquad \int_0^\infty \exp\left(-Z(v^k - kiv)\right) dv = I_1 + I_2 + I_3.$$

Consider first I_1. We have, when v lies on (1),

$$v = x v_0 e^{i\alpha}, \quad 0 \leqq x \leqq \xi \leqq 1,$$

$$\begin{aligned} -Z(v^k - kiv) &= -Ziv_0\left(x^k e^{ki\alpha} - kxe^{i\alpha}\right)^{13)} \\ &= |Z| e^{-(\frac{1}{2}\pi - \frac{1}{2}\pi b + b\psi)i}\left(x^k e^{ki\alpha} - kxe^{i\alpha}\right) \\ &= |Z| x\left(x^{k-1} e^{(k\alpha - \Psi)i} - k e^{(\alpha - \Psi)i}\right), \end{aligned}$$

12) We have $|-b\psi + \frac{1}{8}kb\pi| < \frac{7}{16}\pi$ for $k \geqq 3$.
13) Since $v_0^{k-1} = i$.

where

$$\Psi = \tfrac{1}{2}\pi(1-b)+b\psi.$$

Hence

$$\begin{aligned}\Re(-Z(v^k-kiv)) &= |Z|x(x^{k-1}\cos(k\alpha-\Psi)-k\cos(\alpha-\Psi))\\ &= |Z|x((x^{k-1}\cos k\alpha-k\cos\alpha)\cos\Psi+(x^{k-1}\sin k\alpha-k\sin\alpha)\sin\Psi).\end{aligned}$$

Now $0<\Psi<\dfrac{1}{2}\pi$ (since $k\geqq 3$), $0<\alpha\leqq\dfrac{\pi}{4k}$. Hence

$$\begin{aligned}\Re(-Z(v^k-kiv)) &\leqq |Z|x((1-k\cos\alpha)\cos\Psi+(\sin k\alpha-k\sin\alpha)\sin\Psi)\\ &\leqq |Z|x(-A'\cos\Psi-A''\sin\Psi)\leqq -A|Z|x.\end{aligned}$$

Therefore

$$\begin{aligned}I_1 &= \int_0^{v_1}\exp(-Z(v^k-kiv))\,dv\\ &= \left[\frac{1}{kiZ}\exp(-Z(v^k-kiv))\right]_0^{v_1}+\frac{1}{i}\int_0^{v_1}v^{k-1}\exp(-Z(v^k-kiv))\,dv\\ &= \frac{i}{kZ}+O\left(\frac{1}{|Z|}e^{-A\xi|Z|}\right)+O\left(\int_0^{\xi}x^{k-1}e^{-Ax|Z|}dx\right),\end{aligned}$$

$$I_1 = \frac{i}{kZ}+O\left(\frac{1}{|Z|}e^{-A|Z|}\right)+O(|Z|^{-k}). \tag{7.23}$$

When v lies on (2), we have $v=v_0-ye^{-\frac{1}{8}\pi ib}$, where $0\leqq y\leqq c$, and

$$-Z(v^k-kiv) = -Z\left((v_0^k-kiv_0)+\binom{k}{2}e^{-\frac{1}{4}\pi ib}v_0^{k-2}y^2-\binom{k}{3}e^{-\frac{3}{8}\pi ib}v_0^{k-3}y^3+\cdots\right),$$

the term in y vanishing since $v_0^{k-1}=i$. This is

$$-Z(v_0^k-kiv_0)-A|Z|\exp\left(\left(\tfrac{1}{2}\pi(1-b)-\tfrac{1}{4}\pi b-b\psi\right)i\right).y^2(1+O(y)).$$

Now

$$\tfrac{1}{2}\pi-\tfrac{1}{4}\pi b\geqq\tfrac{1}{2}\pi(1-b)-\tfrac{1}{4}\pi b-b\psi\geqq-\tfrac{1}{4}\pi b \quad (k\geqq 3).$$

Hence

$$\begin{aligned}\Re(-Z(v^k-kiv)) &\leqq \Re(-Z(v_0^k-kiv_0))-A|Z|y^2(1+O(y))\\ &\leqq \Re(-Z(v_0^k-kiv_0))-A|Z|y^2\end{aligned} \tag{7.24}$$

for $0\leqq y<A_2$. We choose $c=\frac{1}{2}\operatorname{Min}(A_1,A_2)$; (7.24) then holds when v is on (2), and

$$\begin{aligned}I_2 &= \int_{v_1}^{v_0}\exp(-Z(v^k-kiv))\,dv = O\left(|\exp(-Z(v_0^k-kiv_0))|\int_0^c e^{-A|Z|y^2}dy\right)\\ &= O\left(\left|Z^{-\frac{1}{2}}\exp(-Z(v_0^k-kiv_0))\right|\right).\end{aligned} \tag{7.25}$$

Finally, on (3) we have $v = v_0 + y e^{-\frac{1}{8}\pi i b}$, where $y \geqq 0$, and

$$-Z(v^k - kiv) = -Z(v_0^k - kiv_0) - Z \sum_{r=2}^{k} \binom{k}{r} e^{-\frac{1}{8} r b \pi i} v_0^{k-r} y^r.$$

It is easily verified that

$$\left|\arg\left(Z e^{-\frac{1}{8} r b \pi i} v_0^{k-r}\right)\right| < \tfrac{1}{2}\pi$$

for $k \geqq 3$, $3 \leqq r \leqq k$. Hence

$$\Re(-Z(v^k - kiv)) \leqq \Re\left(-Z(v_0^k - kiv_0) - A Z e^{-\frac{1}{4} b \pi i} v_0^{k-2} y^2\right).$$

Thus

$$\text{(7.26)} \qquad I_3 = e^{-\frac{1}{8} b \pi i} \int_0^\infty \exp(-Z(v^k - kiv))\, dy$$

$$= O\left(|\exp(-Z(v_0^k - kiv_0))| \int_0^\infty e^{-A|Z|y^2}\, dy\right)$$

$$= O\left(\left|Z^{-\frac{1}{2}} \exp(-Z(v_0^k - kiv_0))\right|\right),$$

as before. Now

$$-Z(v_0^k - kiv_0) = (k-1) Z i v_0 = -(k-1)|Z| e^{-i\Psi},$$

so that

$$\text{(7.27)} \qquad \Re(-Z(v_0^k - kiv_0)) = -A|Z| \cos \Psi.$$

From (7.25), (7.26) and (7.27) it follows that

$$\text{(7.28)} \qquad I_2 + I_3 = O\left(|Z|^{-\frac{1}{2}} \exp(-A|Z| \cos \Psi)\right);$$

and from (7.21), (7.22), (7.23) and (7.28) that

$$\int_0^\infty \exp(-(Yu^k - 2m\pi i u))\, du = \frac{kZ}{2m\pi}(I_1 + I_2 + I_3)$$

$$= \frac{kZ}{2m\pi}\left(\frac{i}{kZ} + O\left(\frac{1}{|Z|} e^{-A|Z|}\right) + O(|Z|^{-k}) + O\left(|Z|^{-\frac{1}{2}} \exp(-A|Z| \cos \Psi)\right)\right)$$

$$= \frac{i}{2m\pi} + O\left(\frac{|Z|^{1-k}}{m}\right) + O\left(\frac{|Z|^{\frac{1}{2}}}{m} \exp(-A|Z| \cos \Psi)\right).$$

Substituting for Z in terms of Y, and observing that the ratio $\cos \Psi : \cos \psi$ remains superior to a positive bound for all values of ψ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, we obtain the result of the lemma.

7. 3. Combining the results of Lemma 6, we see that if x lies on a major arc then

(7.31) $\varphi(m) = O\left(m^{-\frac{1}{2}(1-b)}|Y|^{-\frac{1}{2}b}E\right) + O(m^{-k-1}|Y|)$,

(7.32) $\chi(m) = \dfrac{1}{2m\pi} + \chi_1(m)$,

(7.33) $\chi_1(m) = O\left(m^{-\frac{1}{2}(1-b)}|Y|^{-\frac{1}{2}br}E\right) + O(m^{-k-1}|Y|)$.

Behaviour of $f(x)$ on a major arc (*continued*).

8. 1. We have, from (7.15) and (7.32),

$$f(x) = \varphi_{p,q} + \Phi_{p,q},$$

when

(8.11) $\varphi_{p,q} = 2\Gamma(1+a)S_{p,q}Y^{-a} = 2\Gamma(1+a)\dfrac{S_{p,q}}{q}\left(\log\dfrac{1}{X}\right)^{-a}$

and

$$\Phi_{p,q} = 4\sum_{m=1}^{\infty} S_{p,q,m}\varphi(m) + 4\sum_{m=1}^{\infty} S'_{p,q,m}\chi_1(m) + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}S'_{p,q,m}$$

$$= O\left(q^{\varkappa+\varepsilon}\sum_{m=1}^{\infty}(|\varphi(m)| + |\chi_1(m)|)\right) + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}S'_{p,q,m}.$$

By (7.31) and (7.33), this is

$$O\left(q^{\varkappa+\varepsilon}|Y|^{-\frac{1}{2}b}\sum_{m=1}^{\infty} m^{-\frac{1}{2}(1-b)}E\right) + O\left(q^{\varkappa+\varepsilon}|Y|\sum_{m=1}^{\infty} m^{-k-1}\right)$$

$$+ \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}S'_{p,q,m} = \Phi_1 + \Phi_2 + \Phi_3,$$

say. Now, if $q > 1$,

$$\sum_{m=M+1}^{\infty}\frac{1}{m}S'_{p,q,m} = \sum_{h=0}^{q-1} e_q(h^k p)\sum_{m=M+1}^{\infty}\frac{1}{m}\sin\frac{2mh\pi}{q}$$

$$= \sum_{h=1}^{q-1} e_q(h^k p)\, O\left(\frac{1}{M\sin\dfrac{\pi}{q}}\right) = O\left(\frac{q^2}{M}\right);$$

if $q = 1$, the sum has the value 0. Hence

(8.12) $\Phi_3 = \dfrac{2}{\pi}\displaystyle\sum_{m=1}^{q^2}\frac{1}{m}S'_{p,q,m} + O(1) = O\left(q^{\varkappa+\varepsilon}\sum_{m=1}^{q^2}\frac{1}{m}\right) + O(1)$

$$= O\left(q^{\varkappa+\varepsilon}\right).$$

Clearly also

(8.13) $\Phi_2 = O\left(q^{\varkappa+\varepsilon}\right)$.

8. 2. There remains Φ_1 to be considered. We have

$$Y = q^k(\nu - i\theta) = |Y| e^{i\psi},$$

so that

$$|Y| = q^k \nu \sec\psi \sim \frac{q^k}{n\cos\psi}.$$

Hence

(8. 21) $$E = O(\exp(-A m^{1+b} q^{-1-b} n^b (\cos\psi)^{1+b})),$$

and it follows from the definition of Φ_1 that

$$\Phi_1 = O\left(q^{\varkappa+\varepsilon} |Y|^{-\frac{1}{2}b} \sum_{m=1}^{\infty} m^{-\frac{1}{2}(1-b)} \exp(-A m^{1+b} q^{-1-b} n^b (\cos\psi)^{1+b})\right).$$

We have $0 \leqq \theta^2 < A q^{-2} n^{-2+2a}$, and so

$$\cos^2\psi = \frac{\nu^2}{\nu^2+\theta^2} > \frac{\nu^2}{\nu^2 + A q^{-2} n^{-2+2a}} > \frac{1}{1 + A q^{-2} n^{2a}}$$

$$\geqq \frac{1}{(1+A) q^{-2} n^{2a}} = A q^2 n^{-2a},$$

$$q^{-1-b} n^b (\cos\psi)^{1+b} > A q^{-1-b} n^b (q n^{-a})^{1+b} = A.$$

Hence, as $|Y| > A q^k n^{-1}$, we have

(8. 22) $$\Phi_1 = O\left\{q^{\varkappa+\varepsilon}\left(\frac{q^k}{n}\right)^{-\frac{1}{2}b} \sum_1^{\infty} e^{-A m^{1+b}}\right\} = O\left(n^{\frac{1}{2}b} q^{\varkappa+\varepsilon-\frac{1}{2}kb}\right).$$

From (8. 12), (8. 13) and (8. 22) we have

(8. 23) $$\Phi_{p,q} = O\left(q^{\varkappa+\varepsilon}\right) + O\left(n^{\frac{1}{2}b} q^{\varkappa+\varepsilon-\frac{1}{2}kb}\right) = O\left(n^{a\varkappa+\varepsilon}\right)$$

since $q^k \leqq n$.

8. 3. We may summarize our conclusions, as regards the behaviour of $f(x)$, as follows: *we have*

$$f(x) = O\left(n^{a\varkappa+\varepsilon}\right)$$

on a minor arc, and

$$f(x) = \varphi_{p,q} + O\left(n^{a\varkappa+\varepsilon}\right)$$

on a major arc, $\varphi_{p,q}$ *being defined by* (8. 11).

Proof of Theorem B.

9. 1. Returning now to the integral (2. 21), we have

$$r_{k,s}(n) = \frac{1}{2\pi i}\int_\Gamma \frac{(f(x))^s}{x^{n+1}}\,dx = \sum \frac{1}{2\pi i}\int_{\xi_{p,q}} \frac{(f(x))^s}{x^{n+1}}\,dx,$$

the summation extending over all arcs $\xi_{p,q}$ of the Farey dissection of order $N = [n^{1-a}]$. We write this in the form

$$(9.11)\quad r_{k,s}(n) = \sum \frac{1}{2\pi i}\int_{\mathfrak{M}} \frac{(f(x))^s}{x^{n+1}}\,dx + \sum \frac{1}{2\pi i}\int_{\mathfrak{m}} \frac{(f(x))^s}{x^{n+1}}\,dx$$
$$= S_1 + S_2,$$

$\mathfrak{M}$ and $\mathfrak{m}$ denoting typical major and minor arcs of the dissection. It follows at once from Lemma 3 that

$$(9.12)\qquad S_2 = O\left(n^{sa\varkappa+\varepsilon}\right)\text{ [14]}.$$

As regards S_1 we observe that, on an arc $\mathfrak{M}$,

$$f^s = (\varphi_{p,q} + \Phi_{p,q})^s = (\varphi + \Phi)^s = \varphi^s + O(\mathrm{Max}\,(|\varphi|^{s-1}\,|\Phi|,\,|\Phi|^s)).$$

Hence

$$\frac{1}{2\pi i}\int_{\mathfrak{M}} \frac{f^s}{x^{n+1}}\,dx = \frac{1}{2\pi i}\int_{\mathfrak{M}} \frac{\varphi^s}{x^{n+1}}\,dx + J_{p,q} = I_{p,q} + J_{p,q},$$

where

$$(9.13)\quad J_{p,q} = O\left(\int_{\mathfrak{M}} \frac{|\varphi|^{s-1}\,|\Phi|}{|x|^{n+1}}\,|dx|\right) + O\left(\int_{\mathfrak{M}} \frac{|\Phi|^s}{|x|^{n+1}}\,|dx|\right)$$
$$= O(\varrho_{p,q}) + O(\sigma_{p,q}),$$

say. It is plain that

$$(9.14)\qquad \sum \sigma_{p,q} = O\left(n^{sa\varkappa+\varepsilon}\right).$$

Also

$$\varphi = O(|S_{p,q}|\,|Y|^{-a}) = O\left(q^{\varkappa+\varepsilon}(q^k|\nu - i\theta|)^{-a}\right)$$
$$= O\left(q^{\varkappa-1+\varepsilon}(\nu^2+\theta^2)^{-\frac{1}{2}a}\right).$$

From this equation and (8.23) we deduce

$$\varrho_{p,q} = O\left(n^{a\varkappa+\varepsilon} q^{(s-1)(\varkappa-1)+\varepsilon}\int_{-\infty}^{\infty} \frac{d\theta}{(\nu^2+\theta^2)^{\frac{1}{2}a(s-1)}}\right)$$
$$= O\left(n^{a\varkappa+as-a-1+\varepsilon} q^{(s-1)(\varkappa-1)+\varepsilon}\right),$$

provided $s > k+1$. Now the series $\sum\limits_{q=1}^{\infty}\sum\limits_{p=0}^{q-1} q^{(s-1)(\varkappa-1)+\varepsilon}$ is convergent if $(s-1)(1-\varkappa) > 2$ or $s > 2K+1$, and supposing, as

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14) This formula is not true uniformly in s, and may therefore be held to violate our general principle (§ 2.2) as to the use of O and o. But the formula falls into line with the general principle so soon as we fix our attention on any particular $s = s(k)$; and this is sufficient for our argument.

we shall do henceforth, that this condition is satisfied, we have

$$\sum \varrho_{p,q} = O\left(n^{a\varkappa + as - a - 1 + \varepsilon}\right)^{15)};$$

and so, by (9.13) and (9.14),

$$\sum J_{p,q} = O\left(n^{sa\varkappa + \varepsilon}\right) + O\left(n^{a\varkappa + as - a - 1 + \varepsilon}\right),$$

$$(9.15)\quad r_{k,s}(n) = S_1 + S_2 = \sum I_{p,q} + O\left(n^{sa\varkappa + \varepsilon}\right) + O\left(n^{a\varkappa' + as - a - 1 + \varepsilon}\right).$$

9. 2. We now perform two transformations on $I_{p,q}$.

(i) The functions

$$\varphi^s_{p,q} = (2\Gamma(1+a))^s \left(\frac{S_{p,q}}{q}\right)^s \left(\log\frac{1}{X}\right)^{-sa}$$

and

$$F_{p,q} = \frac{(2\Gamma(1+a))^s}{\Gamma(sa)} \left(\frac{S_{p,q}}{q}\right)^s \sum_{n=1}^{\infty} n^{sa-1} X^n = C\left(\frac{S_{p,q}}{q}\right)^s F_{sa}(X)$$

differ by a function regular at $X = 1$ and *a fortiori* bounded near $X = 1$. Further, it is plain that

$$(2\Gamma(1+a))^s \left(\log\frac{1}{X}\right)^{-sa} - CF_{sa}(X) = O(1)$$

uniformly in p and q. Hence, if we replace $\varphi^s_{p,q}$, in $I_{p,q}$, by $F_{p,q}$, we introduce an error

$$O\left(\sum\left(\frac{S_{p,q}}{q}\right)^s\right) = O\left(\sum q^{-s(1-\varkappa)+\varepsilon}\right) = O(1),$$

provided only $s(1-\varkappa) > 2$, $s > 2K$, a condition which we have already assumed to be satisfied. This error is plainly trivial in comparison with the error terms already present in (9.15).

(ii) We next replace the arc $\xi_{p,q}$, in $I_{p,q}$, by the complete circle Γ. The error thus introduced is

$$\frac{1}{2\pi i} C \sum\left(\frac{S_{p,q}}{q}\right)^s \int_{\eta_{p,q}} \frac{F_{sa}(X)}{x^{n+1}}\, dx = O\left(\sum q^{-s(1-\varkappa)+\varepsilon} \int_{\eta_{p,q}} |F_{sa}(X)|\,|dx|\right).$$

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15) It will be useful later (though irrelevant for the purposes of this memoir) to point out that this formula is still valid when $s = 2K+1$, since then

$$\sum_{q=1}^{n^a} \sum_{p=0}^{q-1}{}' q^{(s-1)(\varkappa-1)+\varepsilon} = O\left(\sum_{q=1}^{n^a} q^{-1+\varepsilon}\right) = O(n^\varepsilon).$$

The interest of this lies in the fact that $2K+1 = 9$ when $k = 3$.

But, if $X = Re^{i\theta}$, we have, uniformly in p and q,

$$F_{sa}(X) = O(|1-X|^{-a}) = O\left(\left((1-R)^2+4R\sin^2\frac{1}{2}\theta\right)^{-\frac{1}{2}sa}\right)$$
$$= O\left(\left(\frac{1}{n^2}+\theta^2\right)^{-\frac{1}{2}sa}\right).$$

If θ_0 is the angular coordinate of one end of $\xi_{p,q}$, $n\theta_0 > A\dfrac{n^a}{q} > A$. Hence our integral is the sum of two, each of which is of the form

$$O\left(\int_{\theta_0}^{\infty}\left(\frac{1}{n^2}+\theta^2\right)^{-\frac{1}{2}sa}d\theta\right) = O\left(n^{sa-1}\int_{n\theta_0}^{\infty}(1+w^2)^{-\frac{1}{2}sa}dw\right)$$
$$= O(n^{sa-1}(n\theta_0)^{-sa+1}) = O\left(n^{sa-1}\left(\frac{n^a}{q}\right)^{-sa+1}\right).$$

Hence the total error introduced is

$$O\left(n^{(sa-1)(1-a)}\sum_{p,q} q^{sa-1-s(1-\varkappa)+\varepsilon}\right)$$
$$= O\left(n^{(sa-1)(1-a)}\sum_{q\leqq n^a} q^{sa-s(1-\varkappa)+\varepsilon}\right) = O\left(n^{sa\varkappa+2a-1+\varepsilon}\right)$$
$$= O\left(n^{sa\varkappa+\varepsilon}\right),$$

since $k>2$, $a<\frac{1}{2}$. We have thus

$$(9.21)\qquad r_{k,s}(n) = \frac{C}{2\pi i}\sum\left(\frac{S_{p,q}}{q}\right)^s\int_{\Gamma}\frac{F_{sa}(X)}{x^{n+1}}dx + O\left(n^{sa\varkappa+\varepsilon}\right)$$
$$+ O\left(n^{a\varkappa+as-a-1+\varepsilon}\right).$$

9. 3. There is now no difficulty in completing the proof. In the first place

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{F_{sa}(X)}{x^{n+1}}dx = n^{sa-1}e_q(-np),$$

so that the first term on the right hand side of (9.21) is

$$Cn^{sa-1}\sum\left(\frac{S_{p,q}}{q}\right)^s e_q(-np) = Cn^{sa-1}S.$$

The series may be regarded as limited by $q\leqq n^a$, or as extended to infinity, indifferently; for the difference is

$$O\left(n^{sa-1}\sum_{q>n^a} q^{-s(1-\varkappa)+1+\varepsilon}\right) = O\left(n^{sa-1}(n^a)^{-s(1-\varkappa)+2+\varepsilon}\right)$$
$$= O\left(n^{sa\varkappa+2a-1+\varepsilon}\right) = O\left(n^{sa\varkappa+\varepsilon}\right).$$

If s is sufficiently large, $S > \frac{1}{2}$, by Lemma 2, so that the leading term in (9.21) is effectively of order n^{sa-1}. The third term is always of lower order, since $a\varkappa - a < 0$. Finally, the second is of lower order if

$$sa\varkappa < sa - 1, \quad sa(1-\varkappa) > 1,$$

or

$$s > kK = k2^{k-1}.$$

10. We have thus proved Theorem B. It is plain that Theorem C is an immediate corollary. To deduce Theorem A we have only to take $g = \operatorname{Max}(G, n_0)$, since any number less than n_0 is the sum of at most n_0 positive k-th powers.

We have not, in this memoir, determined an explicit upper bound for any of the numbers g, G_1, G. The whole trend of our analysis is, however, to suggest that the necessary value of $G(k)$ does not exceed

$$k2^{k-1} + 1;$$

so that, for example, every large number is the sum of at most 33 biquadrates, 81 fifth powers, or 193 sixth powers [16]). It is, in fact, possible to prove more than this; but the proofs involve a detailed examination of the singular series which we must postpone to our later memoirs. Our second memoir, which will be concerned particularly with the case $k = 4$ (a case in some ways the most delicate and interesting of all), and in which (going beyond the results which we have indicated) we shall prove that *every large number is the sum of at most 21 biquadrates*, will, we hope, be published shortly in the *Mathematische Zeitschrift*.

16) This result is new for $k = 4$ and $k \geqq 6$, but not for $k = 5$. The best result we can prove for $k = 5$ is $G(5) \leqq 53$, which is new.

Some problems of „Partitio numerorum": II. Proof that every large number is the sum of at most 21 biquadrates.

By

G. H. Hardy in Oxford and J. E. Littlewood in Cambridge.

1. Introduction.

1.1. This memoir is essentially a sequel to one which we published recently in the Göttinger Nachrichten[1]). It could not in any case be intelligible to a reader unacquainted with our earlier memoir; and we shall therefore quote formulae from the latter without further explanation.

In the memoir referred to we laid the foundations of our new method for the solution of Waring's Problem, carrying our analysis just so far as was necessary for the proof of Hilbert's Theorem, the fundamental existence theorem for the numbers $g(k)$ and $G(k)$. Here our object is to find the best possible inequality for the particular number $G(4)$. A good deal of our analysis, however, is valid for a general k, and will be useful to us when we proceed to the corresponding general problem. It will be found that the special interest of the case $k = 4$ is quite sufficient to justify its consideration in a separate memoir.

1.2. It is known that

$$19 \leqq g(4) \leqq 37, \qquad 16 \leqq G(4) \leqq 37,$$

these inequalities, from left to right, being due to Waring, Wieferich, Kempner, and Wieferich respectively. For detailed references we may refer to the dissertations of Kempner[2]) and of Baer[3]). We need men-

[1]) G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio numerorum'; I: A new solution of Waring's Problem, Göttinger Nachrichten 1920, S. 33–54. We shall refer to this memoir as W. P.

[2]) A. J. Kempner, Über das Waringsche Problem und einige Verallgemeinerungen, Inaugural-Dissertation, Göttingen 1912.

[3]) W. S. Baer, Beiträge zum Waringschen Problem, Inaugural-Dissertation, Göttingen 1913.

tion only that the deepest result, viz. $g(4) \leqq 37$, was obtained in 1909 by Wieferich, whose analysis is a refinement upon that by which Landau, in 1907, had proved that $g(4) \leqq 38$. Here we shall prove nothing concerning $g(4)$; but we shall improve the upper bound for $G(4)$ very notably, by proving

Theorem A: $G(4) \leqq 21$.

2. A sharpening of our earlier analysis.

2.1. In § 9.2. of W. P. we proved that, assuming always

$$s \geqq 2K + 1 = 2^k + 1,$$

$$(2.11) \qquad \begin{aligned} r_{k,s}(n) &= C n^{sa-1} S + O(n^{sa\varkappa+\varepsilon}) + O(n^{sa+a\varkappa-a-1+\varepsilon}) \\ &= \varrho_{k,s}(n) + O(n^{sa\varkappa+\varepsilon}) + O(n^{sa+a\varkappa-a-1+\varepsilon}), \end{aligned}$$

where

$$S = \sum \left(\frac{S_{p,q}}{q}\right)^s e_q(-np).$$

It will be necessary now to replace the term $O(n^{sa\varkappa+\varepsilon})$ by a term of lower order[4]).

2.2. It will be found, on an examination of the analysis of W. P., that the critical error term $O(n^{sa\varkappa+\varepsilon})$ arises in two places only. All other errors are of lower order than that of the dominant factor n^{sa-1}, either independently of the value of s, or at any rate when $s \geqq 2K + 1$. The two critical errors arise as follows.

In the first place we have

$$S_2 = \sum \frac{1}{2\pi i} \int_{\mathfrak{m}} \frac{(f(x))^s}{x^{n+1}} dx = O(n^{sa\varkappa+\varepsilon}),$$

where $\mathfrak{m}$ is a typical minor arc of the Farey dissection.

Secondly, when we consider the corresponding sum connected with the major arcs, we are confronted by a sum $\sum \sigma_{p,q}$, where

$$\sigma_{p,q} = \int_{\mathfrak{M}} \frac{|\Phi|^s}{|x|^{n+1}} |dx|$$

and $\mathfrak{M}$ is a typical major arc of the dissection, and we write

$$\sum \sigma_{p,q} = O(n^{sa\varkappa+\varepsilon}).$$

It will be observed that these two errors arise in exactly the same way. The upper bounds are obtained by substituting in the integrals the crude approximations, $f = O(n^{a\varkappa+\varepsilon})$ on a minor arc and $\Phi = O(n^{a\varkappa+\varepsilon})$

[4]) The formula (2.11) would lead only to $G(4) \leqq 33$, in itself a new result.

on a major arc. Here we refine on our previous argument by the use of a single new idea. This idea consists in an appropriate use of a known result, viz. that the number of positive integral solutions of the equation $x^k + y^k = n$ is $O(n^\varepsilon)$ for every $k > 1$ [5]), or, as we may express it in our notation,

$$r_{k,2}(n) = O(n^\varepsilon).$$

2.3. We have

$$\sum_{\nu=0}^{n} r_{k,2}(\nu) = O\Big(\iint\limits_{|x|^k+|y|^k \leq n} dx\,dy\Big) = O(n^{2a}).$$

Hence

$$\sum \int_{\mathfrak{m}} |f(x)|^4 |dx| \leqq \int_{\Gamma} |f(x)|^4 |dx| \leqq \int_0^{2\pi} |f(Re^{i\theta})|^4 d\theta$$

$$= O\Big(\sum (r_{k,2}(\nu))^2 R^{2\nu}\Big).$$

Now

$$\sum_0^n (r_{k,2}(\nu))^2 = O\Big(\sum_0^n r_{k,2}(\nu) \times \operatorname*{Max}_{\nu \leq n} r_{k,2}(\nu)\Big) = O(n^{2a+\varepsilon});$$

and so, since $R = 1 - \dfrac{1}{n}$,

$$\sum \int_{\mathfrak{m}} |f(x)|^4 |dx| = O(n^{2a+\varepsilon}).$$

Hence

$$S_2 = O\Big(\sum \int_{\mathfrak{m}} |f(x)|^4 |f(x)|^{s-4} |dx|\Big)$$

$$= O\Big(n^{(s-4)a\varkappa+\varepsilon} \sum \int_{\mathfrak{m}} |f(x)|^4 |dx|\Big) = O(n^{(s-4)a\varkappa+2a+\varepsilon}).$$

Again, we have

$$\sum \int_{\mathfrak{M}} |\Phi|^4 |dx| = \sum \int_{\mathfrak{M}} |f-\varphi|^4 |dx| = O\Big(\sum \int_{\mathfrak{M}} |f|^4 |dx|\Big) + O\Big(\sum \int_{\mathfrak{M}} |\varphi|^4 dx\Big)$$

$$= O(n^{2a+\varepsilon}) + O\Big(\sum \int_{\mathfrak{M}} |\varphi|^4 |dx|\Big),$$

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[5]) For a formal proof of this result see D. Cauer, Neue Anwendungen der Pfeifferschen Methode zur Abschätzung zahlentheoretischer Funktionen, Inaugural-Dissertation, Göttingen 1914, S. 38. For $k = 2$ (when the result includes *a fortiori* the corresponding results for 4, 6, ...) see E. Landau, Über die Anzahl der Gitterpunkte in gewissen Bereichen, Göttinger Nachrichten, 1912, S. 750.

$$\sum_{\mathfrak{M}} \int |\varphi|^4 |dx| = O\Big(\sum_{q=1}^{n^a} \sum_{p} \int_{-\theta_0}^{\theta_0} \Big|\frac{S_{p,q}}{q}\Big|^4 |\nu - i\theta|^{-4a} d\theta\Big)$$

$$= O\Big(\sum_{q} \sum_{p} q^{-4(1-\varkappa)+\varepsilon} \nu^{-4a} \theta_0\Big) = O\Big(\sum_{q} q \cdot q^{-4(1-\varkappa)+\varepsilon} . n^{4a} . q^{-1} n^{a-1}\Big)$$

$$= \begin{cases} O\Big(n^{5a-1+\varepsilon} \sum_{q} 1\Big) = O(n^{6a-1+\varepsilon}) = O(n^{2a+\varepsilon}) & (k \geqq 4) \\ O\Big(n^{5a-1} \sum_{q} q^{-1+\varepsilon}\Big) = O(n^{5a-1+\varepsilon}) = O(n^{2a+\varepsilon}) & (k = 3). \end{cases}$$

Thus

$$\sum_{\mathfrak{M}} \int |\Phi|^4 |dx| = O(n^{2a+\varepsilon}) \qquad (k > 2),$$

$$\sum \sigma_{p,q} = \sum_{\mathfrak{M}} \int \frac{|\Phi|^s}{|x|^{n+1}} |dx| = O\Big(\sum_{\mathfrak{M}} \int |\Phi|^s |dx|\Big)$$

$$= O\Big(\sum_{\mathfrak{M}} \int |\Phi|^{s-4} |\Phi|^4 |dx|\Big)$$

$$= O\Big(n^{(s-4)a\varkappa+\varepsilon} \sum_{\mathfrak{M}} \int |\Phi|^4 |dx|\Big) = O(n^{(s-4)a\varkappa+2a+\varepsilon}).$$

2. 4. The argument of W. P. showed that

$$r_{k,s}(n) = C n^{sa-1} S + O(n^{\lambda}),$$

where $\lambda < sa - 1$ if $sa\varkappa < sa - 1$, i. e. if $s > kK = k2^{k-1}$. It is now clear that this result holds if only

$$(s-4)a\varkappa + 2a < sa - 1,$$

i. e. if

$$s > (k-2)K + 4.$$

For $k = 4$ these inequalities reduce to $s > 32$ and $s > 20$ respectively, so that the improvement is very substantial. And if only we can establish the existence of a positive constant σ such that

$$|S| > \sigma$$

when $s \geqq 21$, we shall have proved not only Theorem A but the more precise theorem

Theorem B: $r_{4,s}(n) \sim C n^{\frac{1}{4}s-1} S \qquad (s \geqq 21)$.

The proof of this theorem is in fact reduced to a discussion of the singular series S.

3. Factorization of the singular series.

3.1. The discussion of the singular series is greatly simplified by the following fundamental lemma.

Lemma 1. *If*

$$A_q = \sum_p \left(\frac{S_{p,q}}{q}\right)^s e_q(-np),$$

so that

$$S = 1 + A_2 + A_3 + A_4 + \ldots = \sum A_q$$

if $s \geqq 2K+1$, *and if* $(q, q') = 1$, *then*

$$A_{qq'} = A_q A_{q'};$$

and

$$S = \chi_2 \chi_3 \chi_5 \ldots = \Pi \chi_\pi,$$

where π *is prime*[6]) *and*

$$\chi_\pi = 1 + A_\pi + A_{\pi^2} + A_{\pi^3} + \ldots$$

We have

$$S_{pq'^{k-1},q} S_{pq^{k-1},q'} = \sum_h e\left(\frac{h^k pq'^{k-1}}{q}\right) \sum_{h'} e\left(\frac{h'^k pq^{k-1}}{q'}\right),$$

where h and h' describe complete systems of residues to moduli q and q'. But

$$e\left(\frac{h^k pq'^{k-1}}{q} + \frac{h'^k pq^{k-1}}{q'}\right) = e\left(\frac{\mathfrak{h}^k p}{qq'}\right),$$

where $\mathfrak{h} = hq' + h'q$; and, since $(q, q') = 1$, $\mathfrak{h}$ describes a complete system of residues to modulus qq'. Hence

$$S_{p,qq'} = S_{pq'^{k-1},q} S_{pq^{k-1},q'}. \tag{3.11}$$

Next we observe that, if p describes a complete system of residues *prime* to modulus q, and p' a similar system for modulus q', then $\mathfrak{p} = pq' + p'q$ describes a similar system for modulus qq'. Also

$$S_{\mathfrak{p}q'^{k-1},q} = \sum_h e\left(\frac{(pq'+p'q)q'^{k-1}h^k}{q}\right) = \sum_h e\left(\frac{p(q'h)^k}{q}\right) = \sum_h e\left(\frac{ph^k}{q}\right) = S_{p,q}.$$

Hence

$$\begin{aligned}(qq')^s A_q A_{q'} &= \sum_{p,p'} (S_{p,q})^s (S_{p',q'})^s e\left(-n\left(\frac{p}{q} + \frac{p'}{q'}\right)\right) \\ &= \sum_{\mathfrak{p}} (S_{\mathfrak{p}q'^{k-1},q})^s (S_{\mathfrak{p}q^{k-1},q'})^s e\left(-\frac{n\mathfrak{p}}{qq'}\right) \\ &= \sum_{\mathfrak{p}} (S_{\mathfrak{p},qq'})^s e_{qq'}(-n\mathfrak{p}) = (qq')^s A_{qq'};\end{aligned}$$

which proves the lemma.

[6]) The symbol π is used in this sense down to the end of 5.2, after which it is used in the ordinary sense.

4. Rules for the calculation of A_{π^ν}.

4.1. Lemma 1 is true for any value of k. The lemmas which follow are also true generally, provided only that k is not divisible by π. Thus when $k=4$ they hold for $\pi>2$.

The sum A_{π^ν} involves the argument n, and we might write $A_{\pi^\nu}=A_{\pi^\nu}(n)$. When, as will sometimes happen, n is replaced by another argument, this argument will be shown explicitly.

Lemma 2. *If* $(\pi, k)=1$, $\alpha>0$, $0<\mu<k$, *then*

$$A_{\pi^{\alpha k+\mu}}=0$$

or

$$A_{\pi^{\alpha k+\mu}}=\pi^{\alpha(k-s)}A_{\pi^\mu}\left(\frac{n}{\pi^{\alpha k}}\right),$$

according as n is not or is a multiple of $\pi^{\alpha k}$.

(1) We have first

$$S_{p,\pi^{\alpha k+\mu}}=\sum_h e\left(\frac{h^k p}{\pi^{\alpha k+\mu}}\right).$$

We write

$$h=\pi^{\alpha k+\mu-1}z+h' \qquad (0\leqq z<\pi,\ 0\leqq h'<\pi^{\alpha k+\mu-1}),$$

and we obtain

$$S_{p,\pi^{\alpha k+\mu}}=\sum_{h'}\sum_z e\left(\frac{ph'^k}{\pi^{\alpha k+\mu}}+\frac{kph'^{k-1}z}{\pi}\right).$$

The sum with respect to z vanishes unless h' is divisible by π, i. e. $h'=\pi h_1$, where $0\leqq h_1<\pi^{\alpha k+\mu-2}$; in this case the sum is π. Observing that this range of variation of h_1 is, to modulus $\pi^{(\alpha-1)k+\mu}$, equivalent to π^{k-2} descriptions of the range $0\leqq h_1<\pi^{(\alpha-1)k+\mu}$, we obtain

$$S_{p,\pi^{\alpha k+\mu}}=\pi\cdot\pi^{k-2}\sum_{h_1}e\left(\frac{ph_1^k}{\pi^{(\alpha-1)k+\mu}}\right)=\pi^{k-1}S_{p,\pi^{(\alpha-1)k+\mu}}.$$

It should be observed that the preceding argument is valid even when $\mu=0$. We obtain in fact

$$S_{p,\pi^{\alpha k}}=\pi^{\alpha(k-1)} \qquad (\alpha>0) \tag{4.11}$$

and otherwise

$$S_{p,\pi^{\alpha k+\mu}}=\pi^{\alpha(k-1)}S_{p,\pi^\mu}. \tag{4.12}$$

(2) We have now

$$A_{\pi^{\alpha k+\mu}}=\sum_p\left(\frac{S_{p,\pi^{\alpha k+\mu}}}{\pi^{\alpha k+\mu}}\right)^s e\left(-\frac{np}{\pi^{\alpha k+\mu}}\right).$$

We write

$$p = \pi^{\mu} z + p' \qquad (\mu > 0),$$

where $0 \leqq z < \pi^{ak}$ and p' is less than π^{μ} and not divisible by π. We have then, by (4.12),

$$S_{p,\pi^{ak+\mu}} = \pi^{a(k-1)} S_{p,\pi^{\mu}} = \pi^{a(k-1)} S_{p',\pi^{\mu}};$$

and

$$A_{\pi^{ak+\mu}} = \pi^{-as} \sum_{p'} \sum_{z} \left(\frac{S_{p',\pi^{\mu}}}{\pi^{\mu}}\right)^{s} e\left(-\frac{nz}{\pi^{ak}} - \frac{np'}{\pi^{ak+\mu}}\right).$$

The sum with respect to z vanishes unless n is divisible by π^{ak}. If however $n = \pi^{ak}\nu$, where ν is an integer, we have

$$A_{\pi^{ak+\mu}} = \pi^{a(k-s)} \sum_{p'} \left(\frac{S_{p',\pi^{\mu}}}{\pi^{\mu}}\right)^{s} e\left(-\frac{\nu p'}{\pi^{\mu}}\right) = \pi^{a(k-s)} A_{\pi^{\mu}}\left(\frac{n}{\pi^{ak}}\right).$$

4.2. Lemma 3. *If* $(\pi, k) = 1$, $a > 0$,

$$A_{\pi^{ak}} = 0 \qquad (n \not\equiv 0 \pmod{\pi^{ak-1}}),$$
$$A_{\pi^{ak}} = -\pi^{a(k-s)-1} \qquad (n \equiv 0 \pmod{\pi^{ak-1}}, \ n \not\equiv 0 \pmod{\pi^{ak}}),$$
$$A_{\pi^{ak}} = (\pi - 1)\pi^{a(k-s)-1} \qquad (n \equiv 0 \pmod{\pi^{ak}}).$$

By (4.11), we have

$$S_{p,\pi^{ak}} = \pi^{a(k-1)}, \quad A_{\pi^{ak}} = \pi^{-as} \sum_{p} e\left(-\frac{np}{\pi^{ak}}\right).$$

Writing

$$p = \pi z + p' \quad (0 \leqq z < \pi^{ak-1}, \ 0 < p' < \pi),$$

we obtain

$$A_{\pi^{ak}} = \pi^{-as} \sum_{p'} \sum_{z} e\left(-\frac{nz}{\pi^{ak-1}} - \frac{np'}{\pi^{ak}}\right).$$

The sum with respect to z is zero unless n is a multiple of π^{ak-1}. If $n = \pi^{ak-1}\nu$, we have

$$A_{\pi^{ak}} = \pi^{a(k-s)-1} \sum_{p'} e\left(-\frac{\nu p'}{\pi}\right).$$

The last sum is -1 or $\pi - 1$, according as ν is not or is divisible by π. This proves the lemma.

4.3. Lemma 4. *If* $(\pi, k) = 1$, $1 < \mu < k$,

$$A_{\pi^{\mu}} = 0 \qquad (n \not\equiv 0 \pmod{\pi^{\mu-1}}),$$
$$A_{\pi^{\mu}} = -\pi^{\mu-s-1} \qquad (n \equiv 0 \pmod{\pi^{\mu-1}}, \ n \not\equiv 0 \pmod{\pi^{\mu}}),$$
$$A_{\pi^{\mu}} = (\pi - 1)\pi^{\mu-s-1} \qquad (n \equiv 0 \pmod{\pi^{\mu}}).$$

(1) In the equation

$$S_{p,\pi^{\mu}} = \sum_{h} e\left(\frac{ph^{k}}{\pi^{\mu}}\right)$$

we write

$$h = \pi^{\mu-1} z + h' \quad (0 \leqq z < \pi,\ 0 \leqq h < \pi^{\mu-1});$$

and we obtain

$$S_{p,\pi^\mu} = \sum_{h'} \sum_{z} e\left(\frac{p h'^k}{\pi^\mu} + \frac{k p h'^{k-1} z}{\pi}\right).$$

The sum with respect to z vanishes unless $h' \equiv 0 \pmod{\pi}$, or unless $h' = \pi h_1$, where $0 \leqq h_1 < \pi^{\mu-2}$. In this case the exponential is unity (since $\mu < k$), and we obtain

$$S_{p,\pi^\mu} = \pi^{\mu-1}.$$

(2) We have thus

$$A_{\pi^\mu} = \pi^{-s} \sum_{p} e\left(-\frac{np}{\pi^\mu}\right).$$

Writing

$$p = \pi z + p' \quad (0 \leqq z < \pi^{\mu-1},\ 0 < p' < \pi)$$

we obtain

$$A_{\pi^\mu} = \pi^{-s} \sum_{p'} \sum_{z} e\left(-\frac{np'}{\pi^\mu} - \frac{nz}{\pi^{\mu-1}}\right),$$

and the sum with respect to z vanishes unless $n = \pi^{\mu-1}\nu$, where ν is an integer. In this case

$$A_{\pi^\mu} = \pi^{\mu-s-1} \sum_{p'} e\left(-\frac{\nu p'}{\pi}\right),$$

and the sum is -1 or $\pi - 1$, according as ν is not or is divisible by π.

5. The form of χ_π $(k = 4, \pi > 2)$.

5.1. We now suppose $k = 4$, so that all the results of § 4 hold for $\pi > 2$. Taking first the case $\mu = 0$, we have

$$|A_{\pi^{\alpha k}}| \leqq \pi^{\alpha(k-s)},$$

by Lemma 3.

Next, if $\mu = 1$, we have $|A_\pi| < \pi$ and so

$$|A_{\pi^{\alpha k+1}}| < \pi^{\alpha(k-s)+1},$$

by Lemma 2.

Finally, if $1 < \mu < k$, we have

$$|A_{\pi^\mu}| < \pi^{\mu-s},$$

by Lemma 4, and

$$|A_{\pi^{\alpha k+\mu}}| < \pi^{\alpha(k-s)+\mu-s},$$

by Lemma 2.

Thus the terms of χ_π may be exhibited in the form[7])

$$\begin{aligned} & 1 + A_\pi + [\pi^{2-s}] + [\pi^{3-s}] + \dots + [\pi^{k-1-s}] \\ & + \pi^{k-s}([1] + [\pi] + [\pi^{2-s}] + [\pi^{3-s}] + \dots + [\pi^{k-1-s}]) \\ & + \pi^{2(k-s)}([1] + [\pi] + [\pi^{2-s}] + [\pi^{3-s}] + \dots + [\pi^{k-1-s}]) \\ & + \dots \end{aligned}$$

where $[x]$ denotes a number whose modulus is less than x. Hence (provided only $s > k$) we have

$$\chi_\pi = 1 + A_\pi + B_\pi,$$

where

$$|B_\pi| < \frac{\pi^{k-s} - \pi^{2-s}}{\pi - 1} + \frac{\pi^{k-s}}{1 - \pi^{k-s}}\left(1 + \pi + \frac{\pi^{k-s} - \pi^{2-s}}{\pi - 1}\right).$$

Taking now $k = 4$, $s > 20$, we have

$$|B_\pi| < \pi^{-17} + 2\pi^{-17}(1 + \pi + \pi^{-17}) < 7\pi^{-16} < \pi^{-14},$$

$$\chi_\pi = 1 + A_\pi + [\pi^{-14}]. \tag{5.11}$$

5.2. When we come to consider A_π, it is necessary to distinguish different cases.

Suppose first that π is of the form $4m + 3$. Then the residues of h^4 to modulus π are the same as those of h^2, and $S_{p,\pi}$ reduces to an ordinary Gaussian sum. Thus $|S_{p,\pi}| \leqq \sqrt{\pi}$ and

$$|A_\pi| < \pi^{1-\frac{1}{2}s} < \pi^{-9},$$

$$\chi_\pi = 1 + [\pi^{-9}] + [\pi^{-14}] = 1 + [\pi^{-8}] \quad (\pi = 4m + 3).$$

Next, suppose π of the form $4m + 1$. Then[8])

$$|S_{p,\pi}| < 3\sqrt{\pi}$$

and

$$|A_\pi| < \pi\left(\frac{3}{\sqrt{\pi}}\right)^s.$$

If $\pi = 17$, $|A_{17}| < 17\left(\frac{3}{4}\right)^{21} < \frac{1}{10}$,

$$\chi_{17} = 1 + [\tfrac{1}{10}] + [17^{-14}] = 1 + [\tfrac{1}{9}]. \tag{5.21}$$

If $\pi \geqq 29 > 27$, $3^{21} < \pi^7$, $\left(\frac{3}{\sqrt{\pi}}\right)^{21} < \pi^{-3 \cdot 5}$, $|A_\pi| < \pi^{-2 \cdot 5}$,

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[7]) It should be observed that, owing to the vanishing of $A_{\pi^{\alpha k}}$ and $A_{\pi^{\alpha k + \mu}}$ when n does not satisfy certain congruence conditions, χ_π is in all cases a *finite* series; but this is irrelevant for our argument.

[8]) See H. Weber, Lehrbuch der Algebra, Bd. 1, S. 584. In Weber's notation, $S_{p,\pi}$ is one of the numbers

$$\zeta = 4\eta + 1 = \sqrt{n} + (i, \eta) + (-i, \eta).$$

and

$$(5.22) \qquad \chi_\pi = 1 + [\pi^{-2 \cdot 5}] + [\pi^{-14}] = 1 + [\pi^{-2}] \qquad (\pi = 4m + 1 \geqq 29).$$

From Lemma 1, (5.11), (5.21), and (5.22), it follows that

$$S = \chi_2 \chi_5 \chi_{13} (1 + [\tfrac{1}{9}]) \prod_{\pi = 4m+1 \geqq 29} (1 + [\pi^{-2}]) \prod_{\pi = 4m+3} (1 + [\pi^{-8}]).$$

Thus *in order to establish our conclusion when* $s = 21$, *it is only necessary to show that*

$$|\chi_2| > \sigma > 0, \quad |\chi_5| > \sigma, \quad |\chi_{13}| > \sigma.$$

5.3. We find by direct calculation that[9])

$$S_{1,5} = 1 + 4e^{\frac{2\pi i}{5}}, \quad S_{2,5} = 1 + 4e^{\frac{4\pi i}{5}}, \quad \ldots, \quad \ldots,$$

$$|S_{p,5}| \leqq \sqrt{17 + 8\cos\frac{2\pi}{5}} = \sqrt{15 + 2\sqrt{5}}.$$

It is however (as we have to consider the case of 13 also) more convenient to proceed as follows. The numbers $S_{p,5}$ are the roots of the equation[10])

$$(\zeta^2 + 15)^2 = 20(\zeta - 1)^2,$$

from which.

$$\zeta = \pm\sqrt{5} \pm i\sqrt{10 \pm 2\sqrt{5}},$$

$$|\zeta|^2 \leqq 15 + 2\sqrt{5} < 19{\cdot}473,$$

$$|S_{p,5}| = |\zeta| < 4{\cdot}413,$$

$$|A_5| < 4({\cdot}8826)^{21} < {\cdot}291,$$

$$\chi_5 = 1 + [{\cdot}291] + [5^{-14}] = 1 + [{\cdot}3],$$

$$|\chi_5| > {\cdot}7 = \sigma.$$

5.4. Similarly the various values of $S_{p,13}$ are the roots of

$$(\zeta^2 + 39)^2 = 52(\zeta - 3)^2,$$

from which

$$\zeta = \pm\sqrt{13} + i\sqrt{26 + 6\sqrt{13}},$$

$$|\zeta|^2 \leqq 39 + 6\sqrt{13} < 60{\cdot}7,$$

$$|S_{p,13}| = |\zeta| < 7{\cdot}8,$$

$$|A_{13}| < 12({\cdot}6)^{21} < {\cdot}002,$$

so that

$$|\chi_{13}| > \sigma.$$

[9]) From this point onwards π is used in the ordinary sense.

[10]) Weber, l. c., p. 584.

The proof of Theorems A *and* B *is thus reduced to a proof that* $|\chi_2| > \sigma$.

6. Discussion of χ_2.

6.1. The arguments of § 3 fail when $\pi = 2$, and it is necessary to go back to the definitions of $A_2, A_4, \ldots$ The first step is to calculate the sums $S_{p,2^\nu}$. We find by direct calculation that

$$S_{p,2} = 0,\ S_{p,4} = 2(1+e^{\frac{1}{2}p\pi i}),\ S_{p,8} = 4(1+e^{\frac{1}{4}p\pi i}),\ S_{p,16} = 8(1+e^{\frac{1}{8}p\pi i}).$$

If $\nu \geqq 5$,

$$S_{p,2^\nu} = \sum_h e\left(\frac{ph^4}{2^\nu}\right) \qquad (0 \leqq h < 2^\nu).$$

Writing

$$h = 2^{\nu-3}z + h' \quad (0 \leqq z < 8,\ 0 \leqq h' < 2^{\nu-3}),$$

we obtain

$$S_{p,2^\nu} = \sum_{h'}\sum_z e\left(\frac{ph'^4}{2^\nu} + \frac{ph'^3 z}{2}\right);$$

and the sum with respect to z is zero unless h' is even. Supposing $h' = 2h_1$, so that $0 \leqq h_1 < 2^{\nu-4}$, we obtain

$$S_{p,2^\nu} = 8\sum_{h_1} e\left(\frac{ph_1^4}{2^{\nu-4}}\right) = 8S_{p,2^{\nu-4}}.$$

Thus

$$S_{p,2^{4\alpha+\mu}} = 2^{3\alpha}S_{p,2^\mu} \qquad (\alpha > 0,\ 0 < \mu \leqq 4).$$

6.2. Observing that $A_2 = 0$, we write

$$\begin{aligned}\chi_2 &= 1 + (A_4 + A_8 + A_{16}) + (A_{64} + A_{128} + A_{256}) + \cdots\\ &= 1 + (A_4 + A_8 + A_{16}) + B.\end{aligned}$$

For $\alpha > 0$, $2 \leqq \mu \leqq 4$, we have

$$|A_{2^{4\alpha+\mu}}| = \left|\sum_p \left(\frac{S_{p,2^\mu}}{2^{\alpha+\mu}}\right)^s e\left(-\frac{np}{2^{4\alpha+\mu}}\right)\right| < 2^{4\alpha+\mu}(2^{-\alpha})^s = 2^{\mu-(s-4)\alpha},$$

$$|A_{2^{4\alpha+2}} + A_{2^{4\alpha+3}} + A_{2^{4\alpha+4}}| < 28\cdot 2^{-17\alpha},$$

$$|B| < 28\,\frac{2^{-17}}{1-2^{-17}} < 2^{-12} < \cdot 00025,$$

$$\chi_2 = 1 + A_4 + A_8 + A_{16} + [\cdot 00025].$$

6.3. Of the terms A_4, A_8, A_{16} the most important is the last. We have

$$A_{16} = \sum_{p=1,3,\ldots,15}\left(\cos\frac{p\pi}{16}\right)^{21}\exp\left(\frac{21\,p\pi i}{16} - \frac{np\pi i}{16}\right) = \mathfrak{A}_1 + \mathfrak{A}_3 + \mathfrak{A}_5 + \mathfrak{A}_7,$$

where $\mathfrak{A}_1$ is given by $p=1, 15$, $\mathfrak{A}_3$ by $p=3, 13$, and so on. And

$$\left(\cos\frac{\pi}{16}\right)^{21} = -\left(\cos\frac{15\pi}{16}\right)^{21} = \cdot 665350 + [3\cdot 10^{-6}],$$

$$\mathfrak{A}_1 = -(1\cdot 3307 + [10^{-4}])\cos\left(\frac{5\pi}{16} - \frac{n\pi}{8}\right);$$

$$\left(\cos\frac{3\pi}{16}\right)^{21} = -\left(\cos\frac{13\pi}{16}\right)^{21} = \cdot 020736 + [10^{-6}],$$

$$\mathfrak{A}_3 = (\cdot 0415 + [10^{-4}])\cos\left(\frac{\pi}{16} + \frac{3n\pi}{8}\right);$$

$$\left(\cos\frac{5\pi}{16}\right)^{21} = -\left(\cos\frac{11\pi}{16}\right)^{21} = [5\cdot 10^{-6}],$$

and so also for $\left(\cos\frac{7\pi}{16}\right)^{21}$. Thus

$$\mathfrak{A}_5 + \mathfrak{A}_7 = [2\cdot 10^{-5}].$$

Similarly we may write

$$A_8 = \sum_{1,3,5,7}\left(\cos\frac{p\pi}{8}\right)^{21}\exp\left(\frac{21p\pi i}{8} - \frac{np\pi i}{4}\right) = \mathfrak{A}_1' + \mathfrak{A}_3,$$

where $\mathfrak{A}_1'$ is given by $p=1, 7$, and so on: and

$$\left(\cos\frac{\pi}{8}\right)^{21} = -\left(\cos\frac{3\pi}{8}\right)^{21} = \cdot 189636 + [10^{-6}],$$

$$\mathfrak{A}_1' = -(\cdot 3793 + [10^{-4}])\cos\left(\frac{3\pi}{8} + \frac{n\pi}{4}\right);$$

$$\left(\cos\frac{3\pi}{8}\right)^{21} = -\left(\cos\frac{5\pi}{8}\right)^{21} = [10^{-8}],$$

$$\mathfrak{A}_3' = [10^{-7}].$$

Finally

$$A_4 = \sum_{1,3}\left(\cos\frac{p\pi}{4}\right)^{21}\exp\left(\frac{21p\pi i}{4} - \frac{n\pi i}{2}\right) = [2^{-9\cdot 5}] = [\cdot 0014].$$

Collecting our results, we may write

$$\chi_2 = 1 - 1\cdot 3307\cos\left(\frac{5\pi}{16} - \frac{n\pi}{8}\right) + \cdot 0415\cos\left(\frac{\pi}{16} + \frac{3n\pi}{8}\right)$$
$$- \cdot 3793\cos\left(\frac{3\pi}{8} + \frac{n\pi}{4}\right) + 3[\cdot 0001] + [\cdot 0017],$$

and the total possible error is $[\cdot 002]$.

6.4. We have now to verify that the sum of the first four terms of χ_2 is in all cases greater than $\cdot 002$. It is easy to see that the least favourable cases are those in which $\cos\left(\frac{5\pi}{16} - \frac{n\pi}{8}\right)$ has its greatest possible value, viz. $\cos\frac{\pi}{16}$. This happens when $n \equiv 2, 3 \pmod{16}$. We have then

$$\begin{aligned}\chi_2 &= 1 - 1{\cdot}3307 \cos\frac{\pi}{16} - {\cdot}0415 \cos\frac{3\pi}{16} + {\cdot}3793 \cos\frac{\pi}{8} + [{\cdot}002] \\ &= 1 - 1{\cdot}3051 - {\cdot}0345 + {\cdot}3504 + 3\,[{\cdot}0001] + [{\cdot}002] \\ &= 0108 + [{\cdot}0023] > {\cdot}0085 = \sigma > 0.\end{aligned}$$

It will easily be verified that, when n has any other residue to modulus 16, the margin is much greater.

7. Conclusion.

7.1. We have now proved Theorem B when $s = 21$, and Theorem A is an obvious corollary. It is not immediately obvious that, if Theorem B is true for $s = 21$, it is also true for $s > 21$. All our arguments are valid for $s \geqq 21$, except those of §§ 6. 3—6. 4; but the numerical discussion of these two paragraphs has, strictly, to be repeated for each value of s in question. Our own calculations refer only to the cases $s = 21, 31, 33$, in which we have, at various times, been particularly interested. No point of principle is involved, and the calculations in other cases may be left to anyone who may be sufficiently interested in the matter to make them[11]).

It is evident that we may, with the help of the singular series, study as closely as we wish the variations of $r_{4,5}(n)$ as n assumes various residues to modulus 16. It is clear, for example, that the numbers $16m + 2$ and $16m + 3$ are, to put it roughly, less readily expressible by 21 biquadrates than any other numbers, and something like 200 times less readily expressible than the numbers $16m + 10$ and $16m + 11$.

There is no difficulty in applying the methods of this paper to the proof that

$$G(k) \leqq (k-2)\,2^{k-1} + 5 \tag{7.11}$$

for any ***particular*** value of k, as for example 3, 5, 6 or 7. We find thus that $G(3) \leqq 9$, $G(5) \leqq 53$, $G(6) \leqq 133$, and $G(7) \leqq 325$. The first of these inequalities is not new[12]), and in fact Landau has proved that $G(3) \leqq 8$: but the numbers 53, 133, 325 compare very favourably with the 58, 478, 3806 at present known. The proof that (7. 11) is true ***generally,*** however, presents certain algebraical difficulties, of complication rather than of principle, and we must postpone it to a later memoir. We have not indeed worked out this proof in detail, the analysis which we possess carrying us only so far as the less favourable inequality

$$G(k) \leqq k\,2^{k-1} + 1$$

ndicated by our earlier researches.

[11]) See however the following note of Herr Ostrowski.

[12]) The accompanying asymptotic formula is of course new.

We conclude with one final remark. It might well be supposed that the proof of (7. 11) for (say) $k = 7$ or 13 would be more difficult than for $k = 4$. This is not so; the proof for $k = 4$ is, in essentials, more delicate and critical than for any other value of k. The fact is that *it is only for* $k = 4$ *that our inequality expresses something near the ultimate truth.* It is known that $G(4) \geqq 16$, and, the difference between 16 and 21 is comparatively small: this corresponds to the facts that *the critical factor of* § 6 *nearly vanishes in the least favourable case,* and that there is a term in χ_2 which is sometimes actually greater than the leading term 1. When k is larger, our value is much too high, and the singular series tends (for such values of s as are contemplated in our analysis) to be dominated completely by its leading term.

(Eingegangen im Januar 1920.)

Some problems of 'Partitio Numerorum': IV. The singular series in Waring's Problem and the value of the number $G(k)$.

By

G. H. Hardy in Oxford and J. E. Littlewood in Cambridge.

1. Introduction.

1. 1. In this memoir we continue the investigations initiated in two earlier memoirs bearing a similar title, and complete the proof of all the assertions which they contain[1]). We shall assume throughout that the reader is acquainted with the notation and terminology of these memoirs.

The fundamental theorem of Hilbert[2]) asserts the existence of the numbers $g(k)$ and $G(k)$. In our first memoir we proved that, if

$$a = \frac{1}{k}, \qquad K = 2^{k-1}, \qquad \varkappa = 1 - \frac{1}{K}, \qquad s > 2K + 1,$$

then

$$r_{k,s}(n) = C n^{sa-1} S + O(n^{sa\varkappa+\epsilon}), \tag{1.11}$$

where S is the 'singular series'

$$S = \sum \left(\frac{S_{p,q}}{q}\right)^s e_q(-np). \tag{1.12}$$

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[1]) G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum': I. A new solution of Waring's Problem, Göttinger Nachrichten 1920, S. 33—54; II. Proof that every large number is the sum of at most 21 biquadrates, Mathematische Zeitschrift 9 (1921), S. 14—27.

The third memoir of the series (Some problems of 'Partitio Numerorum': III. On the expression of a number as a sum of primes) will appear shortly in the Acta Mathematica. The problems considered in this memoir are of a somewhat different character. We refer to these memoirs as P. N. 1, P. N. 2, P. N. 3.

[2]) D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen, Göttinger Nachrichten 1909, S. 17—36: reprinted with certain changes in Mathematische Annalen, 67 (1909), S. 281—300.

The sum of the series is positive, and indeed greater than $\frac{1}{2}$, if s is sufficiently large; and $sa\varkappa < sa - 1$ if $s > kK$. Thus

$$(1.13) \qquad r_{k,s}(n) \sim C n^{sa-1} S,$$

as $n \to \infty$, for all large enough values of s, say for $s \geqq G_1(k)$. It is plain that Hilbert's theorem follows as a corollary.

Important simplifications of our method have been effected by Landau[3]) and Weyl[4]). These improvements relate to our treatment of the 'major arcs'. In particular Weyl has shown that, if we are concerned with an existence theorem only, so that it is not important to obtain the best possible upper bound for $G(k)$, the rather difficult analysis which we used may be replaced by an argument of a much more elementary character.

We proved nothing in this memoir about the values of $G_1(k)$ or $G(k)$, though our analysis suggested very forcibly that

$$(1.14) \qquad G(k) \leqq G_1(k) \leqq kK + 1 = s_0.$$

In order to prove this it is necessary to examine the singular series more closely, and to prove that

$$(1.15) \qquad S > \sigma = \sigma(k, s) > 0$$

for $s \geqq s_0$. This would be sufficient; but in fact, as Herr Ostrowski has shown[5]), the truth of (1.15) for $s = s_0$ will involve

$$(1.151) \qquad S > \sigma = \sigma(k) > 0$$

(the value of σ being independent of s) for $s \geqq s_0$. In our second memoir, however, we effected an improvement in (1.11), showing that

$$(1.16) \qquad r_{k,s}(n) = C n^{sa-1} S + O(n^{(s-4)a\varkappa + 2a + \varepsilon})$$

(a better result if only $k > 2$). If now we can prove that (1.15), and therefore (1.151), is true for $s > (k-2)K + 4$, we shall have proved that

$$(1.17) \qquad G(k) \leqq G_1(k) \leqq (k-2)K + 5.$$

This we proved before when $k = 4$, in some ways the most interesting case. It is the general proof of (1.17) that is our primary object now.

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[3]) E. Landau, Zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Göttinger Nachrichten 1921, S. 88—92.

[4]) H. Weyl, Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Göttinger Nachrichten 1922.

[5]) A. Ostrowski, Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Mathematische Zeitschrift, 9 (1921), S. 28—34. We return to this point in § 6. 3.

Our principal theorem then, will be:

Theorem 1. *There is a positive number* $\sigma = \sigma(k, s)$ *such that* $S > \sigma$ *for* $s \geqq (k-2)K + 5$, *so that*

$$r_{k,s}(n) \sim C n^{sa-1} S$$

for all such values of s. *In particular,* $r_{k,s}(n)$ *is positive for all such values of* s *and all sufficiently large values of* n, *so that*

$$G(k) \leqq (k-2)K + 5.$$

1. 2. We have in any event to undertake a detailed examination of the singular series; and we shall push our analysis a good deal further than is necessary for our immediate purpose. We do so primarily because the analysis is interesting in itself. But it must be remembered also that the inequality (1. 17) is, in all probability, far short of the actual truth. It is not unlikely that the order of the error term in (1. 16), which is the obstacle to further progress at present, may before long be materially reduced. The discussion of the singular series, for values of s smaller than those contemplated in Theorem 1, will then become of immediate importance, as every improvement in (1. 16) will give a corresponding improvement in the value of $G(k)$.

It may be useful if we summarise at this stage the existing state of knowledge as regards the values of $g(k)$ and $G(k)$. This is exhibited in the following table.

$k =$	2	3	4	5	6	7	8
$g(k) \leqq$	4	9	37	58	478	3806	31353
$g(k) \geqq \left[\left(\frac{3}{2}\right)^k\right] + 2^k - 2 =$	4	9	19	37	73	143	279
$G(k) \leqq$	4	8	37	58	478	3806	31353
$G(k) \leqq (k-2)2^{k-1} + 5 =$	(5)	(9)	21	53	133	325	773
$G(k) \geqq$	4	4	16	6	9	8	32

The numbers in the first line are the upper bounds for $g(k)$ which have been obtained by elementary arguments, and are due in order to Lagrange, Wieferich, Wieferich, Baer, Baer, Wieferich, and Kempner respectively[6]). Those in the third line are the corresponding

[6]) The names are those of the authors who found the actual numbers quoted. The proofs of 'Waring's Theorem' for the cases in question are due to Lagrange, Maillet, Liouville, Maillet, Fleck, Wieferich, and Hurwitz (and Maillet) respectively. For detailed references see A. J. Kempner, Über das Waringsche Problem und einige Verallgemeinerungen. Inaugural-Dissertation, Göttingen 1912, and W. S. Baer, Beiträge zum Waringschen Problem, Inaugural-Dissertation.

upper bounds for $G(k)$, and are identical with the numbers in the first except when $k = 3$. The inequality $G(3) \leqq 8$ is due to Landau[7]).

The fourth line contains the upper bounds given by Theorem 1. It will be observed that the numbers for $k = 2$ and $k = 3$ are inferior to those already known, but that there is a very substantial improvement for all larger values of k.

The second and fifth lines contain the best known lower bounds for $g(k)$ and $G(k)$ respectively. It was observed by Euler, and later by Bretschneider[8]), that the number $2^k l - 1$, where l is determined by

$$3^k = 2^k l + m, \qquad 0 < m < 2^k,$$

requires $l + 2^k - 2$ powers; and this observation gives the numbers tabulated. The numbers 4, 9, 19 are mentioned by Waring, but there is nothing to show that he had recognised the general law[9]).

The numbers in the fifth line are more interesting and require further explanation. It was proved by Hurwitz[10]) and Maillet[11]) that

$$G(k) \geqq k + 1$$

for every k; and in some cases, *e. g.* for $k = 3$, 5 and 7, no more than this is known.

In other cases it is possible to prove a good deal more by the consideration of simple congruence relations. The simplest case is $k = 4$. Every biquadrate is congruent to 0 or to 1 to modulus 16, so that a number $16m + 15$ requires at least 15 biquadrates. Thus (as was observed by Landau) $G(k) \geqq 15$, and Kempner, considering numbers $16^\beta . 31$,

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Göttingen 1913. The numbers for $k = 7$ and $k = 8$ could no doubt be substantially reduced.

Proofs of the existence of $g(k)$, from which an upper bound for $g(k)$ could be calculated, have also been given for $k = 10$ (I. Schur), $k = 12$ (Kempner) and $k = 14$ (Kempner).

[7]) E. Landau, Über eine Anwendung der Primzahltheorie auf das Waringsche Problem in der elementaren Zahlentheorie, Mathematische Annalen, **66** (1909), S. 102—105.

[8]) See Kempner, loc. cit., S. 44—45.

[9]) Waring asserts quite explicitly, not merely that $g(k)$ exists, but that $g(2) = 4$, $g(3) = 9$, $g(4) = 19$, 'et sic deinceps'. Nothing is known, so far as we are aware, inconsistent with the view that the numbers in question are the actual values of $g(k)$ for every k.

[10]) A. Hurwitz, Über die Darstellung der ganzen Zahlen als Summen von n-ter Potenzen ganzer Zahlen, Mathematische Annalen, **65** (1908), S. 424—427.

[11]) E. Maillet, Sur la décomposition d'un entier en une somme de puissances huitièmes d'entiers, Bulletin de la société mathématique de France, **36** (1908), p. 69–77.

where β is large, improved this inequality to $G(4) \geqq 16$. He also proved that $G(k) \geqq 4k$ whenever k is a power of 2, and that $G(6) \geqq 9$. This is the origin of the remaining numbers in our table. Again, every ninth power is congruent to 0, 1, or -1, to modulus 27, so that a number $27m \pm 13$ requires at least 13 ninth powers: thus $G(9) \geqq 13$.

Considerations of this character concerning cubes lead only to the Hurwitz-Maillet inequality; and when $k=5$ or $k=7$ the resulting inequalities are entirely trivial, for any residue to modulus 25 can be generated by 3 fifth powers, and any residue to modulus 49 by 4 seventh powers. It will be found that these simple facts have a very interesting bearing on the structure of our singular series.

2. The formal theory of the singular series.

2.1. The singular series is absolutely convergent for sufficiently large values of s,[12]) and is then expressible as an infinite product

(2.11) $$S = 1 + A_2 + A_3 + \ldots = \sum A_q = \chi_2\chi_3\chi_5 \cdots = \Pi\chi_\pi,$$

where π is a prime and

(2.12) $$\chi_\pi = 1 + A_\pi + A_{\pi^2} + \ldots = \sum A_{\pi^\lambda}.$$ [13])

The sum χ_π is a finite sum, for A_{π^λ} is always zero from a certain value of λ onwards[14]).

The question of the absolute convergence of the series and product will be discussed more precisely later. Our immediate object is to determine the form of the factors χ_π.

2.2. We suppose that $q = \pi^\lambda \, (\lambda \geqq 1)$, and we denote by $\nu(\xi, q, n)$ the number of solutions of the congrence

(2.21) $$\sum_{r=1}^{s} x_r^k \equiv n \pmod{q}$$

for which

(2.211) $$0 \leqq x_r < \xi \qquad (r = 1, 2, \ldots, s).$$

We write

(2.22) $$\nu(q, q, n) = M(q, n) = M(q).$$ [15])

Finally, we denote by

(2.23) $$N(q, n) = N(q)$$

[12]) P. N. 1, S. 40.

[13]) P. N. 2, S. 18.

[14]) P. N. 2, S. 22 (f. n. 7). This will also appear incidentally later (S. 374).

[15]) When it is unnecessary to show explicitly the dependence of M on n.

the number of solutions of (2.21) for which $0 \leqq x_r < q$ $(r \leqq s)$, and for which not every x is divisible by π. Such a solution we may call a *primitive* solution.

Following Landau, we write '$y \mid z$' and '$y \nmid z$' for 'z is divisible by y' and 'z is not divisible by y'. We shall find it convenient, moreover, to have a special notation to express '$y^r \mid z$, $y^{r+1} \nmid z$', *i. e.* 'y^r is the highest power of y that divides z'. In these circumstances we shall write '$y^r \| z$'.

This being so, the value of χ_π is given, in terms of the numbers N, by the following theorem.

Theorem 2. *Suppose that*

(2.24) $$k > 2, \qquad \pi^\theta \| k \quad (\theta \geqq 0), \qquad (\pi^k)^\beta \| n \quad (\beta \geqq 0),$$[16]

and let

(2.25) $$\varphi = \theta + 1 \quad (\pi > 2), \qquad \varphi = \theta + 2 \quad (\pi = 2).$$

Then

(2.26) $$\chi_\pi = B\pi^{\varphi(1-s)} N(\pi^\varphi, 0) + \pi^{\beta(k-s)+\varphi(1-s)} N\left(\pi^\varphi, \frac{n}{\pi^{\beta k}}\right),$$

where

(2.2611) $$B = 0 \qquad (\beta = 0),$$

(2.2612) $$B = 1 + \pi^{k-s} + \pi^{2(k-s)} + \pi^{(\beta-1)(k-s)} \qquad (\beta > 0).$$

The proof of this theorem rests on a series of lemmas.

2.3. Lemma 1. *If*

$$\pi^\theta \| k \quad (\theta \geqq 0), \qquad q = \pi^\lambda, \qquad \lambda > \theta + 1 \quad (\pi > 2), \qquad \lambda > \theta + 2 \quad (\pi = 2),$$

(2.31) $$x = \xi + \alpha \pi^{\lambda - \theta - 1},$$

then

(2.32) $$x^k \equiv \xi^k + \frac{k}{\pi^\theta} \alpha \xi^{k-1} \pi^{\lambda - 1} \qquad (\operatorname{mod} q).$$

We have

$$x^k = \sum_{r=0}^{k} \binom{k}{r} \alpha^r \xi^{k-r} \pi^{r(\lambda-\theta-1)}.$$

The terms $r = 0, 1$ are those which occur in (2.32).

Suppose then $r \geqq 2$. The index of the highest power of π that divides $r!$ is

$$\left[\frac{r}{\pi}\right] + \left[\frac{r}{\pi^2}\right] + \ldots < \frac{r}{\pi - 1}.$$

---

[16]) $(\pi^k)^\beta \| n$ means, of course, $\pi^{\beta k} \mid n$, $\pi^{(\beta+1)k} \nmid n$. Its meaning is different from that of $\pi^{\beta k} \| n$, which would mean $\pi^{\beta k} \mid n$, $\pi^{\beta k + 1} \nmid n$.

Hence the r-th term is divisible by π^c, where

$$c > \theta - \frac{r}{\pi - 1} + r(\lambda - \theta - 1) = \lambda + \frac{\pi - 2}{\pi - 1} r + (r-1)(\lambda - \theta - 2) - 2.$$

If $\pi > 2$, $c - \lambda > \frac{1}{2} r - 2 \geqq -1$. If $\pi = 2$, $\lambda \geqq \theta + 3$, and so $c - \lambda > r - 1 - 2 \geqq -1$. In either case $c - \lambda > -1$, or $c - \lambda \geqq 0$.

2.4. **Lemma 2:** $\sum_{\lambda=0}^{\mu} A_{\pi^\lambda}(n) = \pi^{\mu(1-s)} M(\pi^\mu, n)$.

Writing, as usual, $q = \pi^\lambda$, we have

$$A_q = A_q(n) = \sum_p \left(\frac{S_{p,q}}{q}\right) e_q(-np)$$

$$= q^{-s} \sum_p \sum_{x_1, x_2, \ldots, x_s = 0}^{q-1} e_q(p(x_1^k + x_2^k + \ldots + x_s^k - n)) = q^{-s} \sum_{x_1, x_2, \ldots, x_s} c_q(X),$$

where $X = x_1^k + x_2^k + \ldots + x_s^k - n$ and $c_q(X)$ is Ramanujan's sum[17])

$$c_q(X) = \sum_p e_q(pX).$$

If $\lambda = 1$,

$$q = \pi, \ c_\pi(X) = -1 \quad (\pi \nmid X), \ c_\pi(X) = \pi - 1 \quad (\pi \mid X),$$

and

$$(2.41) \quad A_\pi = \pi^{-s}\Big(\sum_x (-1) + \sum_{\pi \mid X} \pi\Big) = \pi^{-s}(-\pi^s + \pi M(\pi)) = \pi^{1-s} M(\pi) - 1.$$

If $\lambda > 1$,

$$c_{\pi^\lambda}(X) = 0 \quad (\pi^{\lambda-1} \nmid X), \quad c_{\pi^\lambda}(X) = -\pi^{\lambda-1} \quad (\pi^{\lambda-1} \| X),$$

$$c_{\pi^\lambda}(X) = \pi^{\lambda-1}(\pi - 1) \quad (\pi^\lambda \mid X),$$

$$A_{\pi^\lambda} = \pi^{-\lambda s}\Big(\sum_{\pi^{\lambda-1} \mid X} (-\pi^{\lambda-1}) + \sum_{\pi^\lambda \mid X} \pi^\lambda\Big)$$

$$= \pi^{\lambda - \lambda s} M(\pi^\lambda) - \pi^{\lambda - \lambda s - 1} \nu(\pi^\lambda, \pi^{\lambda-1}, n).$$

If now

$$x_r = \xi_r + \alpha_r \pi^{\lambda-1} \quad (0 \leqq \xi_r < \pi^{\lambda-1}, \ 0 \leqq \alpha_r < \pi),$$

we have

$$\sum x_r^k \equiv \sum \xi_r^k \pmod{\pi^{\lambda-1}},$$

and so

$$\nu(\pi^\lambda, \pi^{\lambda-1}, n) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_s} \nu(\pi^{\lambda-1}, \pi^{\lambda-1}, n) = \pi^s M(\pi^{\lambda-1}),$$

$$(2.42) \qquad A_{\pi^\lambda} = \pi^{\lambda(1-s)} M(\pi^\lambda) - \pi^{(\lambda-1)(1-s)} M(\pi^{\lambda-1}).$$

---

[17]) See S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, Transactions of the Cambridge Philosophical Society, 22 (1918), pp. 259–276; G. H. Hardy, Note on Ramanujan's function $c_q(n)$, Proceedings of the Cambridge Philosophical Society, 20 (1921), pp. 263–271; and P. N. 3.

The lemma follows from (2.41) and (2.42). As corollaries we have

Lemma 3: $\chi_\pi \geqq 0$.

Lemma 4: *If n is representable in any manner as a sum of s (positive, negative, or zero) k-th powers, then* $\chi_\pi > 0$.

Lemma 3 is an immediate consequence of Lemma 2. To prove Lemma 4 we have only to observe that $A_{\pi^\lambda} = 0$ from a certain value of λ onwards[18]), and that, under the hypothesis of the lemma, $M(\pi^\lambda) > 0$ for every λ.

2.5. Lemma 5. *If* $\pi^\theta | k$ $(\theta \geqq 0)$, *then*

$$N(\pi^\mu, m) = \pi^{(\mu-\varphi)(s-1)} N(\pi^\varphi, m), \tag{2.51}$$

where φ is defined as in Theorem 2, $\mu \geqq \varphi$, and m is arbitrary.

We may suppose $\mu > \varphi$, and write

$$x_r = \xi_r + \alpha_r \pi^{\mu-\theta-1} \quad (0 \leqq \xi_r < \pi^{\mu-\theta-1},\ 0 \leqq \alpha_r < \pi^{\theta+1}). \tag{2.52}$$

Let $k = \pi^\theta k_0$. Then $(k_0, \pi) = 1$. Also

$$x_r^k \equiv \xi_r^k + k_0 \alpha_r \xi_r^{k-1} \pi^{\mu-1} \pmod{\pi^\mu}, \tag{2.53}$$

by Lemma 1. If now

$$m = m_1 + m_2 \pi^{\mu-1} \qquad (0 \leqq m_1 < \pi^{\mu-1}), \tag{2.54}$$

the congruence

$$\sum x_r^k \equiv m \qquad (\text{mod } \pi^\mu,\ 0 \leqq x_r < \pi^\mu), \tag{2.55}$$

s equivalent to the pair of congruences

$$\sum \xi_r^k \equiv m_1 \equiv m \qquad (\text{mod } \pi^{\mu-1},\ 0 \leqq \xi_r < \pi^{\mu-\theta-1}), \tag{2.551}$$

and

$$\sum k_0 \alpha_r \xi_r^{k-1} \equiv m_2 - \frac{\Sigma \xi_r^k - m_1}{\pi^{\mu-1}} \qquad (\text{mod } \pi,\ 0 \leqq \alpha_r < \pi^{\theta+1}). \tag{2.552}$$

In what follows we take into consideration only primitive solutions of (2.55) and (2.551). In such a solution of (2.551) some ξ, say ξ_1, is not divisible by π. This being so, the values of $\alpha_2, \alpha_3, \ldots$ in (2.552) may be assigned arbitrarily, and then, since $(k_0 \xi_1^{k-1}, \pi) = 1$, the value of α_1 will be determined uniquely to modulus π. There will therefore be π^θ possible values of α_1 less than $\pi^{\theta+1}$, and $\pi^\theta (\pi^{\theta+1})^{s-1} = \pi^{(\theta+1)s-1}$ sets of α's associated with every solution of (2.551). That is to say we have

$$N(\pi^\mu, m) = \pi^{(\theta+1)s-1} N_1, \tag{2.56}$$

---

[18]) See S. 369, footnote [14]).

where $N(\pi^{\mu}, m)$ is the number of primitive solutions of (2.55) and N_1 the number of primitive solutions of (2.551).

Again, $N(\pi^{\mu-1}, m)$ is the number of primitive solutions of

(2.57) $$\sum x_r^k \equiv m \quad (\operatorname{mod} \pi^{\mu-1},\ 0 \leqq x_r < \pi^{\mu-1}).$$

If here we write

$$x_r = \xi_r + \alpha_r \pi^{\mu-\theta-1} \quad (0 \leqq \xi_r < \pi^{\mu-\theta-1},\ 0 \leqq \alpha_r < \pi^{\theta}),$$

and use Lemma 1 and the hypothesis $\mu > \varphi$, we obtain

$$x_r^k \equiv \xi_r^k \quad (\operatorname{mod} \pi^{\mu-1}).$$

Hence

(2.58) $$N(\pi^{\mu-1}, m) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_s} N_1 = \pi^{\theta s} N_1.$$

From (2.56) and (2.58) we deduce

(2.59) $$N(\pi^{\mu}, m) = \pi^{s-1} N(\pi^{\mu-1}, m) \quad (\mu > \varphi),$$

and the lemma follows immediately.

Proof of Theorem 2.

2.61. Let ν be the integer such that $\pi^{\beta k+\nu} \| n$, so that $0 \leqq \nu < k$; let

(2.611) $$\lambda_0 = \operatorname{Max}(\beta k + \nu + 1,\ \beta k + \varphi);$$

and suppose that $\lambda \geqq \lambda_0$.

We divide the solutions (primitive or imprimitive) of

(2.612) $$\sum x_r^k \equiv n \quad (\operatorname{mod} \pi^{\lambda},\ 0 \leqq x_r < \pi^{\lambda}),$$

into classes as follows. In the first class we put the primitive solutions, $N(\pi^{\lambda}, n)$ in number; in the second class the solutions in which every x is divisible by π but not every x by π^2; in the third those in which every x is divisible by π^2 but not every x by π^3; and so on.

The second class of solutions is correlated with the class of primitive solutions of

(2.613) $$\sum y_r^k \equiv \frac{n}{\pi^k} \quad (\operatorname{mod} \pi^{\lambda-k},\ 0 \leqq y_r < \pi^{\lambda-1}).$$

If we write

$$y_r = \xi_r + \alpha_r \pi^{\lambda-k} \quad (0 \leqq \xi_r < \pi^{\lambda-k},\ 0 \leqq \alpha_r < \pi^{k-1}),$$

then

$$\sum y_r^k \equiv \sum \xi_r^k \quad (\operatorname{mod} \pi^{\lambda-k}),$$

and the number of primitive solutions of (2.613) is plainly $\pi^{(k-1)s}$ times the number of similar solutions of

$$\sum \xi_r^k \equiv \frac{n}{\pi^k} \quad (\operatorname{mod} \pi^{\lambda-k},\ 0 \leqq \xi_r < \pi^{\lambda-k}),$$

or is

$$\pi^{(k-1)s} N\left(\pi^{\lambda-k}, \frac{n}{\pi^k}\right).$$

Similarly the number of solutions of the $(\alpha+1)$-th class, where $\alpha \leqq \beta$, is

$$\pi^{\alpha(k-1)s} N\left(\pi^{\lambda-\alpha k}, \frac{n}{\pi^{\alpha k}}\right). \tag{2.615}$$

There are no solutions of any higher class, since $(\beta+1)k \geqq \beta k+\nu+1$ and $\pi^{\beta k+\nu+1} \nmid n$. Hence, if $\lambda \geqq \beta k+\nu+1$, and so certainly if $\lambda \geqq \lambda_0$, we have

$$M(\pi^\lambda, n) = \sum_{\alpha=0}^{\beta} \pi^{\alpha(k-1)s} N\left(\pi^{\lambda-\alpha k}, \frac{n}{\pi^{\alpha k}}\right). \tag{2.616}$$

2.62. Again, if $\lambda - \alpha k \geqq \varphi$, and so certainly if $\lambda \geqq \lambda_0$ and $\alpha \leqq \beta$, we have, by Lemma 5,

$$N\left(\pi^{\lambda-\alpha k}, \frac{n}{\pi^{\alpha k}}\right) = \pi^{(\lambda-\alpha k-\varphi)(s-1)} N\left(\pi^\varphi, \frac{n}{\pi^{\alpha k}}\right). \tag{2.621}$$

Making this substitution in (2.616), and multiplying by $\pi^{\lambda(1-s)}$, we obtain

$$\begin{aligned} \pi^{\lambda(1-s)} M(\pi^\lambda, n) &= \sum_{\alpha=0}^{\beta} \pi^{\lambda(1-s)} \cdot \pi^{\alpha(k-1)s} \cdot \pi^{(\lambda-\alpha k-\varphi)(s-1)} N\left(\pi^\varphi, \frac{n}{\pi^{\alpha k}}\right) \\ &= \sum_{\alpha=0}^{\beta} \pi^{\alpha(k-s)+\varphi(1-s)} N\left(\pi^\varphi, \frac{n}{\pi^{\alpha k}}\right). \end{aligned} \tag{2.622}$$

If $\alpha < \beta$ and $\varphi \leqq k$, $\frac{n}{\pi^{\alpha k}}$ is divisible by π^φ, and we may replace it in N by 0. If $\pi > 2$, $\varphi = \theta + 1 \leqq 2^\theta \leqq \pi^\theta \leqq k$. If $\pi = 2$, $\varphi = \theta + 2 \leqq 2^\theta \leqq k$ unless $\theta = 0$ or $\theta = 1$, in which cases $\varphi \leqq 3 \leqq k$. Hence we may replace every N in (2.622), except that for which $\alpha = \beta$, by 0.

It follows that the right hand side of (2.622), is equal, when $\lambda \geqq \lambda_0$, to the value for χ_π given in Theorem 2. It is also independent of λ, and therefore, by Lemma 2, equal to

$$\lim_{\lambda\to\infty} \pi^{\lambda(1-s)} M(\pi^\lambda, n) = \chi_\pi. \tag{2.623}$$

This completes the proof of the theorem. We may observe that we have shown incidentally that

$$A_{\pi^\lambda} = 0 \qquad (\lambda > \lambda_0). \tag{2.624}$$

3. Some properties of the sums $S_{p,q}$.

3.1. In this section we establish certain properties of the Gaussian sums

$$(3.11) \qquad S_{p,q} = S_{p,q,k} = \sum_{j=0}^{q-1} e_q(j^k p)$$

which will be useful for the further study of the singular series[19]. We have not attempted to make the theory complete, though we have developed it a little further than is absolutely necessary.

We denote by

$$\chi = \chi_\varkappa = \chi_\varkappa(m) \qquad (1 \leqq \varkappa \leqq h = \varphi(q))$$

the h Dirichlet's ‚characters' to modulus q.[20]) χ_1 is the principal character, and $\bar{\chi}_\varkappa$ is the character conjugate to $\chi_\varkappa$. We shall be concerned only with the case $q = \pi^\lambda$, where $\pi > 2$ and $\lambda \geqq 1$.

It will be convenient to write

$$(3.12) \qquad S'_{p,q} = \sum_{j=0}^{q-1} \chi_1(j)\, e_q(j^k p) = \sum_{(j,q)=1} e_q(j^k p).$$

It is plain that, if $\lambda \leqq k$,

$$(3.13) \qquad S_{p,q} = S'_{p,q} + \sum_{\pi | j} 1 = S'_{p,q} + \pi^{\lambda-1}.$$

3.2. Lemma 6. *If $(l, q) = 1$ then*

$$\sum_\varkappa \bar{\chi}_\varkappa(l)\, \chi_\varkappa(m) = 0$$

unless $m \equiv l \pmod{q}$, in which case the sum is h.

The result is obvious if $(m, q) > 1$. If $(m, q) = 1$, we determine m from the congruence $mm' \equiv 1 \pmod{q}$. We have then

$$\bar{\chi}_\varkappa(l)\, \chi_\varkappa(m) = \bar{\chi}_\varkappa(l)\, \bar{\chi}_\varkappa(m') = \bar{\chi}_\varkappa(lm'),$$

and

$$\sum_\varkappa \bar{\chi}_\varkappa(lm') = 0$$

unless $lm' \equiv 1$ or $m \equiv l$, in which case the sum is h.

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[19]) What we do is, in effect, to develop from our own point of view certain portions of the theory of the division of the circle (*Kreisteilung*). It is not unlikely that the substance of our analysis is to be found elsewhere; but it is not altogether easy to extract, from the classical accounts of the theory, the particular parts which we require.

[20]) A systematic account of the theory will be found in Landau's *Handbuch*, 1 (Zweites Buch).

We write

$$(2.21)\qquad \delta = (h, k) = (\varphi(q), k) = (\pi^{\lambda-1}(\pi - 1), k).$$

3.3. Lemma 7. *There are just* δ *characters* $\chi_\varkappa$ *which possess the property*

$$(3.31)\qquad \chi_\varkappa^k = \chi_1.$$

These characters are given by

$$(3.32)\qquad \chi_\varkappa(l) = e\left(\frac{\varrho z}{\delta}\right),$$

where $\varrho = 0, 1, 2, \ldots, \delta - 1$ *and* z *is the index of* l.

We have generally

$$(3.33)\qquad \chi_\varkappa(l) = e\left(\frac{yz}{h}\right),$$

where y is the index which specifies $\varkappa$.[21] The necessary and sufficient condition for (3.31) is that $kyz \equiv 0 \pmod h$ for every z, or that

$$(3.34)\qquad ky \equiv 0 \pmod{h}.$$

From (3.34) we deduce

$$(3.35)\qquad \frac{ky}{\delta} \equiv 0 \quad \left(\operatorname{mod} \frac{h}{\delta}\right),$$

which has the single solution $y \equiv 0$ to modulus $\frac{h}{\delta}$. Thus (3.35) has the δ solutions

$$y \equiv \frac{\varrho h}{\delta} \qquad (\varrho = 0, 1, \ldots, \delta - 1)$$

to modulus h. These are all solutions of (3.34), and are plainly the only solutions.

We shall call the characters $\chi' = \chi_{\varkappa'}$ which satisfy (3.31) the *special* characters. It is clear that $\bar\chi_{\varkappa'}$ is a special character.

Lemma 8. *We have*

$$(3.36)\qquad \sum_{\varkappa'} \bar\chi_{\varkappa'}(l) = 0 \ (\delta \nmid z), \qquad \sum_{\varkappa'} \bar\chi_{\varkappa'}(l) = \delta \ (\delta \mid z).$$

For

$$\sum_{\varkappa'} \bar\chi_{\varkappa'}(l) = \sum_{\varrho=0}^{\delta-1} e\left(-\frac{\varrho z}{\delta}\right).$$

Lemma 9. *Suppose that* $q = \pi^\lambda$ $(\lambda > 1)$, *and that* $k \mid \pi - 1$, *so that* $\delta = k$. *Then*

$$(3.37)\qquad \sum_{l} e_q(lp) = 0,$$

[21] Landau, S. 401–402.

if $(p, q) = 1$ *and the summation is extended over those residues* l *of* q *for which* $\delta \mid z$.

We denote by

$$G = g + m\pi$$

the primitive root $(\operatorname{mod} q)$ to which the indices refer, g being a primitive root $(\operatorname{mod} \pi)$.[32]

Suppose first that $\delta = k = \pi - 1$. Then the indices of the l's in question are

$$0, \quad \pi - 1, \quad 2(\pi - 1), \quad \ldots, \quad (\pi^{\lambda-1} - 1)(\pi - 1).$$

Suppose that z_1 and z_2 are any two of these $\pi^{\lambda-1}$ indices, $z_2 > z_1$, and l_1 and l_2 the corresponding values of l. Then

$$l_2 - l_1 \equiv G^{z_1}(G^{z_2 - z_1} - 1) = G^{z_1}(G^{\mu\delta} - 1) \qquad (\operatorname{mod} \pi),$$

where μ is an integer, and

$$G^{\mu\delta} - 1 \equiv g^{\mu\delta} - 1 \equiv 0 \qquad (\operatorname{mod} \pi).$$

Hence $l_2 - l_1 \equiv 0 \ (\operatorname{mod} \pi)$. On the other hand, l_1 and l_2 are incongruent to modulus q, since $\mu\delta = z_2 - z_1 < \pi^{\lambda-1}(\pi - 1)$ and G is a primitive root for q. It follows that the l's in question are the numbers of the arithmetical progression

$$1, \quad \pi + 1, \quad 2\pi + 1, \quad \ldots, \quad (\pi^{\lambda-1} - 1)\pi + 1,$$

so that

$$\sum_l e_q(-lp) = e_q(-p) \sum_{r=0}^{\pi^{\lambda-1}-1} e\left(-\frac{rp}{\pi^{\lambda-1}}\right) = 0.$$

The lemma is therefore proved when $\delta = \pi - 1$. The extension to the general case is immediate. The indices of the l's in question are now

$$0, \ \delta, \ 2\delta, \ldots, \ \pi - 1, \ldots, \ \pi^{\lambda-1}(\pi - 1) - \delta$$

and form $\frac{\pi - 1}{\delta}$ arithmetical progressions of the type

$$A, \ A + \pi - 1, \ldots, \ A + (\pi^{\lambda-1} - 1)(\pi - 1),$$

where A is one of $0, \delta, 2\delta, \ldots, \pi - 1 - \delta$. The l's corresponding to the indices contained in any one of these progressions form an arithmetical pregression of difference π, and the sum of the lemma splits up into $\frac{\pi - 1}{\delta}$ sums which vanish individually.

[32]) Landau, *Handbuch*, S. 394.

3.4. Lemma 10. *We have*

$$S'_{p,q} = \sum_{l=0}^{q-1} e_q(lp) \sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l), \tag{3.41}$$

the summation with respect to $\varkappa'$ extending over all special characters.

We may plainly restrict l to values prime to q. If $(l, q) = 1$, $(m, q) = 1$, we have, by Lemma 6,

$$\sum_l e_q(lp) \sum_\varkappa \bar{\chi}_\varkappa(l) \chi_\varkappa(m) = \sum_{l=m} \sum_\varkappa + \sum_{l \not\equiv m} \sum_\varkappa = h\, e_q(mp).$$

Hence, if j runs through values less than and prime to q,

$$S'_{p,q} = \sum_j e_q(j^k p) = \frac{1}{h} \sum_j \sum_l \sum_\varkappa e_q(lp) \bar{\chi}_\varkappa(l) \chi_\varkappa(j^k)$$
$$= \frac{1}{h} \sum_l e_q(lp) \sum_\varkappa \bar{\chi}_\varkappa(l) \sum_j (\chi_\varkappa(j))^k.$$

The sum with respect to j is zero unless $\chi_\varkappa$ is special, when it is h: whence the lemma.

Lemma 11. *If $q = \pi^\lambda$ $(1 \leqq \lambda \leqq k)$ and $\delta = (h, k)$, then*

$$S'_{p,q,k} = S'_{p,q,\delta}. \tag{3.42}$$

This is an immediate consequence of Lemma 10. For the right hand side of (3.41) involves k only in so far as the special characters are fixed by k, and is therefore unaltered when k is replaced by δ.

Lemma 12. *If $q = \pi^\lambda$ $(1 < \lambda \leqq k)$ and $\pi \nmid k$, then*

$$S_{p,q,k} = \pi^{\lambda - 1}.\text{[23]} \tag{3.43}$$

It is plain from (3.13) that what we have to prove is

$$S'_{p,q,k} = 0, \tag{3.44}$$

or, by Lemma 11,

$$S'_{p,q,\delta} = 0. \tag{3.45}$$

By Lemmas 10 and 8, we have

$$S'_{p,q,\delta} = \sum_l e_q(lp) \sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l) = \delta \sum_l e_q(lp),$$

where the last summation is restricted to values of l whose indices are multiples of δ; and this sum is zero, by Lemma 9 [24]).

[23]) This has been proved already, in a different manner, in P. N. 2, S. 19—21; but it is interesting to see how the result arises from our present point of view.

[24]) Since $\delta \mid \pi - 1$ when $\pi \nmid k$.

3.5. Lemma 13. *If* $\lambda = 1$, $q = \pi$, *and* $\delta = 1$, *then*

$$S_{p,q,k} = 0. \tag{3.51}$$

But if $\delta > 1$ *then*

$$S_{p,q,k} = \sum_{\varkappa'} \tau_{\bar{\varkappa}'} \chi_{\varkappa'}(p), \tag{3.52}$$

where

$$\tau_{\bar{\varkappa}} = \sum_{l} e_q(l) \bar{\chi}_{\varkappa}(l), \tag{3.53}$$

and the summation with respect to $\varkappa'$ *extends over the special characters* χ', *exclusive of the principal character* χ_1. *Also*

$$|S_{p,q,k}| \leqq (\delta - 1)\sqrt{q}. \tag{3.54}$$

We may regard (3.52) as including (3.51), since its right hand side disappears when $\delta = 1$.

We have, by (3.13) and (3.41),

$$S_{p,q,k} = 1 + S'_{p,q,k} = 1 + \sum_{l} e_q(lp)\bar{\chi}_1(l) + \sum_{l} e_q(lp) \sum_{\varkappa'} \bar{\chi}_{\varkappa'}(l),$$

where the principal character is now excluded from the summation with respect to $\varkappa'$, and l runs from 0 to $q-1$. The sum of the first two terms is

$$1 + c_q(p) = 1 + \mu(q) = 0.$$

The third term is

$$\sum_{\varkappa'} \chi_{\varkappa'}(p) \sum_{l} e_q(lp) \bar{\chi}_{\varkappa'}(lp).$$

Since lp runs through the residues of q when l does so, the inner sum is $\tau_{\bar{\varkappa}'}$, whence the result of the lemma.

Finally, to prove (3.54), we have only to observe that, q being prime, $\bar{\chi}_{\varkappa}$ is primitive (*eigentlich*)[25]), and

$$|\tau_k| = \sqrt{q}.$$

4. The behaviour of χ_π for large values of π.

4.1. In this section we are concerned with large values of π, and may suppose $\pi > k$, so that $\theta = 0$, $\varphi = 1$. The O's which occur refer to the passage of π to infinity; the constants which they imply depend upon k and s, but not upon n.

We suppose that $k \geqq 3$.

Lemma 14. *We have*

$$A_\pi = \pi^{-s} \sum_{p} e_\pi(-np) \Big(\sum_{\varkappa'} \tau_{\bar{\varkappa}'} \chi_{\varkappa'}(p)\Big)^s, \tag{4.11}$$

[25]) Landau, *Handbuch*, S. 479.

where the summation with respect to $\varkappa'$ extends over all special characters other than the principal character.

This follows at once from (3.52).

Lemma 15. *If $s \geqq 1$, $\beta = 0$, then*

$$\chi_\pi = 1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s}). \tag{4.12}$$

We suppose first that $\pi \nmid n$, so that $\nu = 0$. Then

$$\chi_\pi = 1 + A_\pi. \tag{4.13}$$

Here we replace A_π by the right hand side of (4.11). Any product of χ's is a χ and so, when we expand by the multinomial theorem and invert the order of summation, we obtain

$$A_\pi = \pi^{-s} \sum_1 T \sum_p \chi(p) e_\pi(-np),$$

where T is a product of s τ's, χ a product of s χ's, and the number of terms in $\sum_1$ is $O(1)$. The inner sum is $O(\sqrt{\pi})$ for every χ and all values of n in question[26]), and so

$$A_\pi = O(\pi^{-s} \cdot (\sqrt{\pi})^s \cdot \sqrt{\pi}) = O(\pi^{\frac{1}{2}-\frac{1}{2}s}),$$

which proves the lemma when $\pi \nmid n$.

Next suppose $\pi \mid n$, so that $0 < \nu < k$. In this case $\lambda_0 = \nu + 1$ and

$$\chi_\pi = 1 + A_\pi + \sum_2^{\nu+1} A_{\pi^\lambda}. \tag{4.14}$$

Now $S_{p,\pi^\lambda} = \pi^{\lambda-1}$ for $2 \leqq \lambda \leqq \nu + 1 \leqq k$, by Lemma 12; and so

$$A_{\pi^\lambda} = \pi^{-s} \sum_p e_{\pi^\lambda}(-np) = \pi^{-s} c_{\pi^\lambda}(n),$$

$$A_{\pi^\lambda} = \pi^{\lambda-s-1}(\pi - 1) \quad (2 \leqq \lambda \leqq \nu), \qquad A_{\pi^\lambda} = -\pi^{\lambda-s-1} \quad (\lambda = \nu + 1),$$

$$\sum_2^{\nu+1} A_{\pi^\lambda} = -\pi^{1-s}.$$

Thus

$$\chi_\pi = 1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s}) - \pi^{1-s} = 1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s}).$$

This completes the proof of Lemma 15.

If n is fixed, $\pi \nmid n$ from a certain value of π onwards. Hence we obtain

Theorem 3. *The singular series $S = \Sigma A_q$, and the product $P = \Pi \chi_\pi$, are absolutely convergent for $s \geqq 4$, and $S = P$.*

---

[26]) It is -1 if χ is the principal character, and the product of a χ and a τ if χ is non-principal (and so primitive: Landau, *Handbuch*, S. 480).

4.2. Lemma 16. *If* $s \geqq 1$ *then*

(4.21) $1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s}) < \chi_\pi < (1 + \pi^{k-s} + \ldots + \pi^{\beta(k-s)})(1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s}))$.

This is proved already if $\beta = 0$, and we may suppose $\beta > 0$. From Theorem 2 we have, on the one hand

(4.22) $$\chi_\pi(n) \geqq \pi^{1-s} N(\pi, 0),$$

and on the other

(4.23) $$\chi_\pi(n) \leqq (1 + \pi^{k-s} + \ldots + \pi^{(\beta-1)(k-s)})\pi^{1-s} N(\pi, 0) + \pi^{\beta(k-s)+1-s} N(\pi, n'),$$

where $n' = \frac{n}{\pi^{\beta k}}$. Since neither π nor n' is divisible by π^k, we have

$$\pi^{1-s} N(\pi, 0) = \pi^{1-s} N(\pi, \pi) = \chi_\pi(\pi), \qquad \pi^{1-s} N(\pi, n') = \chi_\pi(n'),$$

and each of these is, by Lemma 15, of the form $1 + O(\pi^{\frac{1}{2}-\frac{1}{2}s})$. Thus (4.21) follows from (4.22) and (4.23).

As a corollary we have

Lemma 17. *If* $s \geqq k + 2$ *then* $\chi_\pi = 1 + O(\pi^{-2})$.

5. The numbers γ_π, $\Gamma(k)$.

5.1. Given k and π, and any positive integer m, there are two possibilities. Either (i) there is a number

(5.11) $$h_\pi = h(k, s, \pi) > 0$$

such that

(5.12) $$\chi_\pi \geqq h_\pi$$

for $s \geqq m$ and all values of n, or (ii) there is no such number. We define

$$\gamma_\pi = \gamma(k, \pi)$$

as the least value of m for which (i) is true, and $\Gamma(k)$ by

(5.13) $$\Gamma(k) = \operatorname*{Max}_{\pi} \gamma_\pi.$$

Further, we define

$$\gamma'_\pi = \gamma'(k, \pi).$$

as the least value of m such that

(5.14) $$\chi_\pi > 0$$

for $s \geqq m$ and all values of n.

It is evident that $\gamma'_\pi \leqq \gamma_\pi$.

Lemma 18. *If* $\chi_\pi > 0$ *for all sufficiently large values of* n, *then* $\chi_\pi > 0$ *for all values of* n.

In proving this Lemma we leave out of account for the moment the special case $k = 4$, $\pi = 2$. That the result is still true in this case will appear incidentally later.

It is easy to see that, apart from the exceptional case, $\varphi < k$. Thus if $\pi > 2$, $\varphi = \theta + 1 \leqq 2^\theta < \pi^\theta \leqq k$.

If $\pi = 2$, $\theta \geqq 3$, then $\varphi = \theta + 2 < 2^\theta \leqq k$.

If $\pi = 2$, $\theta = 0$, k is odd and $\varphi = 2 < 3 \leqq k$.

If $\pi = 2$, $\theta = 1$, then k is oddly even and $\varphi = 3 < 6 \leqq k$.

If $\pi = 2$, $\theta = 2$, then $\varphi = 4 < 6 \leqq k$, unless $k = 4$.

Thus $\varphi < k$ in every case except that in which $k = 4$, $\pi = 2$, when $\varphi = k$.

Now let

$$n = \pi^\varphi m + n' \qquad (0 \leqq n' < \pi^\varphi).$$

If $n' \neq 0$ then $\beta = 0$ (since $\varphi < k$) and so, by Theorem 2,

$$\chi_\pi(n) = \pi^{\varphi(1-s)} N(\pi^\varphi, n) = \pi^{\varphi(1-s)} N(\pi^\varphi, n') = \chi_\pi(n').$$

But $\chi_\pi(n) > 0$ for large values of m, and therefore $\chi_\pi(n') > 0$. It follows that $\chi_\pi > 0$ for all values of n that are not divisible by π^φ.

Again, if $(m, \pi) = 1$, we have, by Theorem 2,

$$\chi_\pi(\pi^\varphi m) = \pi^{\varphi(1-s)} N(\pi^\varphi, 0),$$

since $\varphi < k$. The left hand side is positive if m is large, and so $N(\pi^\varphi, 0) > 0$. Hence, whatever be the value of m (prime to π).

$$\chi_\pi(\pi^\varphi m) \geqq \pi^{\varphi(1-s)} N(\pi^\varphi, 0) > 0.$$

It follows that $\chi_\pi > 0$ also when $n' = 0$, which proves the lemma.

5.2. Lemma 19. *The necessary and sufficient condition that*

$$N(\pi^\varphi, n) > 0, \tag{5.21}$$

for every n, *is that* $s \geqq \gamma_\pi$. *Further,*

$$\gamma'_\pi = \gamma_\pi \tag{5.22}$$

except when $k = 4$, $\pi = 2$, *in which case*

$$\gamma_2 = 16, \qquad \gamma'_2 = 15. \tag{5.23}$$

Leaving aside the exceptional case, so that $\varphi < k$, let $s \geqq \gamma'_\pi$. Then $\chi_\pi(\pi^\varphi) > 0$. But $\beta = 0$ when $n = \pi^\varphi$ (since $\varphi < k$), and so

$$\chi_\pi(\pi^\varphi) = \pi^{\varphi(1-s)} N(\pi^\varphi, \pi^\varphi) = \pi^{\varphi(1-s)} N(\pi^\varphi, 0).$$

Hence

$$N(\pi^\varphi, 0) > 0.$$

If on the other hand $n \not\equiv 0 \pmod{\pi^\varphi}$, then $\beta = 0$ (since $\varphi \leqq k$). Hence

$$\chi_\pi = \pi^{\varphi(1-s)} N(\pi^\varphi, n)$$

and

$$N(\pi^{\varphi}, n) > 0.$$

Thus $s \geqq \gamma'_\pi$ is a sufficient condition that (5.21) should hold for every n.

Next, suppose that (5.21) holds for $s = s_1$ and every n. Then it holds, *a fortiori*, for $s \geqq s_1$ and every n, and the N's that occur in Theorem 2 are both positive. Hence

$$\chi_\pi \geqq \pi^{\varphi(1-s)} \qquad (s \geqq s_1)$$

and so

$$s_1 \geqq \gamma_\pi \geqq \gamma'_\pi.$$

It follows, first that $s \geqq \gamma'_\pi$ is both necessary and sufficient for (5.21), and secondly that $s \geqq \gamma'_\pi$ involves $s \geqq \gamma_\pi$, *i. e.* that $\gamma'_\pi = \gamma_\pi$.

If $k = 4$, $\pi = 2$, then $2^{\varphi} = 16$. Now x^4 is congruent to 0 or to 1 to modulus 16, according as x is even or odd. It follows that $N(16, n) > 0$ for $s \geqq 16$ and every n; that

$$N(16, n) > 0 \quad (16 \nmid n), \qquad N(16, 0) = 0$$

when $s = 15$; and that $N(16, 15) = N(16, 0) = 0$ when $s < 15$. Finally it follows, from Theorem 2, that

$$\chi_2 > h_2 \quad (s \geqq 16), \qquad \chi_2 > 0 \quad (s = 15),$$

$$\chi_2(16^{\beta}.\,15) = 2^{\beta(4-15)+4(1-15)} N(16, 15) = 2^{-11(\beta+1)} \qquad (s = 15),\ ^{27})$$

and

$$\chi_2(16^{\beta}.\,15) = 0 \qquad (s < 15).$$

Since $2^{-11(\beta+1)} \to 0$ when $\beta \to \infty$, these results embody (5.23). Incidentally we see that Lemma 18 is still true in the exceptional case.

5.3. Theorem 4: $G(k) \geqq \Gamma(k)$.

Leaving aside for the moment the exceptional case $k = 4$, $\pi = 2$, suppose that $s \geqq G(k)$. Then any sufficiently large n is the sum of s k-th powers, so that $\chi_\pi > 0$ for every π and all sufficiently large values of n. Hence, by Lemma 18, $\chi_\pi > 0$ for every π and every n, so that $s \geqq \gamma_\pi$. It follows that $G(k) \geqq \gamma_\pi$ for every π, which proves the theorem, apart from the exceptional case. In this case $\gamma_2 = 16$, and the result is still true, since $G(4) \geqq 16$ [28]).

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[27]) $N(16, 15) = 8^{15}$ when $s = 15$, since each x may have any one of the values $1, 3, 5, \ldots, 15$.

[28]) The lower bound Γ for G is associated with the vanishing of the singular series S for $s \leqq \Gamma - 1$, *except when* $k = 4$. When $k = 4$, $\Gamma = 16$, and the series is positive for $s = 15$, but assumes arbitrarily small values for suitable values of n.

It should be observed that our proof (see § 5. 5 below) that

$$G(\pi^{\theta}(\pi - 1)) \geqq \gamma_\pi = \pi^{\varphi} \qquad (\pi > 2)$$

(Fortsetzung der Fußnote 28 auf nächster Seite)

5.4. Lemma 20. *Suppose that* $\pi^\theta \mid k$, *and that* φ *is defined as in Theorem* 2. *Further, suppose that*

$$k = \pi^\theta \varepsilon k_0, \tag{5.41}$$

where

$$\varepsilon = (\pi^{-\theta} k, \pi - 1), \tag{5.42}$$

and

$$d = \frac{\pi - 1}{\varepsilon}; \tag{5.43}$$

so that $\varepsilon \mid \pi - 1$ *and* $(k_0, d) = 1$. *Then*

$$\gamma_\pi \leqq c = c_\pi = c(k, \pi) = \frac{\pi^\varphi - 1}{\pi - 1}\varepsilon + 1. \tag{5.44}$$

We write $\varrho = \pi^\varphi$. We must distinguish the cases $\pi > 2$ and $\pi = 2$.

(i) If $\pi > 2$, $\varphi = \theta + 1$. We suppose that G is a primitive root $(\bmod\ \varrho)$. We divide the residues to modulus ϱ into classes as follows. Consider first the residues n_0 prime to ϱ. If v is the index of n_0, we have

$$n_0 \equiv G^v \equiv G^{m_0 \psi_0 + e} \pmod{\varrho},$$

where

$$\psi_0 = \frac{\Phi(\varrho)}{d} = \frac{\Phi(\pi^\varphi)}{d} = \frac{\pi^{\varphi - 1}(\pi - 1)}{d} = \pi^{\varphi - 1}\varepsilon, \text{ [29]}$$

m_0 has one or other of the d values $0, 1, \ldots, d - 1$, and e one or other of the ψ_0 values $0, 1, \ldots, \psi_0 - 1$. The d values of n_0 with a common e we class together and call the numbers

$$\alpha_e^0 \qquad (e = 0, 1, \ldots, \psi_0 - 1);$$

the class of numbers α_e^0 with a fixed e we call C_e^0.

Next, consider the residues n_i for which $\pi^i \mid n_i$, where $0 < i < \varphi$. We have

$$n_i \equiv \pi^i N_i,$$

where the N_i's are the $\Phi(\pi^{\varphi - i})$ numbers less than and prime to $\pi^{\varphi - i}$. As G is also a primitive root to modulus $\pi^{\varphi - i}$, we can write

$$N_i \equiv G^{m_i \psi_i + e} \pmod{\pi^{\varphi - i}},$$

$$n_i \equiv \pi^i N_i \equiv \pi^i G^{m_i \psi_i + e} \pmod{\pi^\varphi},$$

---

is essentially the same as Kempner's proof (see pp. 45—46 of his Inaugural-Dissertation) that

$$G(2^\theta) \geq 2^\varphi = 2^{\theta + 2}.$$

His proof too fails when $k = 4$, and he has to appeal to the structure of the particular number 31.

[29]) We write $\Phi(\varrho)$ for Euler's function usually denoted by $\varphi(\varrho)$, as φ is used here in a different sense.

where

$$\psi_i = \frac{\Phi(\pi^{\varphi-i})}{d} = \frac{\pi^{\varphi-i-1}(\pi-1)}{d} = \pi^{\varphi-i-1}\varepsilon,$$

m_i has again one or other of the values $0, 1, \ldots, d-1$, and e one or other of the values $0, 1, \ldots, \psi_i - 1$. The ψ_i new classes obtained in this manner we denote by

$$C_e^i \qquad (e = 0, 1, \ldots, \psi_i - 1),$$

and a typical member of C_e^i by α_e^i.

Finally, the single number 0 is the sole member α_0^φ of a class C_0^φ. The total number of classes into which the residues are divided is

$$\psi_0 + \psi_1 + \ldots + \psi_{\varphi-1} + 1 = \frac{\pi^\varphi - 1}{d} + 1 = c_\pi = c.$$

We may denote the whole system of classes, in the order in which they have been defined, by $C_0, C_1, \ldots, C_r, \ldots, C_c$, and a typical member of C_r by α_r.

The class C_0 consists of the residues of k-th powers of numbers x prime to π. For

$$k = \pi^\theta \varepsilon k_0 = k_0 \frac{\pi^\theta(\pi-1)}{d} = k_0 \psi_0.$$

Also $x \equiv G^t$ for some t (since $(x, \pi) = 1$), and

$$x^k \equiv G^{t k_0 \psi_0} = G^{m_0 \psi_0},$$

so that x^k is an α_0. Moreover we can choose t so that $t k_0$ has an arbitrary residue m_0 to modulus d, since $(k_0, d) = 1$, so that every α_0 is an x^k.

Finally, to complete the properties of the classes which are immediately relevant, (1) 1 belongs to C_0, (2) $\alpha_0 \alpha_r$, where α_0 and α_r are any members of C_0 and C_r respectively, belongs to C_r, and (3) $\alpha_0 \alpha_r$, where α_r is a given member of C_r, can be identified with any member of C_r by choice of α_0.

Of these properties (1) is obvious. To prove (2) we observe that, if

$$c \equiv n_0 \equiv G^{m_0 \psi_0}, \qquad \alpha_r \equiv n_i \equiv \pi^i G^{m_i \psi_i + e},$$

then

$$\alpha_0 \alpha_r \equiv \pi^i G^{m_0 \psi_0 + m_i \psi_i + e}$$

is an α_r, since $\psi_i \mid \psi_0$. Finally

$$m_0 \psi_0 + m_i \psi_i = (\pi^i m_0 + m_i)\psi_i,$$

and we can choose m_0 so that $\pi^i m_0 + m_i$ shall have an arbitrary residue (mod d), since $(\pi, d) = 1$; hence $\alpha_0 \alpha_r$ can be identified with any member of C_r.

5. 5. To prove Lemma 20 it is enough, by Lemma 19, to show that

$$N(\pi^\varphi, n) > 0 \tag{5. 51}$$

for $s \geqq c$ and every n. And the necessary and sufficient condition for (5.51) is that every n should be congruent (mod π^φ) to the sum of at most c numbers α_0. If any α_r is the sum of not more than c α_0's, then so, by (2) and (3) of the last paragraph, is every α_r. In these circumstances we shall say that C_r is representable, and what we have to prove is that this is so for all the c values of r.

Suppose that $1 \leqq c' \leqq c$. *Then there are at least c' different classes representable by not more than c' α_0's.* For, in the first place, this is true when $c' = 1$. Suppose that it is true for $c' = \bar{c} < c$ but false for $c' = \bar{c} + 1$, and let $\bar{C}$ be a typical class representable by $\bar{c}$ α_0's, and C_r a $\bar{C}$. Then α_r belongs to a $\bar{C}$, and therefore, since no new classes become representable when $\bar{c}$ is changed to $\bar{c}+1$, $\alpha_r + 1$ belongs to a $\bar{C}$. Similarly $\alpha_r + 1 + 1 = \alpha_r + 2$ belongs to a $\bar{C}$, and, repeating the argument, *every* residue (mod ϱ) belongs to a $\bar{C}$, which is a contradiction.

Taking $c' = c$ we see that c distinct classes, and therefore all residues (mod ϱ), are representable by c α_0's, which proves the lemma, when $\pi > 2$.

(ii) There remains the case $\pi = 2$, in which $\varphi = \theta + 2$, $\varepsilon = d = 1$, $c = \pi^\varphi = \varrho$. In this case there is nothing to prove, for any residue (mod ϱ) is representable by at most ϱ 1's.

A particularly interesting case is that in which $d = 1$, $\varepsilon = \pi - 1$. In this case

$$k = \pi^\theta (\pi - 1) k_0,$$

where k_0 is prime to π. Here

$$\gamma_\pi \leqq \pi^\varphi = \pi^{\theta+1} \quad (\pi > 2), \qquad \gamma_2 \leqq 2^\varphi = 2^{\theta+2} \qquad (\pi = 2).$$

If $\pi > 2$, $\gamma_\pi = \pi^\varphi$. For

$$x^k = x^{\pi^\theta (\pi-1) k_0} \equiv 1 \qquad (\operatorname{mod} \pi^\varphi),$$

so that 1 is the only α_0. Hence $N(\pi^\varphi, 0) = 0$ if $s < \pi^\varphi$, and $\gamma_\pi \geqq \pi^\varphi$, by Lemma 19. In particular

$$\gamma_\pi = \pi = k + 1$$

if $k = \pi - 1$. Thus $\gamma_5 = 5$ if $k = 4$, $\gamma_7 = 7$ if $k = 6$.

If $\pi = 2$, $k = 2^\theta k_0$. Suppose first that $\theta > 0$. Then

$$x^{2^\theta} \equiv 1 \qquad (\operatorname{mod} 2^{\theta+2}),$$

and so $x^k \equiv 1$ (mod 2^φ). Except when $k = 4$ our argument above applies, and we obtain

$$\gamma_2 = 2^\varphi = 2^{\theta+2} \qquad (\theta > 0).$$

The result still holds when $k = 4$, since then $\gamma_2 = 16 = 2^4$.

The argument fails if $\theta=0$ (so that k is odd). Here $\varrho=2^2=4$; -1 is a k-ic residue (mod 4); and 0, 1, 2, 3 are all representable by at most two of the numbers ± 1. Thus

$$\gamma_2=2=2^{\theta+1} \qquad (\theta=0).$$

5. 6. In general it is possible to go a little further than in Lemma 20.

Lemma 21. *Suppose that* $d_1 \mid d$, *where* $d_1>1$. *Then*

$$\gamma_\pi \leqq \mathrm{Max}(d_1, c-1). \tag{5. 61}$$

Since $d_1 \mid \pi-1$, (5. 61) gives in particular

$$\gamma_\pi \leqq \mathrm{Max}(\pi-1, c-1)$$

in all cases and,

$$\gamma_\pi \leqq \mathrm{Max}(k-1, c-1)$$

if $\theta>0$.

To prove Lemma 21, suppose that $1 \leqq c' \leqq c$, and let $\nu(c')$ be the number of classes, other than the class C_c (containing the residue 0 only), that are representable by not more than c' α_0's. Then

$$\nu(c'+1) \geqq \mathrm{Min}(\nu(c')+1, c-1). \tag{5. 62}$$

For, if (5. 62) is false $\nu(c'+1)=\nu(c')<c-1$. Let $\bar{C}$ be a typical class of the $\nu(c')$ classes, and C_r a $\bar{C}$. Then, if α_r belongs to C_r, α_r+1 must belong to a $\bar{C}$ or to C_c, since no new classes, other than perhaps C_c, are representable by $c'+1$ α_0's. If $\alpha_r+1\equiv 0$, α_r+2 belongs to C_0, and therefore to a $\bar{C}$. If α_r+1 belongs to a $\bar{C}$, α_r+2 must belong to a $\bar{C}$ or to C_c. Repeating the argument, we see that every residue, other than 0, belongs to a $\bar{C}$, which is a contradiction.

From (5. 62) it follows that

$$\nu(c-1) \geqq c-1,$$

so that all residues, 0 perhaps excepted, are representable by at most $c-1$ α_0's. It remains to consider the residue 0. Let $d=\eta d_1$ and

$$\alpha_0' \equiv G^{\eta\psi_0} \qquad (\mathrm{mod}\ \varrho).$$

Then $\alpha_0' \not\equiv 1$, since $\eta\psi_0<\varphi(\varrho)$ and G is a primitive root (mod ϱ),

$$(\alpha_0')^{d_1} \equiv G^{d\psi_0}=G^{\Phi(\varrho)} \equiv 1 \qquad (\mathrm{mod}\ \varrho),$$

$$1+\alpha_0+(\alpha_0')^2+\ldots+(\alpha_0')^{d_1-1}=\frac{1-(\alpha_0')^{d_1}}{1-\alpha_0'}\equiv 0 \qquad (\mathrm{mod}\ \varrho),$$

and 0 is representable by d_1 α_0's, which completes the proof of the lemma.

Suppose in particular that $d_1=d=2$, so that $\pi>2$ and

$$k=\frac{1}{2}\pi^\theta(\pi-1)k_0.$$

In this case the α_0's are the two numbers ± 1, and

$$\gamma_\pi \geqq \frac{1}{2}(\pi^\varphi - 1).$$

But

$$c - 1 = \frac{1}{2}(\pi^{\theta+1} - 1) = \frac{1}{2}(\pi^\varphi - 1),$$

so that

$$\gamma_\pi = \frac{1}{2}(\pi^\varphi - 1) = c - 1.$$

Thus in this case also we can determine γ_π exactly.

5. 7. It is convenient to sum up our results concerning the cases $d = 1$ and $d = 2$ in a separate lemma.

Lemma 22. *If $k = \pi^\theta(\pi - 1)k_0$, where $\pi > 2$ and k_0 is prime to π, then*

$$\gamma_\pi = \pi^{\theta+1}. \tag{5.71}$$

If $k = 2^\theta k_0$, where $\theta > 0$ and k_0 is odd, then

$$\gamma_2 = 2^{\theta+2} \tag{5.72}$$

If k is odd, then $\gamma_2 = 2$.

If $k = \frac{1}{2}\pi^\theta(\pi - 1)k_0$, where $\pi > 2$ and k_0 is prime to π, then

$$\gamma_\pi = \frac{1}{2}(\pi^{\theta+1} - 1). \tag{5.73}$$

5. 8. We know that $G(k) \geqq \Gamma(k) = \operatorname{Max}\gamma_\pi$. Thus, when k is given, every value of γ_π gives a lower bound for $G(k)$. These, when less than $k + 2$, add nothing to our knowledge of $G(k)$, since $G(k)$ is always greater than k. There is therefore a special interest in determining as systematically as possible all cases in which

$$\gamma_\pi > k + 1.$$

Lemma 23. *We have*

$$\gamma_\pi \leqq k + 1 \tag{5.81}$$

unless (α) $k = 2^\theta$ $(\theta > 0)$, $\pi = 2$, *when* $\gamma_2 = 2^{\theta+2} = 4k$,

(β) $k = 2^\theta 3$ $(\theta > 0)$, $\pi = 2$, *when* $\gamma_2 = 2^{\theta+2} = \frac{4}{3}k$,

or (γ) $k = \pi^\theta \varepsilon$ $(\theta > 0)$, *where* $\pi > 2$ *and* $\varepsilon \mid \pi - 1$.

In cases (α) *and* (β) (5. 81) *is false; in case* (γ) *it may be true or false.*

We write $k = \pi^\theta \varepsilon k_0$, as in Lemma 20. If $\theta = 0$, $\pi > 2$, then

$$\gamma_\pi \leqq c = \varepsilon + 1 \leqq k + 1,$$

by Lemma 20. If $\theta = 0$, $\pi = 2$, then $\gamma_2 = 2$ by Lemma 22. Thus we need only consider cases in which $\theta > 0$.

Suppose first $\pi > 2$. If $k_0 > 1$, we have

$$\gamma_\pi \leqq c = \frac{\pi^{\theta+1}-1}{\pi-1}\varepsilon + 1 < \frac{2(\pi^{\theta+1}-\pi^\theta)}{\pi-1}\varepsilon + 1 \leqq \pi^\theta \varepsilon k_0 + 1 = k + 1.$$

Thus (5. 81) is true unless $k_0 = 1$, $k = \pi^\theta \varepsilon$, which is case (γ).

Next suppose $\pi = 2$, $k = 2^\theta k_0$. If $k_0 > 3$, we have

$$\gamma_2 = 2^{\theta+2} = 4\frac{k}{k_0} < k + 1.$$

Thus (5. 81) is true unless $k_0 = 1$ or 3, cases (α) and (β).

The case in which $k = 6$ is interesting as falling under both (β) and (γ). If $\pi = 3$, $k = 3.2 = \pi(\pi-1)$, $\varepsilon = \pi - 1$, $d = 1$, and $\gamma_3 = 3^2 = 9$. And $\gamma_2 = 2^3 = 8$.

In case (γ), (5. 81) may be true or false. Thus it is true when $k = 3$, $\pi = 3$, for then $\gamma_3 = 4$. But it is false when $k = 6$, $\pi = 3$.

5. 9. We must now collect our results and state them as theorems concerning $\Gamma(k)$. We shall say that k is exceptional if it has one of the forms in (α), (β), or (γ) of Lemma 23.

Theorem 5. *If k is not exceptional, then*

$$\Gamma(k) \leqq k + 1.$$

This is an immediate corollary of Lemma 23.

Theorem 6. *If $\theta > 1$ then $\Gamma(2^\theta) = 2^{\theta+2}$.*

Theorem 7. *If $\theta > 1$ then $\Gamma(2^\theta 3) = 2^{\theta+2}$.*

Theorem 8. $\Gamma(6) = 9$.

These theorems follow from Lemma 23, when we observe that the numbers in question in each case exceed $k + 1$.

Theorem 9. *If $\pi > 2$, $\theta > 0$, then $\Gamma(\pi^\theta(\pi-1)) = \pi^{\theta+1}$. This equality holds also when $\theta = 0$, provided that $k = \pi - 1$ is not exceptional.*

The second part follows from Theorem 5 and Lemma 22. We may therefore suppose $\theta > 0$. We have already seen that $\gamma_\pi = \pi^{\theta+1}$, which is greater than $k+1$. If π_1 is a prime other than π, $\gamma_{\pi_1} \leqq k + 1$ unless $\pi_1 = 2$, $\pi^\theta(\pi-1) = 2^{\theta_1}$, or $\pi_1 = 2$, $\pi^\theta(\pi-1) = 2^{\theta_1}3$, or $\pi_1 > 2$, $\pi^\theta(\pi-1) = \pi_1^{\theta_1}\varepsilon_1$, where $\varepsilon_1 \mid \pi_1 - 1$.

It is easy to see that the first and third alternatives are impossible, and that the second can occur only when $\pi = 3$, $\theta = 1$, $k = 6$. In this case the result has been proved already; in all other cases we have $\gamma_{\pi_1} < \gamma_\pi$ and $\Gamma(k) = \gamma_\pi = \pi^{\theta+1}$.

Theorem 10. *If $\pi > 2$, $\theta > 0$, then*

$$\Gamma\left(\tfrac{1}{2}\pi^\theta(\pi-1)\right) = \tfrac{1}{2}(\pi^{\theta+1}-1).$$

Here $\gamma_\pi = \frac{1}{2}(\pi^{\theta+1} - 1)$, since $d = 2$. This is greater than $k+1$ except when $\pi = 3$, $\theta = 1$, $k = 3$, when the two numbers are equal. Moreover $\frac{1}{2}\pi^\theta(\pi - 1)$ cannot be equal to 2^{θ_1}, $2^{\theta_1}3$, or $\pi_1^{\theta_1}\varepsilon_1$, where $\pi_1 \neq \pi$, $\theta_1 > 0$, $\varepsilon_1 \mid \pi_1 - 1$. Hence $\gamma_{\pi_1} \leqq \gamma_\pi$ and $\Gamma(k) = \gamma_\pi$.

Theorem 11. *If $\pi > 2$ and $k = \pi^\theta \varepsilon$, where $\theta > 0$, $\varepsilon \mid \pi - 1$, then*

$$\Gamma(k) \leqq \mathrm{Max}(\gamma_\pi, k+1).$$

It may be verified at once that $\pi^\theta \varepsilon$ cannot be of any of the forms $2^{\theta_1}, 2^{\theta_1}3, \pi_1^{\theta_1}\varepsilon_1$, except when $\pi = 3, \theta = 1, \varepsilon = 2, k = 6$. In this case $\Gamma(k) = \gamma_3 = 9$. The result follows from Lemma 23.

Theorem 12. *In all cases*

$$\Gamma(k) \leqq 4k.$$

The sign of equality occurs if and only if $k = 2^\theta$ $(\theta \geqq 2)$.

Theorem 13. *In all cases*

$$\Gamma(k) < (k-2)2^{k-1} + 5.$$

This theorem, which is included in Theorem 12 except when $k = 3$, is inserted only because it is what we require for the proof of Theorem 1. Our actual bounds for $\Gamma(k)$ are much better.

When $k = 3$, $\Gamma(3) = 4 < 9 = 1 \cdot 4 + 5$.

It may help to elucidate the results which we have obtained if we show in tabular form the actual values of $\Gamma(k)$ for a number of values of k.

$k =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\Gamma(k) =$	4	16	5	9	4	32	13	12	11	16	6	14	15	64	6	27

$k =$	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\Gamma(k) =$	4	25	24	23	23	32	10	26	40	29	29	30	5	128

The values of $\Gamma(k)$ for $k = 3, 4, 6, 8, 9, 10, 12, 16, 18, 20, 21, 24, 27$ and 28 are given by the actual theorems and lemmas which we have proved; the determination of the remaining values demands further calculations into which we cannot enter here.

6. The behaviour of the singular series when $s \geqq \Gamma(k)$.

6.1. Theorem 15. *Suppose that $k > 2$ and $s_1 = \mathrm{Max}(\Gamma(k), 4)$. Then*

$$S > \sigma \tag{6.11}$$

for $s \geqq s_1$ and all values of n.

By Lemma 16, we have

$$\chi_\pi > 1 - \sigma\pi^{\frac{1}{2} - \frac{1}{2}s} \qquad (s \geqq s_1).$$

Hence there is a $\pi_0 = \pi_0(k, s)$ such that

$$\chi_\pi > 1 - \sigma \pi^{\frac{1}{2} - \frac{1}{2}s} > 1 - \sigma \pi^{-\frac{3}{2}} \qquad (s \geqq s_1,\ \pi \geqq \pi_0);$$

and so

$$\prod_{\pi \geqq \pi_0} \chi_\pi > \sigma \qquad (s \geqq s_1). \tag{6.12}$$

But $\chi_\pi > \sigma$ if $\pi < \pi_0$ and $s \geqq \Gamma(k)$, and so

$$\prod_{\pi < \pi_0} \chi_\pi > \sigma \qquad (s \geqq s_1); \tag{6.13}$$

and (6.11) follows from (6.12) and (6.13).

It is plain that our main purpose is now accomplished; with Theorems 13 and 15, the proof of Theorem 1 is completed.

6.2. It is of some interest also to obtain an upper bound for S.

Theorem 16. *If* $s \geqq k + 2$ *then*

$$S < \sigma. \tag{6.21}$$

For, by Lemma 16,

$$\chi_\pi < (1 + \sigma \pi^{-2})(1 + \sigma \pi^{-\frac{3}{2}}) < 1 + \sigma \pi^{-\frac{3}{2}};$$

and the result follows immediately.

Theorem 17. *If* $s \geqq k > 3$, *then*

$$S < n^\epsilon \tag{6.22}$$

for all sufficiently large values of n.

By Lemma 16

$$\chi_\pi < (1 + \sigma \pi^{-\frac{3}{2}}) \varrho_\pi,$$

where $\varrho_\pi = 1$ unless $\pi^k | n$, and then $\varrho_\pi = 1 + \beta$. It is plain that

$$\prod \varrho_\pi \leqq \prod_{\pi | n} (1 + a) = d(n),$$

where $\pi^a | n$. As $d(n) = O(n^\epsilon)$, the theorem follows.

The interest of this theorem lies in the resulting equation

$$\varrho_{k,k}(n) = O(n^\epsilon). \tag{6.23}$$

There is some reason for supposing that

$$r_{k,k}(n) = O(n^\epsilon), \tag{6.24}$$

an equation from which very important consequences would follow. This equation would cease to be plausible if (6.23) at any rate were not true.

6.3. In conclusion, we return for a moment the equations (1.15) and (1.151). As we remarked before, the equation (1.15) is sufficient for

our present purpose; but it is interesting to bring the remark of Ostrowski into relation with our analysis.

Suppose that

$$N(\pi^{\varphi}, n) \geqq 1$$

for every n and for $s = s_0$. There is then a primitive solution of

$$x_1^k + x_2^k + \dots + x_{s_0}^k \equiv n \pmod{\pi^{\varphi}}$$

for every n. Consider now the similar congruence in which s_0 is replaced by $s > s_0$. Of the x's, the last $s - s_0$ may then be selected arbitrarily, and there will be at least one primitive solution of the ensuing congruence in the first s_0. Hence

$$N_s(\pi^{\varphi}, n) \geqq \pi^{\varphi(s-s_0)}.$$

It follows that the inequalities which we have used, of the type

$$\chi_\pi \geqq \pi^{\varphi(1-s)};$$

may be replaced by inequalities of the type

$$\chi_\pi \geqq \pi^{\varphi(1-s)} \pi^{\varphi(s-s_0)} = \pi^{\varphi(1-s_0)};$$

and our numbers $h_\pi = h(k, \pi, s)$ and $\sigma = \sigma(k, s)$ by numbers of the type $h_\pi = h(k, \pi, s_0) = h(k, \pi)$, and $\sigma = \sigma(k, s_0) = \sigma(k)$. It is however unnecessary to develop this remark further at the moment.

We add, finally, that the number $\Gamma(k)$ has a simple and interesting arithmetical interpretation. In fact $\Gamma(k)$ is: *the least number m such that every arithmetical progression contains an infinity of numbers which are sums of m k-th powers.*

(Eingegangen am 31. Oktober 1921.)

CORRECTIONS

In the table on p. 186 the value of $\Gamma(30)$ should be 31; see 1928, 4 (footnote on p. 540).

In various footnotes (e.g. 14 on p. 165) there are references to other pages of the present paper. In these, the numbers are incorrect and the correct numbers can be found by subtracting 204.

On p. 172, for (2.21) read (3.21).

On p. 175, in the formula above the heading of § 4, for τ_k read τ_χ.

On p. 176 and later, Lemmas 15, 16, 17 need correction: see § 3 of 1925, 1.

On p. 188, the final statement needs modification when $k = 4$; for this, see 1925, 1 (p. 7) and 1928, 4.

Some problems of 'Partitio Numerorum' (VI): Further researches in Waring's Problem[1]).

By

G. H. Hardy in Oxford and J. E. Littlewood in Cambridge.

1. Introduction.

1.1. In this memoir we continue the researches of the first, second and fourth memoirs of the series.

The memoir falls into three parts. In the first (§§ 2—6) we are concerned with properties of 'almost all' positive integers n. We say that almost all numbers n possess the property P if the number of numbers less than n, for which P is false, is $o(n)$ when n is large. Thus almost all numbers are composite.

We suppose throughout that $k > 2$. We denote by

$$r(n) = r_s(n) = r_{k,s}(n)$$

the coefficient of x^n in

$$(f(x))^s = \left(1 + 2\sum_1^{\infty} x^{m^k}\right)^s. \tag{1.11}$$

We write

$$a = \frac{1}{k}, \quad K = 2^{k-1}, \quad \varkappa = 1 - \frac{1}{K}, \quad \mathfrak{C} = \frac{(2\,\Gamma(1+a))^s}{\Gamma(sa)}; \tag{1.12}$$

$$\varrho(n) = \varrho_s(n) = \varrho_{k,s}(n) = \mathfrak{C}\, n^{sa-1} \sum_{q=1}^{\infty} \mathfrak{A}_q(n) = n^{sa-1}\,\mathfrak{S}(n), \tag{1.13}$$

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[1]) G. H. Hardy and J. E. Littlewood, Some problems of "Partitio Numerorum": (I) A new solution of Waring's Problem (Göttinger Nachrichten, 1920, 33–54); (II) Proof that every large number is the sum of at most 21 biquadrates (Mathematische Zeitschrift 9 (1921), 14–27); (IV) The singular series in Waring's Problem, and the value of the number $G(k)$ (ibidem 12 (1922), 161–188). We refer to these memoirs as P. N. 1, P. N. 2, P. N. 4.

We shall also have occasion to refer to the fifth memoir (P. N. 5), viz. 'A further contribution to the study of Goldbach's Problem (Proc. London Math. Soc. (2) 22 (1923), 46–56). This memoir, though concerned with a different problem, contains, in a different setting, several of the essential ideas of our present analysis.

where

$$\mathfrak{A}_q(n) = \sum_p \left(\frac{S_{p,q}}{q}\right)^s e_q(-np), \tag{1.14}$$

$$e(x) = e^{2\pi i x}, \qquad e_q(x) = e\left(\frac{x}{q}\right), \qquad S_{p,q} = \sum_{h=0}^{q-1} e_q(ph^k), \tag{1.15}$$

$p = 0$ for $q = 1$, while otherwise p runs through the numbers less than and prime to q; and

$$\sigma(n) = \sigma_s(n) = \sigma_{k,s}(n) = r_{k,s}(n) - \varrho_{k,s}(n). \tag{1.16}$$

Our argument involves a number of parameters, and of letters used in conventional senses, so that our system of notation requires very careful explanation.

Our principal variables are n, k, s, ε, and δ. Of these, ε is an arbitrary positive number; δ is also positive, and both ε and δ are to be thought of as small. The choice of δ, which has to be made so as to satisfy the varying requirements of our analysis, is always subsequent to that of k, s, and ε.

We use the letters A, B, C, with or without suffixes, to denote positive numbers whose value varies from one occurrence to another. When no suffix is used, A is an absolute constant (such as 2); $B = B(k)$ is a function of k only; and $C = C(k, s)$ a function of k and s.

Sometimes, however, A, B, C will depend on other parameters, in which case these parameters will be indicated explicitly by suffixes. Thus

$$B_\varepsilon = B(k, \varepsilon), \qquad C_{\varepsilon,\delta} = C(k, s, \varepsilon, \delta).$$

In *exponents*, we use c instead of C. When c occurs not in an exponent, our conventions do not apply, and all the variables on which c depends are shown by suffixes.

The symbols O, o refer to the passage of n to infinity. Thus $f = O(\varphi)$, where f and φ are functions of n (and in general of k, s or other parameters also) means that $|f| < M\varphi$. Here M is a number of one of the types A, B, C; and it is to be understood, unless the contrary is stated, that M involves *all parameters whose occurence is possible in the nature of the case.* If any parameter is missing, we say that the result holds uniformly in that parameter.

Similarly $f = o(\varphi)$ means that $\frac{f}{\varphi} \to 0$ when $n \to \infty$, or that, if η is any positive number, then $|f| < \eta\varphi$ for $n \geqq n_0(\eta, \alpha, \beta, \ldots)$, where $\alpha, \beta, \ldots$ are some of the parameters k, s, ε, δ; and again it is to be understood, unless the contrary is stated, that all possible parameters are involved.

1.2. We prove first

Theorem 1. *If*

$$s \geqq \left(\frac{1}{2} k - 1\right) K + 3 \tag{1.21}$$

then

$$\sum_{m=1}^{n} (\sigma(m))^2 = \sum_{m=1}^{n} (r(m) - \varrho(m))^2 = O(n^{2sa-1-c}). \tag{1.22}$$

From this we deduce

Theorem 2. *If* $k \neq 4$, *then almost all numbers are sums of* $(\frac{1}{2} k - 1)\ K + 3$ *non-negative k-th powers; and in particular of* 5 *cubes,* 27 *fifth powers,* 67 *sixth powers,* 163 *seventh powers,* 387 *eighth powers,* 899 *ninth powers, and* 2051 *tenth powers. The number of non-representable numbers is in all cases* $O(n^{1-c})$.

The result for cubes is considerably in advance of anything proved before, the utmost that is known being that a finite proportion of numbers are sums of 7 cubes [2]).

The case $k = 4$ is abnormal, since then $(\frac{1}{2} k - 1) K + 3 = 11 < 15$, while the singular series $\mathfrak{S}(n)$ vanishes, for all numbers of appropriately chosen arithmetical progressions, for any value of s less than 15. In this case we prove

Theorem 3. *Almost all numbers are sums of* 15 *biquadrates.*

In this theorem, 15 is the correct number, and cannot be replaced by any smaller number. If $G_1(k)$ is the smallest value of s for which almost all numbers are sums of s non-negative k-th powers, then

$$G_1(4) = 15, \tag{1.23}$$

while $G_1(3)$ is either 4 or 5, and $G_1(5)$ may be any number from 6 to 27 inclusive.

1.3. In the second part of the memoir (§ 7) we return to $G(k)$. The best upper bounds known for $G(3)$, $G(4)$, ..., $G(k)$, ..., are 8, 21, 53, 133, ..., $(k-2)K + 5$, ... Our object is to improve these numbers (apart from $G(3)$, with which we can do nothing) by combining the results of the first part with a simple elementary idea. We prove

Theorem 4. *If* $k > 3$ *then*

$$G(k) \leqq \left(\frac{1}{2} k - 1\right) K + k + 5 + [\zeta_k], \tag{1.31}$$

[2]) That is to say, there is a constant A such that more than An numbers less than n are sums of 7 cubes. This result is due to Baer (W. S. Baer, Über die Zerlegung der ganzen Zahlen in sieben Kuben, Mathematische Annalen 74 (1913), 511—514).

where

$$\zeta_k = \frac{(k-2)\log 2 - \log k + \log(k-2)}{\log k - \log(k-1)}. \tag{1.32}$$

In particular, all large numbers are sums of 19 *biquadrates*, 41 *fifth powers,* 87 *sixth powers,* 193 *seventh powers,* 425 *eighth powers,* 949 *ninth powers, and* 2113 *tenth powers.*

We can still prove nothing new for cubes; but the result for biquadrates is particularly interesting. All numbers from a certain number N_4 onwards are sums of 19 biquadrates, and it would be possible to determine a numerical bound for N_4. Now $g(4) \geqq 19$ (since $79 = 4\cdot 2^4 + 15\cdot 1^4$ actually requires 19 biquadrates). The determination of the actual value of $g(4)$, whether it be in fact 19 or some higher number, is thus theoretically reducible to a problem of computation.

1.4. The third part (§ 8) stands on a different footing, since it depends on an unproved hypothesis, viz.

Hypothesis K. *The number of representations of n by k k-th powers is $O(n^\epsilon)$ for every positive ϵ.*

This hypothesis (a separate hypothesis for each value of k) is equivalent to

$$r_{k,k}(n) = O(n^\epsilon). \tag{1.41}$$

It is true when $k = 2$[3]), but has not been proved for any higher value of k. Its consequences in Waring's Problem are very striking, and it is well worth while to investigate them in anticipation of a proof.

We prove first

Theorem 5. *If Hypothesis K is true, then* (1.22) *is true for* $s \geqq k+1$.

From this follows

Theorem 6. *If Hypothesis K is true,* $k \neq 4$, *and*

$$\Gamma_1(k) = \mathrm{Max}(\Gamma(k), k+1), \tag{1.42}$$

where $\Gamma(k)$ is the least value of s for which the singular series is always positive, then almost all numbers are sums of $\Gamma_1(k)$ non-negative k-th powers. In particular almost all numbers are (on Hypothesis K) sums of 4 *cubes,* 6 *fifth powers,* 9 *sixth powers,* 8 *seventh powers,* 32 *eighth powers,* 13 *ninth powers, and* 12 *tenth powers. The number of non-representable numbers is in any case $O(n^{1-c})$.*

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[3]) E. Landau, Über die Anzahl der Gitterpunkte in gewissen Bereichen, Göttinger Nachrichten (1912), 687—771 (750).

It was proved by Hurwitz[4]) that

$$G_1(k) \geqq k+1,$$

and by us[5]) that

$$G_1(k) \geqq \Gamma(k),$$

except when $k=4$. Hence, except in this case,

$$G_1(k) \geqq \Gamma_1(k).$$

Combining this inequality with Theorem **6**, we obtain

Theorem 7. *If Hypothesis K is true, and $k \neq 4$, then*

$$G_1(k) = \Gamma_1(k). \tag{1.43}$$

As $\Gamma(k)$ can be calculated, with sufficient labour, for any k, the problem of $G_1(k)$ would be solved completely if Hypothesis K were proved.

Finally we prove

Theorem 8. *If Hypothesis K is true, then*

$$r_{k,s}(n) = \varrho_{k,s}(n) + O(n^{s\alpha-1-c}) \tag{1.44}$$

for $s \geqq 2k+1$;

from which follows

Theorem 9. *If Hypothesis K is true, and*

$$\Gamma_2(k) = \operatorname{Max}(\Gamma(k), 2k+1),\ ^{6)} \tag{1.45}$$

then all large numbers are sums of $\Gamma_2(k)$ non-negative k-th powers; and in particular of 7 *cubes,* 16 *biquadrates,* 11 *fifth powers,* 13 *sixth powers,* 15 *seventh powers,* 32 *eighth powers,* 19 *ninth powers, and* 21 *tenth powers.*

We show also that when (as is usually the case), $\Gamma(k) \leqq k+1$, Theorem **9** may be deduced in an elementary manner from Theorem **6**[7]).

It can hardly be doubted that the numbers of Theorem **9** are far closer to the realities of the problem than any given before. It is not too much to say that, if Hypothesis K were proved, Waring's Problem would be within measurable distance of a final solution.

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[4]) A. Hurwitz, Über die Darstellung der ganzen Zahlen als Summen von n-ten Potenzen ganzer Zahlen, Mathematische Annalen **65** (1908), 424—427.

[5]) See P. N. 4, 179 (Theorem 4). The ground of the inequality $G(k) \geqq \Gamma(k)$ being (except for $k=4$) the existence of a forbidden arithmetical progression, the same inequality holds for $G_1(k)$.

[6]) In fact $\Gamma_2(k) = 2k+1$ except when k is a power of 2, in which case it is $4k$.

[7]) This deduction is still possible in the exceptional cases, but the proof requires elaboration.

k		2	3	4	5	6	7	8	9	10
k	$=$	2	3	4	5	6	7	8	9	10
$g(k)$	$\leqq$	4	9	37	58	478	3306	31353	— [8]	140004
$g(k) \geqq \left[\left(\frac{3}{2}\right)^k\right] + 2^k - 2$	$=$	4	9	19	37	73	143	279	548	1079
$G(k) \leqq \begin{matrix} (k-2)\,2^{k-2} + k + 5 + [\zeta_k] \text{[8a]} \; (k > 3) \\ 4 \;\; (k=2), \quad 8 \;\; (k=3) \end{matrix}$	$=$	4	8	19	41	87	193	425	949	2113
$G(k) \gtreqless \mathbf{Max}\,(k+1,\; \Gamma(k))$	$=$	4	4	16	6	9	8	32	13	12
$G_1(k) \leqq \begin{matrix} (k-2)\,2^{k-2} + 3 \;\; (k \neq 2, 4) \\ 4 \;\; (k=2), \quad 15 \;\; (k=4) \end{matrix}$	$=$	4	5	15	27	67	163	387	899	2051
$G_1(k)$	$\geqq$	4	4	15	6	9	8	32	13	12
$\Gamma(k)$	$=$	4	4	16	5	9	4	32	13	12
On Hypothesis K $G(k) \leqq \begin{matrix} \mathbf{Max}\,(2k+1,\; \Gamma(k)) \;\; (k > 2) \\ 4 \;\; (k=2) \end{matrix}$	$=$	4	7	16	11	13	15	32	19	21
$G_1(k) = \begin{matrix} \mathbf{Max}\,(k+1,\; \Gamma(k)) \;\; (k \neq 4) \\ 15 \;\; (k=4) \end{matrix}$	$=$	4	4	15	6	9	8	32	13	12

[8]) No upper bound for $g(9)$ is known.

[8a]) See (1.32) for the definition of ζ_k.

It may be useful that we should exhibit, in the form of a table, all that is known about the numbers $g(k)$, $G(k)$, $G_1(k)$, $\Gamma(k)$ for $2 \leqq k \leqq 10$. We may repeat that $g(k)$, $G(k)$, $G_1(k)$ are the least numbers s such that *all, all large, almost all* numbers are sums of s k-th powers; while $\Gamma(k)$ is, *except for* $k=4$, (a) the least number s for which the singular series is positive for every n, and (b) the least number s such that every arithmetical progression contains an infinity of representable numbers. When $k=4$, $\Gamma(k)=16$, while (a) and (b) are true for $s \geqq 15$, but for no lower value of s.[9])

2. Further definitions.

2.1. We must begin by defining a number of additional symbols.

In our earlier memoirs we used the 'Farey dissection' of order $N=[n^{1-a}]$, the 'major arcs' $\mathfrak{M}$ and 'minor arcs' $\mathfrak{m}$ of the dissection being defined by $q \leqq n^a$ and $q > n^a$ respectively. It was then shown by Weyl[10]) and Landau[11]) that our argument could be simplified by using a different dissection, viz. that of order $[n^{1-\alpha}]$, where $\alpha = \frac{aK}{K+1}$. Here we use two dissections, each different from either of those used before.

In the proof of Theorems 1—4 we use the dissection of order

$$N_1 = [n^{1-a_1}], \tag{2.11}$$

where

$$a_1 = a_1(k, \delta) = a - \delta \tag{2.12}$$

and δ is positive and small. The choice of δ will be subsequent to that of k, s, ε, so that δ will then become a function of those variables, tending to zero with ε. The major and minor arcs will be defined by

$$q \leqq n^{a_1}, \qquad q > n^{a_1} \tag{2.13}$$

and denoted by $\mathfrak{M}_1$, $\mathfrak{m}_1$.

In the proof of Theorems 5—9 (those which depend upon Hypothesis K) we use a quite different dissection, viz. that of order

$$N_2 = [n^{1-3\alpha}], \tag{2.14}$$

the major and minor arcs $\mathfrak{M}_2$, $\mathfrak{m}_2$ being defined by

$$q \leqq n^{\alpha}, \qquad q > n^{\alpha}. \tag{2.15}$$

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9) See P. N. 4, 179, f. n. 28). This point about $k=4$ is overlooked on p. 188 (last sentence).

10) H. Weyl, Bemerkung über die Hardy-Littlewoodschen Untersuchungen zum Waringschen Problem, Göttinger Nachrichten 1921, 189—192.

11) E. Landau, Zum Waringschen Problem, Hilbert Festschrift (1922), 423–451 [Math. Zeitschr. 12 (1921), 219—247].

Here α is a number of the type c, which will be chosen small enough to fulfil the demands of our analysis.

2.2. We use $\xi = \xi_{p,q}$, as hitherto, for a 'Farey arc' in general. We write

(2.211) $$x = |x| e^{i\psi} = e^{-\frac{1}{n}+i\psi}$$

and (on the arc $\xi_{p,q}$)

(2.212) $$x = e_q(p) e^{-y}, \qquad y = \frac{1}{n} + i\theta, \qquad \theta = \frac{2p\pi}{q} - \psi.$$

We write also

(2.22) $$\varphi = \varphi_{p,q} = 2\Gamma(1+a) \frac{S_{p,q}}{q} \frac{1}{y^a}.$$

(2.23) $$g(z) = \sum_{m=1}^{\infty} m^{sa-1} z^m \qquad (|z| < 1),$$

(2.24) $$F(x) = F(x, k, s) = \sum_{m=1}^{\infty} \varrho_s(m) x^m = \sum_{m=1}^{\infty} m^{sa-1} \mathfrak{S}(m) x^m,$$

(2.25) $$F_{p,q} = \mathfrak{C} \left(\frac{S_{p,q}}{q}\right)^s \sum_{m=1}^{\infty} m^{sa-1} (x e_q(-p))^m = \mathfrak{C} \left(\frac{S_{p,q}}{q}\right)^s g(x e_q(-p)).$$

Finally, we write

(2.26) $$\mathfrak{S}_1(n) = \mathfrak{C} \sum_{q \leqq \nu} \mathfrak{A}_q(n), \qquad \mathfrak{S}_2(n) = \mathfrak{C} \sum_{q > \nu} \mathfrak{A}_q(n),$$

so that

(2.261) $$\mathfrak{S}(n) = \mathfrak{S}_1(n) + \mathfrak{S}_2(n);$$

and

(2.27) $$F_1(x) = \sum_{m=1}^{\infty} m^{sa-1} \mathfrak{S}_1(m) x^m, \qquad F_2(x) = \sum_{m=1}^{\infty} m^{sa-1} \mathfrak{S}_2(m) x^m,$$

so that

(2.271) $$F(x) = F_1(x) + F_2(x).$$

Then

(2.281) $$F_1 = \mathfrak{C} \sum_{p,q \leqq \nu} \sum_m \left(\frac{S_{p,q}}{q}\right)^s m^{sa-1} (x e_q(-p))^m = \sum_{p,q \leqq \nu} F_{p,q},$$ [11a])

and similarly

(2.282) $$F_2 = \sum_{p,q \geqq \nu} F_{p,q}.$$

We shall take

(2.29) $$\nu = n^{\beta};$$

β will be chosen later.

We suppose always that $n > 1$.

[11a]) This notation means that we sum for $q \leqq \nu$ and for all values of p associated with each such q.

3. Lemmas concerning the singular series.

3.1. We have frequently to appeal in what follows to the results and arguments of P. N. 4, and we must begin by revising slightly a portion of that memoir. An incorrect argument in the proof of Lemma 15, to which our attention was called by Prof. Landau, affects the results of Lemmas 15, 16 and 17, and the proofs, though not the results, of Theorems 15, 16 and 17. It happens that the correction is relevant to the present memoir.

Lemma 1[12]. *We have, for* $s > 0$,

$$|\mathfrak{A}_{\mathfrak{p}}(n)| < C\mathfrak{p}^{\frac{1}{2}-\frac{1}{2}s} \qquad (\mathfrak{p} \nmid n), \tag{3.111}$$

$$|\mathfrak{A}_{\mathfrak{p}}(n)| < C\mathfrak{p}^{1-\frac{1}{2}s} \qquad (\mathfrak{p} \mid n); \tag{3.112}$$

$$|\mathrm{X}_{\mathfrak{p}} - 1| < C\mathfrak{p}^{\frac{1}{2}-\frac{1}{2}s} \qquad (\mathfrak{p} \nmid n), \tag{3.121}$$

$$|\mathrm{X}_{\mathfrak{p}} - 1| < C\mathfrak{p}^{1-\frac{1}{2}s} \qquad (\mathfrak{p} \mid n,\ \mathfrak{p}^k \nmid n), \tag{3.122}$$

$$1 - C\mathfrak{p}^{1-\frac{1}{2}s} < \mathrm{X}_{\mathfrak{p}} < (1 + \mathfrak{p}^{k-s} + \ldots + \mathfrak{p}^{\beta(k-s)})(1 + C\mathfrak{p}^{1-\frac{1}{2}s}) \quad ((\mathfrak{p}^k)^{\beta} \mid n).^{[12a]} \tag{3.13}$$

Also

$$|\mathrm{X}_{\mathfrak{p}} - 1| < C\mathfrak{p}^{-\frac{3}{2}} \qquad (s \geqq k+2). \tag{3.14}$$

Here (3.121) and (3.122) give the corrected form of Lemma 15, (3.13) that of Lemma 16, and (3.14) that of Lemma 17.

The result (3.111) is correctly proved on p. 176; and (3.112) follows by the same argument, the only difference being that the 'inner sum' is the trivial $O(\mathfrak{p})$ instead of $O(\sqrt{\mathfrak{p}})$.[13] The subsequent argument of the text is a correct deduction of (3.122) from (3.112), provided we write $O(\mathfrak{p}^{1-\frac{1}{2}s})$ for $O(\mathfrak{p}^{\frac{1}{2}-\frac{1}{2}s})$ in the two places in which the latter occurs. The other formula (3.121) is, of course, merely a convenient restatement of (3.111), since $\mathrm{X}_{\mathfrak{p}} = 1 + \mathfrak{A}_{\mathfrak{p}}$ when $\mathfrak{p} \nmid n$.

If now we substitute, in the proofs of Lemmas 16 and 17, correct expressions from (3.11), (3.12) in place of incorrect expressions from Lemma 15, and leave the arguments otherwise unchanged, we obtain (3.13) and (3.14).

3.2. The remaining Lemmas and Theorems of P. N. 4 are actually true, but the proofs of Theorems 15, 16, and 17 have to be revised. We require Theorem 15 in the sequel, and restate it.

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[12]) Here $\mathfrak{A}$, $\mathfrak{p}$, X are the A, π, χ of P. N. 4.

[12a]) This notation means that $(\mathfrak{p}^k)^{\beta}$ is the highest power of $\mathfrak{p}^k$ which divides n: see P. N. 4, 166. The β here has of course no connection with that of (2.29).

[13]) Our mistake there lay in failing to distinguish the two cases.

Lemma 2. *If* $s \geqq \operatorname{Max}(\Gamma(k), 4)$ *then*

$$\mathfrak{S}(n) > C. \tag{3.21}$$

The original proof, when corrected, requires the 4 to be replaced by 5. But the result is still valid when $k = 4$.[14] For if $s = \Gamma(k) = 4$, k is odd, since otherwise $\gamma_2 = 2^{\theta+2} > 4$. Then, writing

$$\mathfrak{S} = \mathfrak{C} \prod_{\mathfrak{p} \nmid n} \chi_{\mathfrak{p}} \prod_{\mathfrak{p} \mid n} \chi_{\mathfrak{p}} = \mathfrak{C}\, \Pi_1 \Pi_2,$$

we have

$$\chi_{\mathfrak{p}} > 1 - C\mathfrak{p}^{\frac{1}{2}-\frac{1}{2}s} = 1 - C\mathfrak{p}^{-\frac{3}{2}} > 1 - \mathfrak{p}^{-\frac{5}{4}} \qquad (\mathfrak{p} > C) \tag{3.22}$$

in Π_1, by (3.121). On the other hand, in Π_2, $S_{p,\mathfrak{p}}$ is real, $S_{p,\mathfrak{p}}^4$ is positive or zero, and (since $\mathfrak{p} \mid n$)

$$\mathfrak{A}_{\mathfrak{p}} = \sum S_{p,\mathfrak{p}}^4\, e_{\mathfrak{p}}(-np) = \sum S_{p,\mathfrak{p}}^4 \geqq 0.$$

If now $\beta = 0$ we have[15]

$$\chi_{\mathfrak{p}} = 1 + \mathfrak{A}_{\mathfrak{p}} + \sum_{2}^{\nu+1} \mathfrak{A}_{\mathfrak{p}^\lambda} = 1 + \mathfrak{A}_{\mathfrak{p}} - \mathfrak{p}^{1-s} \geqq 1 - \mathfrak{p}^{-3}; \tag{3.23}$$

while if $\beta > 0$ we have[16]

$$\chi_{\mathfrak{p}} \geqq \chi_{\mathfrak{p}}(\mathfrak{p}) \geqq 1 - \mathfrak{p}^{-3} \tag{3.24}$$

by (3.23), since $\beta = 0$ for $n = \mathfrak{p}$. From (3.22) to (3.24) we obtain

$$\mathfrak{S} > C \prod_{\mathfrak{p} \nmid n,\, \mathfrak{p} < C} \chi_{\mathfrak{p}} \prod_{\mathfrak{p} \nmid n} (1 - \mathfrak{p}^{-\frac{5}{4}}) \prod_{\mathfrak{p} \mid n} (1 - \mathfrak{p}^{-3}) > C,$$

since $\chi_{\mathfrak{p}}$ is positive when $\mathfrak{p}$ is fixed and $s \geqq \Gamma(k)$, and has only a finite number of distinct values for different values of n.

3.3. In the proof of Theorem **16** we have, by (3.13) above,

$$\chi_{\mathfrak{p}} < (1 + \mathfrak{p}^{-2} + \mathfrak{p}^{-4} + \ldots)(1 + C\mathfrak{p}^{-\frac{3}{2}}) < 1 + C\mathfrak{p}^{-\frac{3}{2}},$$

and the theorem follows.

In the proof of Theorem **17** we have $s \geqq k > 3$. Hence

$$\mathfrak{S} \leqq C \prod_{\mathfrak{p} \nmid n} \chi_{\mathfrak{p}} \prod_{\mathfrak{p} \mid n} \chi_{\mathfrak{p}} < C \prod_{\mathfrak{p} \mid n} \chi_{\mathfrak{p}},$$

since

$$\chi_{\mathfrak{p}} < 1 + C\mathfrak{p}^{\frac{1}{2}-\frac{1}{2}s} < 1 + C\mathfrak{p}^{-\frac{3}{2}} \qquad (\mathfrak{p} \nmid n).$$

If $\mathfrak{p} \mid n$, we have, by (3.13),

$$\chi_{\mathfrak{p}} < 1 + C\mathfrak{p}^{1-\frac{1}{2}s} \varrho_{\mathfrak{p}},$$

---

[14]) We owe this observation to Prof. Landau.

[15]) P. N. 4, 176, (4.14) and the equation five lines lower.

[16]) P. N. 4, 177, (4.22).

where $\varrho_{\mathfrak{p}}=1$ unless $\mathfrak{p}^k | n$, and then $\varrho_{\mathfrak{p}}=1+\beta$. Thus

$$\mathrm{X}_{\mathfrak{p}} < \left(1+\frac{C}{\mathfrak{p}}\right)\varrho_{\mathfrak{p}} < \left(1-\frac{1}{\mathfrak{p}}\right)^{-c}\varrho_{\mathfrak{p}},$$

$$\prod_{\mathfrak{p}|n}\mathrm{X}_{\mathfrak{p}} < \left(\frac{n}{\varphi(n)}\right)^{c}\prod_{\mathfrak{p}|n}\varrho_{\mathfrak{p}}.$$

The first factor here is less than n^{ε} for sufficiently large n, and it is proved in the original argument that the same is true of the second. Hence

$$\mathfrak{S} < n^{\varepsilon}$$

for sufficiently large n, the result desired.

3.4. **Lemma 3.** *We have*

$$\left|\frac{S_{p,q}}{q}\right| < Bq^{-a}, \tag{3.41}$$

and

$$|\mathfrak{A}_q(m)| < Cq^{1-sa} \tag{3.42}$$

for $m=1, 2, 3, \ldots$ [17]).

It is enough to prove (3.41), (3.42) being an obvious corollary; and it is enough, in virtue of the multiplication theorem [18]) for $S_{p,q}$, to prove (3.41) (i) when $(q, k)=1$ and (ii) when q is a product of powers of primes which divide k.

(i) If $\mathfrak{p} \nmid k$, we have [19])

$$\left|\frac{S_{p,\mathfrak{p}^{\alpha k+\mu}}}{\mathfrak{p}^{\alpha k+\mu}}\right| = \mathfrak{p}^{-\alpha-1} \leqq (\mathfrak{p}^{\alpha k+\mu})^{-a} \tag{3.431}$$

for

$$\alpha \geqq 0, \qquad 1<\mu\leqq k,$$

and

$$\left|\frac{S_{p,\mathfrak{p}^{\alpha k+1}}}{\mathfrak{p}^{\alpha k+1}}\right| = \mathfrak{p}^{-\alpha}\left|\frac{S_{p,\mathfrak{p}}}{\mathfrak{p}}\right| = (\mathfrak{p}^{\alpha k})^{-a}\left|\frac{S_{p,\mathfrak{p}}}{\mathfrak{p}}\right| \tag{3.432}$$

for $\alpha \geqq 0$. Write now

$$q = q_1 q_2^k Q, \tag{3.44}$$

where Q contains all factors $\mathfrak{p}^r$ of q for which r is not congruent to 1 (mod k), and q_1 is a product of different primes. It follows from the multiplication theorem and the inequalities (3.43) that

$$\left|\frac{S_{p,q}}{q}\right| \leqq (q_2^k Q)^{-a}\left|\frac{S_{p,q_1}}{q_1}\right|, \text{ [20])}$$

[17]) Compare P. N. 5, 49 (Lemma 5). The argument there is simpler.

[18]) P. N. 2, 18.

[19]) P. N. 2, 19–21.

[20]) Here, and in the formulae which follow, the values of the p's, which differ from formula to formula, are irrelevant.

so that it is enough to prove (3.41) when $q = q_1$. But, if $q = q_1 = \Pi \mathfrak{p}_1$, we have [21])

$$\left| \frac{S_{p,\mathfrak{p}_1}}{\mathfrak{p}_1} \right| < \frac{B}{\sqrt{\mathfrak{p}_1}} < \mathfrak{p}_1^{-a}$$

for $\mathfrak{p}_1 > B$, and so

$$(3.45) \qquad \left| \frac{S_{p,q_1}}{q_1} \right| = \Pi \left| \frac{S_{p,\mathfrak{p}_1}}{\mathfrak{p}_1} \right| < \prod_{\mathfrak{p}_1 \leqq B} B \prod_{\mathfrak{p}_1 > B} \mathfrak{p}_1^{-a} < B q_1^{-a}.$$

This is (3.41) for $q = q_1$.

(ii) Suppose that $\mathfrak{p}^\theta | k$, where $\theta > 0$, that

$$\lambda = \alpha k + \mu > k \geqq \theta + 2, \qquad 0 < \mu \leqq k,$$

and

$$h = \mathfrak{p}^{\lambda-\theta-1} z + h'.$$

Then [22])

$$p h^k \equiv p h'^k + p z P h'^{k-1} \mathfrak{p}^{\lambda-1} \pmod{\mathfrak{p}^\lambda},$$

where $(P, \mathfrak{p}) = 1$. Hence

$$S_{p,\mathfrak{p}^\lambda} = \sum_{h=0}^{\mathfrak{p}^\lambda - 1} e\left(\frac{p h^k}{\mathfrak{p}^\lambda}\right) = \sum_{h'=0}^{\mathfrak{p}^{\lambda-\theta-1}-1} e\left(\frac{p h'^k}{\mathfrak{p}^\lambda}\right) \sum_{z=0}^{\mathfrak{p}^{\theta+1}-1} e\left(\frac{p z P h'^{k-1}}{\mathfrak{p}}\right).$$

The inner sum vanishes unless $h' = \mathfrak{p} h_1$, when it is $\mathfrak{p}^{\theta+1}$. Hence

$$S_{p,\mathfrak{p}^\lambda} = \mathfrak{p}^{\theta+1} \sum_{h_1=0}^{\mathfrak{p}^{\lambda-\theta-2}-1} e\left(\frac{p h_1^k}{\mathfrak{p}^{\lambda-k}}\right) = \mathfrak{p}^{k-1} \sum_{h_1=0}^{\mathfrak{p}^{\lambda-k}-1} e\left(\frac{p h_1^k}{\mathfrak{p}^{\lambda-k}}\right) = \mathfrak{p}^{k-1} S_{p,\mathfrak{p}^{\lambda-k}},$$

and

$$(3.46) \qquad \left| \frac{S_{p,\mathfrak{p}^\lambda}}{\mathfrak{p}^\lambda} \right| = (\mathfrak{p}^k)^{-a} \left| \frac{S_{p,\mathfrak{p}^{\lambda-k}}}{\mathfrak{p}^{\lambda-k}} \right| = (\mathfrak{p}^{\alpha k})^{-a} \left| \frac{S_{p,\mathfrak{p}^\mu}}{\mathfrak{p}^\mu} \right|.$$

This equation naturally holds also when $\lambda \leqq k$, with $\alpha = 0$, $\lambda = \mu$. From (3.46) we deduce

$$(3.47) \qquad \left| \frac{S_{p,q}}{q} \right| = \Pi \left| \frac{S_{p,\mathfrak{p}^\lambda}}{\mathfrak{p}^\lambda} \right| = q^{-a} \Pi \left| \frac{S_{p,\mathfrak{p}^\mu}}{\mathfrak{p}^{(1-a)\mu}} \right| < B q^{-a},$$

since the last product has at most B factors, all less than B. This disposes of case (ii), and completes the proof of the lemma.

3.5. **Lemma 4.** *If $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ is a set of C primes, and Q is a number formed from these primes only, then*

$$(3.51) \qquad \sum_{Q > \xi} Q^{-c} < C \xi^{-c} \qquad (\xi > 1).$$

For

$$\sum_{Q > \xi} Q^{-c} < \xi^{-\frac{1}{2}c} \sum_{Q > 0} Q^{-\frac{1}{2}c} < \xi^{-\frac{1}{2}c} \sum_{l_1, l_2, \ldots, l_r \geqq 0} (\mathfrak{p}_1^{l_1} \mathfrak{p}_2^{l_2} \cdots \mathfrak{p}_r^{l_r})^{-\frac{1}{2}c} < C \xi^{-\frac{1}{2}c}.$$

---

[21]) P. N. 4, 175.

[22]) P. N. 4, 166, Lemma 1.

3.6. **Lemma 5** [23]. *The series* $\Sigma \mathfrak{A}_q(m)$ *is absolutely convergent for* $m > 0$, $s \geqq k+1$. *Also*

$$\sum_{q>\nu} |\mathfrak{A}_q(m)| < C\nu^{-c}(d(m))^3 \qquad (s \geqq k+1), \tag{3.61}$$

$$|\mathfrak{S}_2(m)| < C\nu^{-c}(d(m))^3 \qquad (s \geqq k+1), \tag{3.62}$$

$$|\mathfrak{S}_2(m)| < C\nu^{2-sa} \qquad (s \geqq 2k+1). \tag{3.63}$$

The last result is easily proved, for, by (3.42),

$$|\mathfrak{S}_2(m)| < C \sum_{q>\nu} |\mathfrak{A}_q(m)| < C \sum_{q>\nu} q^{1-sa} < C\nu^{2-sa}$$

(since $1 - sa < -1$ if $s > 2k$).

The proof of (3.61), of which (3.62) is an obvious corollary, is much more elaborate. We write

$$q = q'Q = q_1 q_2 Q \tag{3.64}$$

where now q' is the product of all odd factors of q which are prime to k, and q_1 is the product of the primes which divide q' once only [24]; so that

$$\mathfrak{A}_q = \mathfrak{A}_{q_1} \mathfrak{A}_{q_2} \mathfrak{A}_Q. \tag{3.65}$$

In each term of (3.61), one at least of q_1, q_2, Q exceeds $\nu^{\frac{1}{3}}$; and if Σ_1, Σ_2, Σ_3 are the sums of the terms in which (i) $q_1 > \nu^{\frac{1}{3}}$, (ii) $q_2 > \nu^{\frac{1}{3}}$, and (iii) $Q > \nu^{\frac{1}{3}}$, then

$$\sum_{q>\nu} |\mathfrak{A}_q(m)| \leqq \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{3.66}$$

3.7. We write $q_1 = \Pi \mathfrak{p}_1$, $q_2 = \Pi \mathfrak{p}_2^{\lambda}$; and we make three preliminary observations. First

$$|\mathfrak{A}_q| < Cq^{-c} < C \tag{3.71}$$

for all q and m, by (3.12), and in particular when q is q_1, q_2, or Q. Next [25]

$$\mathfrak{A}_{\mathfrak{p}_2^{\lambda}}(m) = 0$$

unless $\mathfrak{p}_2^{\lambda-1} | m$, and *a fortiori* unless $\mathfrak{p}_2^{\lambda} | m^2$; so that $\mathfrak{A}_{q_2}(m) = 0$ unless $q_2 | m^2$. From this last remark it follows that not more than $d(m^2)$, and therefore not more than $(d(m))^2$, different values of q_2 can occur in non-zero terms of (3.61). Finally

$$|\mathfrak{A}_{q_1}| = \Pi |\mathfrak{A}_{\mathfrak{p}_1}| < \Pi(C\mathfrak{p}_1^{1-\frac{1}{2}s}) \leqq \Pi(C\mathfrak{p}_1^{-1}) < C\Pi\mathfrak{p}_1^{-\frac{1}{2}} = Cq_1^{-\frac{1}{2}},$$

---

[23]) Compare P. N. 5, 52, Lemma 6.

[24]) Thus, if $k = 15$ and $q = 2^2 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 11$, $q_1 = 11$, $q_2 = 7^2$, $Q = 2^2 \cdot 3 \cdot 5^2$.

[25]) P. N. 2, 20; P. N. 4, 170.

by (3.11). Hence, if $\xi > 0$, we have $|\mathfrak{A}_{q_1}| < C q_1^{-\frac{1}{8}} |\mathfrak{A}_{q_1}|^{\frac{3}{4}}$ and

$$(3.72) \quad \sum_{q_1 > \xi} |\mathfrak{A}_{q_1}| < C\xi^{-\frac{1}{8}} \sum_{q_1>0} |\mathfrak{A}_{q_1}|^{\frac{3}{4}} < C\xi^{-\frac{1}{8}} \prod_{\mathfrak{p}} (1 + |\mathfrak{A}_p|^{\frac{3}{4}})$$
$$< C\xi^{-\frac{1}{8}} \prod_{\mathfrak{p} \nmid m} (1 + C\mathfrak{p}^{-\frac{9}{8}}) \prod_{\mathfrak{p} \mid m} (1 + C\mathfrak{p}^{-\frac{3}{4}}) < C\xi^{-\frac{1}{8}} \prod_{\mathfrak{p}<C} C \prod_{\mathfrak{p}\mid m} 2$$
$$< C\xi^{-c} d(m).$$

In particular, taking $\xi = \frac{1}{2}$, we have

$$(3.721) \qquad \sum_{q_1>0} |\mathfrak{A}_{q_1}| < C d(m).$$

We can now write down upper bounds for Σ_1, Σ_2 and Σ_3. Consider first Σ_1. Associated with a given q_1, we have at most $(d(m))^2$ values of q_2; and $|\mathfrak{A}_{q_2}| < C$, $|\mathfrak{A}_Q| < CQ^{-c}$. Hence the sum of these terms of Σ_1 is less than

$$C(d(m))^2 |\mathfrak{A}_{q_1}| \sum_Q Q^{-c} < C(d(m))^2 |\mathfrak{A}_{q_1}|,$$

by (3.51); and

$$(3.73) \qquad \Sigma_1 < C(d(m))^2 \sum_{q_1 > \nu^{\frac{1}{3}}} |\mathfrak{A}_{q_1}| < C\nu^{-c} (d(m))^3,$$

by (3.72).

The discussion of Σ_2 is similar, but we use the inequality $|\mathfrak{A}_{q_2}| < Cq_2^{-c}$. We plainly obtain

$$(3.74) \qquad \Sigma_2 < C\nu^{-c}(d(m))^2 \sum_{q_1>0} |\mathfrak{A}_{q_1}| < C\nu^{-c}(d(m))^3,$$

by (3.721). Finally, the discussion of Σ_3 depends on the inequalities used for Σ_1. We have

$$(3.75) \quad \Sigma_3 < C(d(m))^2 \sum_{q_1>0} |\mathfrak{A}_{q_1}| \sum_{Q > \nu^{\frac{1}{3}}} Q^{-c}$$
$$< C(d(m))^3 \sum_{Q>\nu^{\frac{1}{3}}} Q^{-c} < C\nu^{-c}(d(m))^3,$$

by (3.51).

It is clear that (3.61) is a consequence of (3.66) and (3.73)—(3.75).

4. The behaviour of $f(x)$ on a major arc $\mathfrak{M}_1$.

4.1. Our object in this section is to prove

Lemma 6. *On a major arc* $\mathfrak{M}_1$

$$(4.11) \qquad |f - \varphi_{p,q}| < B_{\varepsilon,\delta}\, q^{\varkappa+\varepsilon},$$

so that

$$(4.12) \qquad |f - \varphi_{p,q}| < B_\varepsilon\, q^{\varkappa+\varepsilon}$$

if δ is chosen appropriately as a function of k and ε.

This lemma may be deduced from the analysis of P. N. 1, §§ 8.1—8.2. Every $\mathfrak{M}_1$ is part of an $\mathfrak{M}$ (since $a_1 < a$ and $1 - a_1 > 1 - a$), so that the formulae there given may be used. The sums there called Φ_2 and Φ_3 are of the form required, and it is a question only of the magnitude of Φ_1. In this sum we have now

$$\cos^2 \psi > B q^2 n^{-2a_1}, \text{[26]}$$

$$q^{-1-b} n^b (\cos\psi)^{1+b} > B q^{-1-b} n^b (q n^{-a_1})^{1+b} = B n^{(1+b)\delta} > B n^{\delta},$$

$$|E| < B e^{-B m^{1+b} n^{\delta}}.$$

It follows at once that Φ_1 is also of the form required, and indeed extremely small.

If this course were adopted, Lemmas 7—11, which are required only for the proof of Lemma 6, would become unnecessary, and our analysis would be appreciably shortened.

The objection to this procedure is that it involves the reinstatement of the whole of the analysis of §§ 7—8 of P. N. 1, which is eliminated entirely in Landau's version of the proof. If we are to preserve the simplifications introduced by Landau and Weyl, an independent proof of Lemma 6 is essential.

4.2. **Lemma 7.** *If*

(4.21) $$0 \leqq \mu \leqq q, \qquad m = 0, 1, 2, \ldots$$

then

(4.22) $$|U_{\mu,m}| = \left|\sum_{h=0}^{\mu} h^m e_q(p h^k)\right| < B_\varepsilon q^{m+\varkappa+\varepsilon}.$$

We have[27])

$$|U_{\mu,0}|^K < B_\varepsilon \mu^\varepsilon q^\varepsilon \left(\mu^{K-1} + \frac{\mu^K}{q} + q\mu^{K-k}\right) < B_\varepsilon q^{K-1+\varepsilon},$$

which proves the lemma when $m = 0$. And in general

$$|U_{\mu,m}| \leqq q^m \operatorname*{Max}_{0 \leqq \mu_1 \leqq \mu_2 \leqq q} \left|\sum_{h=\mu_1}^{\mu_2} e_q(p h^k)\right| < B_\varepsilon q^{m+\varkappa+\varepsilon}.$$

Lemma 8. *Suppose that* $0 \leqq h < q$,

(4.23) $$x = e^{-v}, \quad Y = q^k y, \quad t = \frac{h}{q}, \quad V = e^{-Y(v+t)^k}(v+t)^{k-1},$$

(4.24) $$f_h = \sum_{m=0}^{\infty} e^{-(mq+h)^k y} = \sum_{m=0}^{\infty} e^{-\lambda_m y};$$

---

[26]) We use for the moment the notation of P. N. 1, except naturally that, in conformity with the conventions laid down in § 1.1, we write B instead of A.

[27]) Landau, l. c. 226 (Hilfssatz 2).

and that $P(v)$, $P_1(v)$, $P_2(v)$, $\ldots$ *are polynomials in* $(v) = v - [v]$, *defined successively by*

$$\overline{P}(v) = v + 1 - (v) = v + P(v) = c_0 v + P(v), \tag{4.251}$$

$$\overline{P}_1(v) = \int_0^v P(w)\,dw = c_1 v + P_1(v), \tag{4.252}$$

$$\overline{P}_2(v) = \int_0^v P_1(w)\,dw = c_2 v + P_2(v), \quad \ldots, \tag{4.253}$$

so that $P(v) = 1 - (v)$, $P_1(v) = \frac{1}{2}(v) - \frac{1}{2}(v)^2$, $c_0 = 1$, $c_1 = \frac{1}{2}$, *and in general* c_ν *is a function of* ν *only. Then*

$$\begin{aligned} \frac{f_h}{kY} &= \sum_{\mu=0}^{\nu} (-1)^\mu c_\mu \int_0^\infty v V^{(\mu)}\,dv + (-1)^\nu \int_0^\infty P_\nu(v) V^{(\nu)}\,dv \\ &= \sum_{\mu=0}^{\nu} (-1)^\mu c_\mu I_\mu + J_\nu. \end{aligned} \tag{4.26}$$

This lemma merely embodies the definition of certain symbols, and the results of certain formal transformations required for the proof of Lemma 6.

If we define $N(x)$ by

$$N(x) = m + 1 \qquad (\lambda_m \leqq x < \lambda_{m+1}),$$

we have

$$f_h = \sum_0^\infty e^{-\lambda_m v} = y \int_{\lambda_0}^\infty N(u) e^{-uv}\,du.$$

Writing $u = (vq + h)^k = q^k (v + t)^k$, and observing that $N(u) = \overline{P}(v)$, we obtain

$$\begin{aligned} f_h &= kY \int_0^\infty \overline{P}(v) V\,dv = kY \int_0^\infty v V\,dv + kY \int_0^\infty \overline{P}_1'(v) V\,dv \\ &= kY \int_0^\infty v V\,dv - c_1 kY \int_0^\infty v V'\,dv - kY \int_0^\infty P_1(v) V'\,dv. \end{aligned}$$

This is (4.26) with $\nu = 1$, and it is plain that the general formula follows from a repetition of the argument.

Lemma 9. *If*

$$\mathsf{H} = \Re(Y) = \Re(q^k y) = \frac{q^k}{n}, \tag{4.27}$$

then

$$|Y| < \frac{|Y|}{\mathsf{H}^{1-a}} < A n^{-\delta} \tag{4.28}$$

on $\mathfrak{M}_1$.

For $\mathsf{H} \leqq n^{a_1 k-1} < 1$ and

$$\frac{|Y|}{\mathsf{H}^{1-a}} = qn^{1-a}|y| < An^{a_1-a} = An^{-\delta}.$$

4.3. **Lemma 10.** *We can choose* $\nu_0 = \nu_0(k, \delta)$ *so that*

$$|kYJ_\nu| = \left|kY\int_0^\infty P_\nu V^{(\nu)}\,dv\right| < \frac{B_\delta}{n} \tag{4.31}$$

for $\nu = \nu_0$.

Since $|kY| < Ak < B$, we may ignore the outside factor. We take $\nu = \nu_1 k$, where $\nu_1 > 2$.

We write

$$V^{(\nu)} = e^{-Y(v+t)^k}\sum_{l,m} c_{k,l,m}\,Y^l(v+t)^m.$$

Every differentiation of the exponential factor of V introduces a factor Y, and a term in Y^l can be generated only by l such differentiations together with $\nu - l$ which do not bear upon the exponential. Hence the term in Y^l is a multiple of

$$\left(\frac{d}{dv}\right)^{\nu-l}[(-kY)^l(v+t)^{(k-1)(l+1)}],$$

and is zero unless $(k-1)(l+1) \geqq \nu - l$ or

$$l \geqq \frac{\nu-k+1}{k} > \nu_1 - 1.$$

That is to say, ν_1 is the least possible value of l. Associated with a given l we may have values of m from 0 to $(k-1)(l+1)$.

It follows that

$$|V^{(\nu)}| < B_\nu e^{-\mathsf{H}(v+t)^k}\sum_{l=\nu_1}^{\nu}|Y|^l(1+|v+t|^{(k-1)(l+1)}),$$

and so

$$\begin{aligned}\left|\int_0^\infty P_\nu V^{(\nu)}\,dv\right| &< B_\nu\sum_{l=\nu_1}^{\nu}|Y|^l\int_0^\infty e^{-\mathsf{H}(v+t)^k}(1+|v+t|^{(k-1)(l+1)})\,dv\\ &< B_\nu\sum_{l=\nu_1}^{\nu}|Y|^l(\mathsf{H}^{-a}+\mathsf{H}^{-(l+1)(1-a)-a})\\ &< B_\nu\mathsf{H}^{-a}\sum_{l=\nu_1}^{\nu}|Y|^l + \frac{B_\nu}{\mathsf{H}}\sum_{l=\nu_1}^{\nu}\left(\frac{|Y|}{\mathsf{H}^{1-a}}\right)^l\\ &< B_{\nu,\delta}\mathsf{H}^{-1}n^{-\nu_1\delta} < B_{\nu,\delta}n^{1-\nu_1\delta} < \frac{B_\delta}{n}\end{aligned}$$

(by (4.28)), if $\nu_1 = \nu_{1,0}(k,\delta)$, i. e. if $\nu = \nu_0(k,\delta)$.

It should be observed that we can always replace $\nu_0(k,\delta)$ by a larger number of the same type.

4.4. We understand by $Q(Y,t)$ a polynomial in Y and t, whose degree in either variable has an upper bound of the type B_δ and whose coefficients have upper bounds of the same type. Since $|Y|<A$, $0 \leqq t < 1$, we have always $|Q(Y,t)| < B_\delta$. And we understand by R a number whose absolute value is less than $\frac{B_\delta}{n}$.

Lemma 11. *We can choose* $\nu_0 = \nu_0(k,\delta)$ *so that, in addition to* (4.31), *we have*

$$kYI_\mu = kY\int_0^\infty vV^{(\mu)}\,dv = Q(Y,t) + R \tag{4.41}$$

for $1 \leqq \mu \leqq \nu = \nu_0$, *and*

$$kYI_0 = kY\int_0^\infty vV\,dv = \Gamma(1+a)\,Y^{-a} + Q(Y,t) + R. \tag{4.42}$$

We have

$$e^{-Yt^k} = \sum_{m=0}^{m_1} \frac{(-Yt^k)^m}{m!} + r,$$

where

$$|r| < A\,|Y|^{m_1} < A\,(A n^{-\delta})^{m_1} < \frac{B_\delta}{n},$$

if $m_1 = m_1(k,\delta)$ is sufficiently large; and so

$$e^{-Yt^k} = Q(Y,t) + R.$$

If $\mu > 1$

$$\begin{aligned} kYI_\mu &= kY\int_0^\infty vV^{(\mu)}\,dv = kYV^{(\mu-2)}(0) = kYe^{-Yt^k}\,Q(Y,t) \\ &= kYQ(Y,t)\,(Q(Y,t)+R) = Q(Y,t) + R; \end{aligned}$$

while if $\mu = 1$

$$kYI_1 = kY\int_0^\infty vV'\,dv = -kY\int_0^\infty V\,dv = e^{-Yt^k} = Q(Y,t) + R.$$

Finally, if $\mu = 0$,

$$\begin{aligned} kYI_0 &= kY\int_0^\infty v(v+t)^{k-1}e^{-Y(v+t)^k}\,dv \\ &= \int_0^\infty e^{-Y(v+t)^k}\,dv = \int_t^\infty e^{-Yw^k}\,dw = \Gamma(1+a)\,Y^{-a} - \int_0^t e^{-Yw^k}\,dw \\ &= \Gamma(1+a)\,Y^{-a} - \int_0^t (Q(Y,w)+R)\,dw \\ &= \Gamma(1+a)\,Y^{-a} + Q(Y,t) + R. \end{aligned}$$

4.5. We can now prove Lemma 6. We have

$$(4.51)\quad f = -1 + 2\sum_{h=0}^{q-1} e_q(ph^k) \sum_{m=0}^{\infty} e^{-\lambda_m y} = -1 + 2\sum_h e_q(ph^k) f_h.$$

It follows from (4.26), (4.31), (4.41) and (4.42) that

$$(4.52)\quad f_h = \Gamma(1+a) Y^{-a} + Q(Y, t) + R$$

when $\nu = \nu_0$. Combining this with (4.51), we find

$$(4.53)\quad f = 2\Gamma(1+a) S_{p,q} Y^{-a} + \varrho = \varphi_{p,q} + \varrho,$$

where

$$(4.541)\quad \varrho = \varrho_1 + \varrho_2 + \varrho_3, \quad \varrho_1 = -1, \quad \varrho_2 = \sum_h e_q(ph^k) R,$$

$$(4.542)\quad \varrho_3 = \sum_h e_q(ph^k) Q\left(Y, \frac{h}{q}\right).$$

Now

$$(4.55)\quad |\varrho_2| \leqq \sum_h |R| < B_\delta \frac{q}{n} < B_\delta.$$

Further,

$$Q\left(Y, \frac{h}{q}\right) = \sum_{l,m} c_{k,l,m} Y^l \left(\frac{h}{q}\right)^m,$$

where l, m, and the coefficients have upper bounds B_δ. And

$$(4.56)\quad |\varrho_3| = \left| \sum_h e_q(ph^k) \sum_{l,m} c_{k,l,m} Y^l \left(\frac{h}{q}\right)^m \right|$$
$$= \left| \sum_{l,m} c_{k,l,m} Y^l q^{-m} \sum_h h^m e_q(ph^k) \right| < B_{\varepsilon,\delta}\, q^{\varkappa+\varepsilon},$$

by Lemma 7. Lemma 6 follows from (4.53) to (4.56).

5. Further lemmas preliminary to the proof of Theorems 1—4.

5.1. **Lemma 12.** *If* $|z| < 1$ *and*

$$-\pi + C < \Im \log z = \arg z < \pi - C$$

then

$$(5.11)\quad |g(z)| < C |\arg z|^{-sa},$$

$$(5.12)\quad \left| g(z) - \Gamma(sa)\left(\log \frac{1}{z}\right)^{-sa} \right| < C.$$

These are known results.

Lemma 13. *We have*

$$(5.13)\quad \sum \int_{\xi_{p,q}} |F_{p,q} - \varphi_{p,q}^s|^2 |d\theta| < C.$$

This is trivial; for, if $z = x\,e_q(-p) = e^{-\mathsf{Y}}$, we have $y = \log\frac{1}{z}$ and

$$|F_{p,q} - \varphi^s_{p,q}| = \left|\mathfrak{C}\left(\frac{S_{p,q}}{q}\right)^s \{g(x\,e_q(-p)) - \Gamma(sa)\,y^{-sa}\}\right| < C,$$

by Lemma 12.

Lemma 14. *We have*

(5.14) $$\sum_{\mathfrak{M}_1}\int |d\theta| < A n^{2a_1-1} < A n^{2a-1}.$$

For the sum is less than

$$A\sum_{q \leqq n^{a_1}}\sum_p \frac{1}{q n^{1-a_1}} < A n^{a_1-1}\sum_{q \leqq n^{a_1}} 1 < A n^{2a_1-1}.$$

Lemma 15. *If* $s \geqq k+1$, $\nu > 1$, *then*

(5.15) $$\sum_{q>\nu}\sum_p \int_{\xi_{p,q}} |\varphi_{p,q}|^{2s}\,|d\theta| < C\nu^{2-2sa}\,n^{2sa-1}.$$

For

$$\int_\xi |\varphi|^{2s}\,|d\theta| = C\left|\frac{S_{p,q}}{q}\right|^{2s}\int_\xi |y|^{-2sa}\,|d\theta|,$$

and

$$\int_\xi |y|^{-2sa}\,|d\theta| < \int_{-\infty}^{\infty}\frac{d\theta}{\left(\frac{1}{n^2}+\theta^2\right)^{sa}} = n^{2sa-1}\int_{-\infty}^{\infty}\frac{du}{(1+u^2)^{sa}} < n^{2sa-1}\int_{-\infty}^{\infty}\frac{du}{1+u^2} < A n^{2sa-1}.$$

Hence, by Lemma 3,

$$\sum_{q>\nu}\sum_p\int_\xi |\varphi|^{2s}\,|d\theta| < C n^{2sa-1}\sum_{q>\nu} q^{1-2sa} < C\nu^{2-2sa}\,n^{2sa-1}.$$

Lemma 16. *If* $\nu = n^\beta$, *where* $\beta = \beta(k,s) > 0$,[27] *then*

(5.16) $$\int_0^{2\pi} |F_2|^2\,d\psi < C n^{2sa-1-c}$$

if $s \geqq k+1$; *and*

(5.17) $$\int_0^{2\pi} |F_2|^2\,d\psi < C n^{2sa-1-\lambda_1},$$

where

(5.18) $$\lambda_1 = 2\beta(sa-2),$$

if $s \geqq 2k+1$.

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[27]) β is ultimately chosen in three different ways to suit three different arguments.. In each case it is chosen as a function of k and s only. We anticipate this choice and allow ourselves to treat β as a c.

(i) The integral is

$$2\pi \sum (\mathfrak{S}_2(m))^2 m^{2sa-2} |x|^{2m} < C n^{-c} \sum m^{2sa-2} (d(m))^6 |x|^{2m}$$
$$< C_\varepsilon n^{-c} \sum m^{2sa-2+\varepsilon} |x|^{2m} < C_\varepsilon n^{2sa-1-c+\varepsilon},$$

by Lemma 5 (3.62). The inequality (5.16) follows if we take ε to be half the c which precedes it, and then replace c by $2c$.

(ii) If $s \geqq 2k+1$, the integral is also less than

$$C\nu^{4-2sa} \sum m^{2sa-2} |x|^{2m} < C\nu^{4-2sa} n^{2sa-1} < C n^{2sa-1-\lambda_1},$$

by Lemma 5 (3.63).

5.2. Our next lemma, which concerns the value of $F_{P,Q}$ when x lies on an arc $\xi_{p,q}$, requires a word of preliminary explanation. The function $F_{P,Q}$ and the arc $\xi_{P,Q}$ are invariant if P is altered by a multiple of Q, and we obtain the complete set of arcs $\xi_{P,Q}$, for a given Q, if P runs through any complete set of residues prime to Q. In what follows, p, q, Q being fixed, we select for our set of residues that for which

$$Q\left(\frac{p}{q} - \frac{1}{2}\right) \leqq P < Q\left(\frac{p}{q} + \frac{1}{2}\right)$$

(and of course $P = 0$ if $Q = 1$). From this it follows that

$$\left|\frac{p}{q} - \frac{P}{Q}\right| \leqq \frac{1}{2}.$$

Lemma 17[28]). *Suppose that x lies on $\xi_{p,q}$ and that $\xi_{P,Q}$ is a Farey arc other than $\xi_{p,q}$. Then*

(5.21) $$|F_{P,Q}| < C N_1^{sa} |pQ - qP|^{-sa}.$$

We have

(5.22) $$x = |x| e^{i\varphi} = |x| e_q(p) e^{-i\theta} = |x| e_Q(P) e^{-i\omega},$$

where

(5.221) $$\omega = 2\pi\left(\frac{P}{Q} - \frac{p}{q}\right) + \theta,$$

so that $|\omega| \leqq \pi + |\theta| < \frac{3}{2}\pi$ if $N_1 \geqq 4$. Hence

(5.23) $$|F_{P,Q}| = \left|\mathfrak{C}\left(\frac{S_{P,Q}}{Q}\right)^s g(|x| e^{-i\omega})\right| < C Q^{-sa} |\omega|^{-sa},$$

by Lemma 12 (5.11) and Lemma 3 (3.41).

If p, q, and P, Q do not correspond to arcs adjacent in the dissection then

$$\left|2\pi\left(\frac{p}{q} - \frac{P}{Q}\right)\right| \geqq \frac{4\pi}{qQ} \geqq 2\,\frac{2\pi}{qN_1} \geqq 2|\theta|,$$

[28]) Compare P. N. 5, 52 (Lemma 10).

and so

(5.24) $$|\omega| > A\left|\frac{p}{q} - \frac{P}{Q}\right| = \frac{A}{qQ}|pQ - qP|.$$

As $q \leqq N_1$, (5.21) follows from (5.23) and (5.24).

This argument fails when the arcs are adjacent. In this case $|pQ - qP| = 1$. As x is not on $\xi_{P,Q}$,

$$|\omega| > \frac{A}{QN_1} = \frac{A}{QN_1}|pQ - qP|,$$

so that the conclusion still follows.

Lemma 18[29]**.** *If*

(5.25) $$G = \overline{\sum}|F_{P,Q}|,$$

where the sign $\overline{\Sigma}$ implies summation over those pairs (P, Q), distinct from (p, q), for which $Q \leqq \nu = n^\beta$, and $s \geqq k+1$, then

(5.26) $$\sum_{\xi_{p,q}} \int G^2 |d\theta| < C_\varepsilon n^{2sa-1-\lambda_2+\varepsilon+l},$$

where

(5.261) $$\lambda_2 = a(2sa - 1) - 4\beta,$$

and $l = l(k, s, \delta)$ tends to zero with δ; so that

(5.27) $$\sum_{\xi_{p,q}} \int G^2 |d\theta| < C_\varepsilon n^{2sa-1-\lambda_2+\varepsilon}$$

if $\delta = \delta(k, s, \varepsilon)$ is sufficiently small.

Since G is a sum of less than ν^2 terms, we have

$$G^2 < \nu^2 \overline{\sum}|F_{P,Q}|^2 < C\nu^2 N_1^{2sa} \overline{\sum}|pQ - qP|^{-2sa}$$

by Lemma 17. Hence

$$\sum_{\xi_{p,q}} \int G^2 |d\theta| \leqq C\nu^2 N_1^{2sa-1} \sum_{p,q} \frac{1}{q} \overline{\sum_{P,Q}} |pQ - qP|^{-2sa}$$

$$\leqq C\nu^2 N_1^{2sa-1} \sum_{P,Q} \sum_{p,q} \frac{1}{q} |pQ - qP|^{-2sa},$$

where now the inner summation is defined by $q \leqq N_1$, $pQ - qP \neq 0$, and the outer summation by $Q \leqq \nu$. But, if $Pq \equiv h \pmod{Q}$, where $0 \leqq h < Q$, we have

$$\sum_p |pQ - qP|^{-2sa} < \sum_{m=-\infty}^{\infty}{}' |mQ + h|^{-2sa} \text{ [30]} < 2 + 2Q^{-2sa} \sum_{m=1}^{\infty} m^{-2sa} < A,$$

[29] Compare P. N. 5, 52 (Lemma 11).

[30] The dash denoting that any term for which $mQ + h = 0$ is to be omitted.

and so

$$(5.28)\quad \sum_{\xi}\int G^2|d\theta| < C\nu^2 N_1^{2sa-1}\sum_{P,Q}\sum_{q}\frac{1}{q} < C\nu^4 N_1^{2sa-1}\log(N_1+2)$$
$$< Cn^{(2sa-1)(1-a_1)+4\beta}\log n < C_\varepsilon n^{2sa-1-\lambda_2+\varepsilon+l},$$

where $l=(2sa-1)\delta$. This proves (5.26). To prove (5.27), we choose δ so that $l<\varepsilon$, and then replace 2ε by ε.

5.3. **Lemma 19.** *If* $s\geqq k+1$, $\nu=n^\beta$, *and* $\beta<a_1$, *then*

$$(5.31)\quad \sum\int_{\mathfrak{M}_1}|F_1-\varphi^s|^2|d\theta| < C_\varepsilon n^{2sa-1-\lambda_3+\varepsilon+l} < C_\varepsilon n^{2sa-1-\lambda_3+\varepsilon},$$

$$(5.32)\quad \sum\int_{\mathfrak{m}_1}|F_1|^2|d\theta| < C_\varepsilon n^{2sa-1-\lambda_3+\varepsilon+l} < C_\varepsilon n^{2sa-1-\lambda_3+\varepsilon}.$$

Here

$$(5.33)\quad \lambda_3=\mathrm{Min}(2\beta(sa-1),\lambda_2)\geqq \mathrm{Min}(\lambda_1,\lambda_2),$$

λ_1 *and* λ_2 *being defined by* (5.18) *and* (5.261); $l=l(k,s,\delta)$ *tends to zero with* δ; *and the second inequality is in each case secured by an appropriate choice of* δ.

(i) We have

$$F_1=F_{p,q}+\overline{\sum}F_{P,Q},$$

$$(5.341)\quad |F_1-\varphi^s|^2\leqq 2|F_{p,q}-\varphi^s|^2+2G^2,$$

if $q\leqq\nu$, and

$$F_1=\overline{\sum}F_{P,Q},$$

$$(5.342)\quad |F_1-\varphi^s|^2\leqq 2|\varphi|^{2s}+2G^2,$$

if $q>\nu$. Hence

$$(5.35)\quad \sum\int_{\mathfrak{M}_1}|F_1-\varphi^s|^2|d\theta|$$
$$\leqq 2\sum_{q\leqq\nu}\int_{\mathfrak{M}_1}|F_{p,q}-\varphi^s|^2|d\theta|+2\sum_{q>\nu}\int_{\mathfrak{M}_1}|\varphi|^{2s}|d\theta|+2\sum\int_{\mathfrak{M}_1}G^2|d\theta|$$
$$\leqq 2\sum\int_{\xi}|F_{p,q}-\varphi^s|^2|d\theta|+2\sum_{q>\nu}\int_{\xi}|\varphi|^{2s}|d\theta|+2\sum\int_{\xi}G^2|d\theta|$$
$$< C+Cn^{2sa-1-2\beta(sa-1)}+C_\varepsilon n^{2sa-1-\lambda_2+\varepsilon+l},$$

by Lemmas 13, 15 and 18. As $\lambda_1<2\beta(sa-1)$ and $\lambda_3\leqq\lambda_2<2sa-1$, (5.31) follows.

(ii) The proof of (5.32) is similar, but rather simpler. Since $\beta<a_1$, $q>\nu$ on every $\mathfrak{m}_1$. Hence

$$(5.36)\quad \sum\int_{\mathfrak{m}_1}|F_1|^2|d\theta|\leqq 2\sum_{q>\nu}\int_{\xi}|\varphi|^{2s}|d\theta|+2\sum_{q>\nu}\int_{\xi}G^2|d\theta|,$$

and the conclusion follows as before.

5.4. **Lemma 20.** *If* $s \geqq 2$

(5.41) $$s_0 = s_0(k) = \left(\frac{1}{2}k - 1\right)K + 2,$$

and

(5.42) $$\lambda_4 = \frac{2a}{K}(s - s_0),$$

then

(5.43) $$\sum\int_{\mathfrak{m}_1} |f|^{2s} |d\theta| < C_\varepsilon n^{2sa-1-\lambda_4+\varepsilon+l} < C_\varepsilon n^{2sa-1-\lambda_4+\varepsilon},$$

the meaning of l and of the double inequality being the same as before.

It is known[31]) that

$$|f| < B_\varepsilon n^{a - \frac{a_1}{K} + \varepsilon}$$

on $\mathfrak{m}_1$. Hence

$$\sum\int_{\mathfrak{m}_1} |f|^{2s} |d\theta| < C_\varepsilon n^{(2s-4)\left(a - \frac{a_1}{K}\right) + \varepsilon} \sum\int_{\mathfrak{m}_1} |f|^4 |d\theta|$$

$$< C_\varepsilon n^{(2s-4)\left(a - \frac{a_1}{K}\right) + 2a + \varepsilon}. \;^{32})$$

But

$$(2s - 4)\left(a - \frac{a_1}{K}\right) + 2a = 2sa - 1 - \mu,$$

where

$$\mu = \frac{2a}{K}\left(s - \frac{1}{2}kK + K - 2\right) - \frac{(2s-4)\delta}{K} = \lambda_4 - l,$$

which proves the lemma.

5.5. **Lemma 21.** (i) *If* $s \geqq 2k + 1$ *then*

(5.51) $$\sum\int_{\mathfrak{M}_1} |f^s - \varphi^s|^2 |d\theta| < C_{\varepsilon,\delta} n^{2sa-1-\lambda_6+\varepsilon} < C_\varepsilon n^{2sa-1-\lambda_6+\varepsilon},$$

where

(5.511) $$\lambda_6 = \mathrm{Min}(\lambda_4, \lambda_5) = \mathrm{Min}(\lambda_4, 2a),$$

and the second inequality is secured by choice of δ.

(ii) *If* $s \geqq k + 1$ *then*

(5.52) $$\sum\int_{\mathfrak{M}_1} |f^s - \varphi^s|^2 |d\theta| < C_{\varepsilon,\delta} n^{2sa-1-\lambda_6'+\varepsilon} < C_\varepsilon n^{2sa-1-\lambda_6'+\varepsilon},$$

where

(5.521) $$\lambda_6' = \mathrm{Min}(\lambda_4, \lambda_5') = \mathrm{Min}\left(\lambda_4, \frac{2a}{K}\right),$$

and the second inequality is secured by choice of δ.

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[31]) Landau, l. c. 230 (Hilfssatz 4).

[32]) See P. N. 2, 16—17; Landau, l. c. 241.

We have

$$|f^s-\varphi^s|^2 < A_s|f-\varphi|^{2s} + A_s|f-\varphi|^2|\varphi|^{2s-2},$$

(5.53) $$\sum\int_{\mathfrak{M}_1}|f^s-\varphi^s|^2|d\theta| < A_s\sum\int_{\mathfrak{M}_1}|f-\varphi|^{2s}|d\theta| + A_s\sum\int_{\mathfrak{M}_1}|f-\varphi|^2|\varphi|^{2s-2}|d\theta| = A_s\varrho_1 + A_s\varrho_2,$$

say. Now

$$|f-\varphi| < B_{\varepsilon,\delta}\,q^{\varkappa+\varepsilon} < B_{\varepsilon,\delta}\,n^{a_1(\varkappa+\varepsilon)} < B_{\varepsilon,\delta}\,n^{a(\varkappa+\varepsilon)}$$

by Lemma 6, and so

$$\varrho_1 < C_{\varepsilon,\delta}\,n^{2sa\varkappa+\varepsilon}\sum\int_{\mathfrak{M}_1}|d\theta| < C_{\varepsilon,\delta}\,n^{2sa\varkappa+2a-1+\varepsilon},$$

by Lemma 14. Now

$$2sa\varkappa + 2a - 1 \leqq 2sa - 1 - \lambda_4 = 2sa\varkappa - 4a\varkappa + 2a$$

if $4a\varkappa \leqq 1$, which is true (with inequality except when $k=3$). Hence

(5.54) $$\varrho_1 < C_{\varepsilon,\delta}\,n^{2sa-1-\lambda_4+\varepsilon}.$$

Again, using Lemma 6 once more, we have

$$\varrho_2 < \sum\int_{\mathfrak{M}_1} B_{\varepsilon,\delta}\,q^{2\varkappa+\varepsilon}\cdot C\left|\frac{S_{p,q}}{q}\right|^{2s-2}|y|^{-2a(s-1)}\cdot|d\theta|.$$

Now

$$\left|\frac{S_{p,q}}{q}\right| < Bq^{-a}$$

by Lemma 3, and

$$\int_{\mathfrak{M}_1}|y|^{-2a(s-1)}|d\theta| < An^{2a(s-1)-1}$$

(as in the proof of Lemma 15). Hence

(5.55) $$\varrho_2 < C_{\varepsilon,\delta}\,n^{2a(s-1)-1}\sum_{q\leqq n^{a_1}} q^{2\varkappa+1-2a(s-1)+\varepsilon},$$

and so also, replacing $q^{2\varkappa+\varepsilon}$ by its upper bound $n^{2a_1\varkappa+\varepsilon}$,

(5.56) $$\varrho_2 < C_{\varepsilon,\delta}\,n^{2a(s-1)-1+2a_1\varkappa+\varepsilon}\sum_{q\leqq n^{a_1}} q^{1-2a(s-1)}.$$

In order to establish the complete lemma it is enough, after (5.54), to prove that

(5.57) $$\varrho_2 < C_{\varepsilon,\delta}\,n^{2sa-1-2a+\varepsilon} \qquad (s\geqq 2k+1)$$

and

(5.58) $$\varrho_2 < C_{\varepsilon,\delta}\,n^{2sa-1-\frac{2a}{K}+\varepsilon} \qquad (s\geqq k+1).$$

Now $2\varkappa+1-2a(s-1) < 3-4ak = -1$ if $s\geqq 2k+1$, and then (5.57) follows from (5.55). Finally, we may replace a_1 by a in (5.56);

observing then that $1-2a(s-1) \leqq 1-2ak=-1$ when $s \geqq k+1$, (5. 56) gives

$$\varrho_2 < C_{\varepsilon,\delta}\, n^{2as-\frac{2a}{K}+\varepsilon} \log n < C_{\varepsilon,\delta}\, n^{2as-\frac{2a}{K}+\varepsilon},$$

which is (5. 58).

6. Proofs of Theorems 1—3.

6. 1. **Proof of Theorem 1.** We have

$$1+\sum \sigma(m)\, x^m = (f(x))^s - F(x,k,s) = f^s - F,$$

$$1+\sum (\sigma(m))^2 |x|^{2m} = \frac{1}{2\pi}\int_0^{2\pi} |f^s - F|^2\, d\psi;$$

and so

$$(6.11) \quad \sum_{m=1}^{n} (\sigma(m))^2 \leqq e^2 \sum_{m=1}^{n} (\sigma(m))^2 e^{-\frac{2m}{n}} < A\int_0^{2\pi} |f^s - F|^2\, d\psi$$

$$\leqq A\Big(\sum \int_{\mathfrak{M}_1} |f^s - \varphi^s|^2 |d\theta| + \sum \int_{\mathfrak{M}_1} |F_1 - \varphi_s|^2 |d\theta|$$

$$+ \sum \int_{\mathfrak{m}_1} |f|^{2s} |d\theta| + \sum \int_{\mathfrak{m}_1} |F_1|^2 d\theta + \int_0^{2\pi} |F_2|^2 d\psi\Big)$$

$$= A(J_1 + J_2 + J_3 + J_4 + J_5),$$

say. Upper bounds for all these sums have been found in the preceding lemmas. Thus, when δ is chosen appropriately, we have

$$(6.121) \qquad J_1 < C_\varepsilon n^{2sa-1-\lambda_6+\varepsilon} \qquad (s \geqq 2k+1),$$

$$(6.122) \qquad J_1 < C_\varepsilon n^{2sa-1-\lambda_6'+\varepsilon} \qquad (s \geqq k+1),$$

by Lemma 21;

$$(6.13) \qquad J_2 < C_\varepsilon n^{2sa-1-\lambda_3+\varepsilon} \qquad (s \geqq k+1)$$

by Lemma 19 (5. 31);

$$(6.14) \qquad J_3 < C_\varepsilon n^{2sa-1-\lambda_1+\varepsilon} \qquad (s \geqq 2)$$

by Lemma 20;

$$(6.15) \qquad J_4 < C_\varepsilon n^{2sa-1-\lambda_3+\varepsilon} \qquad (s \geqq k+1),$$

by Lemma 19 (5. 32); and

$$(6.161) \qquad J_5 < C n^{2sa-1-\lambda_1} \qquad (s \geqq 2k+1),$$

$$(6.162) \qquad J_5 < C n^{2sa-1-\gamma} \qquad (s \geqq k+1),$$

by Lemma 16, γ being the particular c which occurs in that lemma. Hence, on the one hand,

(6.17) $$\sum_{m=1}^{n}(\sigma(m))^2 < C_\varepsilon n^{2sa-1-\lambda+\varepsilon} \qquad (s \geqq 2k+1),$$

where

(6.171) $$\lambda = \operatorname{Min}(\lambda_1, \lambda_3, \lambda_4, \lambda_6) \geqq \operatorname{Min}(\lambda_1, \lambda_2, \lambda_4, \lambda_5),$$

(6.172) $$\lambda_1 = 2\beta(sa-2), \quad \lambda_2 = a(2sa-1) - 4\beta, \quad \lambda_4 = \frac{2a}{K}(s-s_0),$$
$$\lambda_5 = 2a,$$

and on the other

(6.18) $$\sum_{m=1}^{n}(\sigma(m))^2 < C_\varepsilon n^{2sa-1-\lambda'+\varepsilon} \qquad (s \geqq k+1),$$

where

(6.181) $$\lambda' = \operatorname{Min}(\gamma, \lambda_3, \lambda_4, \lambda_6') = \operatorname{Min}(\gamma, 2\beta(sa-1), \lambda_2, \lambda_4, \lambda_5'),$$

(6.182) $$\lambda_5' = \frac{2a}{K}.$$

The inequality (6.17) is much better than (6.18) when it is valid, that is, when $s \geqq 2k+1$; and it is essential to the proof of Theorem 4. But (6.17) is not valid for $s > s_0$ when $k=3$, since $s_0 = 4$, $2k+1 = 7$; and we therefore use (6.18) in proving Theorem 1.

We note first that $s_0 \geqq k+1$; for if $k=3$, $s_0=4$, and if $k>3$, $s_0 \geqq 2^{k-1}+2 > k+1$. Hence (6.18) holds for $s > s_0$. Also

$$2\beta(sa-1) \geqq 2a\beta, \qquad \lambda_2 = a(2sa-1) - 4\beta > a - 4\beta, \qquad \lambda_4 \geqq \frac{2a}{K}.$$

If we take $\beta = \frac{1}{8}a$ (and suppose δ small enough to secure that $\beta < a_1$), the numbers on the right hand sides of these inequalities are $\frac{1}{4}$, $\frac{1}{2}a$, $\frac{2a}{K}$. All these numbers are c's (indeed independent of s), and so is γ. Hence we may write c for λ' in (6.18). If finally we chose ε appropriately as a function of k and s, we obtain (1.22), and the proof of Theorem 1 is completed.

6.2. **Proof of Theorem 2.** If

(6.21) $$s \geqq s_1 = \operatorname{Max}(\Gamma(k), 4)$$

then, by Lemma 2,

(6.22) $$\varrho(m) > C m^{sa-1}.$$

Now $(\frac{1}{2}k-1)K+3 \geqq 4k \geqq \Gamma(k)$ except for $k=3$ and $k=4$, while $(\frac{1}{2}k-1)K+3 = 5 > 4 = \Gamma(k)$ for $k=3$. Hence (6.22) is true in all cases considered in Theorem 2.

If $r(m) = 0$, then $|\sigma(m)| = \varrho(m) > Cm^{sa-1}$. If then $\mu(n)$ is the number of non-representable numbers between $\frac{1}{2}n$ (exclusive) and n (inclusive), we have

$$\sum_{\frac{1}{2}n}^{n} (\sigma(m))^2 \geqq C\left(\frac{1}{2}n\right)^{2sa-2} \mu(n)$$

and so

$$\mu(n) \leqq C 2^{2sa-2} n^{2-2sa} \sum_{1}^{n} (\sigma(m))^2 < C n^{1-c},$$

by (1. 22). The total number of non-representable numbers less than n is therefore less than

$$C n^{1-c}\left(1 + \left(\tfrac{1}{2}\right)^{1-c} + \left(\tfrac{1}{4}\right)^{1-c} + \ldots\right) < C n^{1-c}.$$

This proves Theorem 2.

6. 3. **Proof of Theorem 3.** When $k = 4$

$$\left(\frac{1}{2}k - 1\right)K + 3 = 11 < 16 = \Gamma(k),$$

and the preceding argument fails. In this case the conclusion of Theorem 2 is false.

Suppose that

$$s = 15, \quad t > 0, \quad \frac{1}{2}n < m \leqq n,$$

and write

$$m = m' \qquad (16^t \nmid m), \qquad m = m'' \qquad (16^t \mid m).$$

Further, suppose that a zero suffix attached to a number implies that it is not representable by 15 biquadrates, and that $\mu(n)$, $\mu'(n)$, $\mu''(n)$ are the numbers of the classes m_0, m_0', m_0''.

We have[33])

$$(6.31) \qquad \varrho(m') > A_t (m')^{\frac{11}{4}}$$

and so

$$|\sigma(m_0')| > A_t (m_0')^{\frac{11}{4}} > A_t n^{\frac{11}{4}}.$$

Hence

$$A_t n^{\frac{11}{2}} \mu'(n) \leqq \sum_{\frac{1}{2}n}^{n} (\sigma(m))^2 < A n^{\frac{13}{2}-a}, \quad ^{34})$$

$$(6.32) \qquad \mu(n) = \mu'(n) + \mu''(n) < n(A_t n^{-a} + 16^{-t}).$$

[33]) P. N. 4, 179.

[34]) We use a momentarily for an indicial A (*i.e.* an absolute c). We may plainly suppose $a < 1$.

If $\nu(n)$ is the total number of non-representable numbers less than n, then, by (6.32),

$$\nu(n) = \mu(n) + \mu\left(\frac{1}{2}n\right) + \mu\left(\frac{1}{4}n\right) + \dots$$
$$< n\left(\frac{2^{1-a}}{2^{1-a}-1} A_t n^{-a} + 2\cdot 16^{-t}\right) < \varepsilon n,$$

by choice first of t and then of n. This proves Theorem 3.

7. Proof of Theorem 4.

7.1. We denote by

$$N_s(n) = N_{k,s}(n)$$

the number of distinct numbers not exceeding n and representable by s or fewer positive k-th powers, and by

$$\nu_s(n) = \nu_{k,s}(n) = n + 1 - N_{k,s}(n)$$

the number not so representable; and by

$$\alpha_s = \alpha_s(k)$$

the largest number ξ such that

$$N_s(n) > C_\varepsilon n^{\xi-\varepsilon}$$

for $n \geqq n_0(k, s, \varepsilon)$. It is plain that α_s exists, does not exceed 1, and is a non-decreasing function of s. Also, as there are $[n^a]+1$ k-th powers which do not exceed n, we have $\alpha_s \geqq a$ for $s \geqq 1$.

Lemma 22. *We have*

$$\alpha_s \geqq 1 - (1-2a)(1-a)^{s-2} \qquad (s \geqq 2). \tag{7.11}$$

First, suppose $s = 2$. Then (7.11) is $\alpha_2 \geqq 2a$, and is an obvious deduction from the formulae

$$\sum_1^n r_2(m) \sim B n^{2a}, \qquad r_2(n) < B_\varepsilon n^\varepsilon.$$

We may suppose then that $s > 2$.

We divide the interval $(1, n)$ into the $\mathfrak{n} = [n^a]$ parts

$$(1^k, 2^k),\ (2^k, 3^k),\ \dots,\ (\mathfrak{n}^k, n)$$

(of which the last may disappear). We denote the interval beginning at j^k by δ_j, and its length by l_j. Then $l_j > k j^{k-1}$, except possibly for the last interval. If then we consider in particular the intervals δ_j included in $(\frac{1}{4}n, \frac{3}{4}n)$, it is clear that there are at least Bn^a such intervals, and that each of them is of length greater than Bn^{1-a}.

In $(0, l_j)$ lie $N_{s-1}(l_j)$ numbers representable by $s-1$ k-th powers; and so more than $N_{s-1}(Bn^{1-a})$ such numbers. Adding each of these to j^k, we obtain more than $N_{s-1}(Bn^{1-a})$ *distinct* numbers, lying in δ_j, and representable by s k-th powers. It follows that

$$N_s(n) \geqq Bn^a N_{s-1}(Bn^{1-a}),$$

and therefore that

$$\alpha_s \geqq a + (1-a)\alpha_{s-1}. \tag{7.12}$$

From this (7.11) follows by induction.

7.2. For the proof of Theorem 4 we work with a new value of the number β of § 6, viz.

$$\beta = \frac{2sa-1}{2s},$$

observing that the necessary condition $\beta < a_1$ is satisfied if $2s\delta < 1$, which is certainly true if δ is sufficiently small. This value of makes

$$\lambda_1 = \lambda_2 = \frac{(2sa-1)(sa-2)}{s}. \tag{7.21}$$

We denote now by $\lambda = \lambda_s = \lambda_{k,s}$ the number defined by (6.171) and (6.172) when β has its new value.

Lemma 23. *If*

$$s \geqq \mathrm{Max}(\Gamma(k), 2k+1, s_0+1) \tag{7.22}$$

then

$$\nu_s(n) < C_\varepsilon n^{1-\lambda+\varepsilon}. \tag{7.23}$$

The proof is similar to that of Theorem 2 (§ 6.2). If m is non-representable

$$(\sigma_s(m))^2 = (r_s(m) - \varrho_s(m))^2 = (\varrho_s(m))^2 > Cm^{2sa-2},$$

$$\nu_s(n) - \nu_s\left(\frac{1}{2}n\right) < Cn^{2-2sa}\sum_{\frac{1}{2}n}^{n}{}'(\sigma_s(m))^2$$

$$< Cn^{2-2sa}.C_\varepsilon n^{2sa-1-\lambda+\varepsilon} < C_\varepsilon n^{1-\lambda+\varepsilon},$$

$$\nu_s(n) < C_\varepsilon n^{1-\lambda+\varepsilon}\left(1 + \left(\frac{1}{2}\right)^{1-\lambda+\varepsilon} + \ldots\right) < C_\varepsilon n^{1-\lambda+\varepsilon}.$$

Lemma 24. *Suppose that s satisfies* (7.22). *Then if $\lambda_s > 1 - \alpha_{s'}$, and therefore certainly if*

$$\lambda_s > (1-2a)(1-a)^{s'-2} \qquad (s' \geqq 2) \tag{7.24}$$

we have

$$G(k) \leqq s + s'. \tag{7.25}$$

In fact, if n is not representable by $s+s'$ powers, there are at least $N_{s'}(n)$ numbers, not exceeding n, which are not representable by s powers. That is,

$$\nu_s(n) \geqq N_{s'}(n), \qquad 1-\lambda_s \geqq \alpha_{s'}$$

in contradiction with the hypothesis $\lambda_s > 1-\alpha_{s'}$.

7.3. We are now in a position to prove Theorem **4**. The case $k=4$ is exceptional, and we suppose first that $k>4$. We shall verify that numbers s and s' exist which satisfy the conditions

$$s+s' = \left(\tfrac{1}{2}k-1\right)K+k+5+[\zeta_k]; \tag{7.31}$$

$$s \geqq \Gamma(k), \qquad s \geqq 2k+1, \qquad s > s_0, \qquad s' \geqq 2; \tag{7.32}$$

$$\lambda_s > (1-2a)(1-a)^{s'-2}. \tag{7.33}$$

We shall prove that these conditions are satisfied by taking

$$s = \left(\tfrac{1}{2}k-1\right)K+k+1 = s_0+k-1, \qquad s' = [\zeta_k]+4. \tag{7.34}$$

In the first place, these values of s and s' satisfy (7.31). Next

$$s_0 > \tfrac{1}{2}(k-2)2^{k-1} > 2^{k-3}k \geqq 4k \geqq \mathrm{Max}(\Gamma(k), 2k+1). \tag{7.35}$$

Thus the first three inequalities (7.32) are satisfied. Also ζ_k is positive, since its numerator is

$$\log \frac{s_0-2}{k} > 0;$$

and so the last inequality (7.32) is also satisfied. It remains only to verify (7.33).

When we substitute the value of s from (7.34) into (7.21) and (6.172), we find

$$\lambda_1 = \lambda_3 = \frac{(2s_0+k-2)(s_0-k-1)}{k^2(s_0+k-1)}, \quad \lambda_4 = \frac{2a(k-1)}{K}, \quad \lambda_5 = 2a.$$

Of these numbers, the least is λ_4. For it is obvious that $\lambda_4 < \lambda_5$; and $s_0 \geqq 4k+1$ if $k \geqq 5$, so that

$$\lambda_1 \geqq \frac{9}{k}\frac{3k}{5k} > 2a = \lambda_5.$$

Hence $\lambda = \lambda_4$, and it is only necessary to verify the inequality

$$\lambda = \lambda_4 = \frac{2a(k-1)}{K} > (1-2a)(1-a)^{[\zeta_k]+2}. \tag{7.36}$$

This is

$$\left(\frac{k}{k-1}\right)^{[\zeta_k]+2} > \frac{(k-2)K}{2(k-1)}$$

or

$$(7.37)\qquad [\zeta_k]+1 > \frac{(k-2)\log 2-\log k+\log(k-2)}{\log k-\log(k-1)}=\zeta_k$$

which is true. Hence (7.33) is true[35].

7.4. It remains to consider the case $k=4$; here $[\zeta_k]=2$ and we have to prove that $G(4)\leqq 19$. The difficulty is that $s_0+k-1=13<\Gamma(k)$; and this difficulty cannot be overcome directly by another choice of s and s'. There are two alternatives.

(i) We may consider each progression l (mod 16) separately as follows. Let $\nu_{s,l}(n)$ be the number of numbers $m\equiv l$, not exceeding n, and not representable by s fourth powers; and $N_{s',l'}(n)$ the number of distinct numbers $m\equiv l'$, not exceeding n, and representable by s' fourth powers.

It may be shown, on the one hand that

$$(7.41)\qquad |\mathfrak{S}(m)|>A \qquad (s=s_0+3=13,\ m\not\equiv 0,\,14,\,15)$$

and so

$$(7.42)\qquad \nu_{s,l}(n)<A_\varepsilon n^{1-\lambda+\varepsilon} \qquad (l\not\equiv 0,\,14,\,15);$$

and on the other that

$$(7.43)\qquad N_{s',l'}(n)>A_\varepsilon n^{\alpha_{s'}-\varepsilon},$$

where $\alpha_{s'}$ satisfies (7.11). Suppose now that arbitrarily large numbers $n\equiv l$ are non-representable by 19 fourth powers. We take $s=13$, $s'=6$, choosing l' so that

$$l-l'\not\equiv 0,\,14,\,15.$$

Then the argument of Lemma 24 shows that there exist at least $N_{s',l'}(u)$ numbers m, less than n, not congruent to 0, 14 or 15, and not representable by s' fourth powers. Thus

$$N_{s',l'}(n)\leqq \nu_{s,l}(n),$$

and so

$$1-\lambda_s\geqq \alpha_{s'}.$$

But this is false, since calculation gives

$$1-\lambda_s=\frac{13}{16}<\frac{431}{512}=1-\frac{1}{2}\left(\frac{3}{4}\right)^4\leqq \alpha_{s'}.$$

The proof of (7.41), however, reopens the question of the arithmetic of the singular series, which we do not wish to discuss here. We therefore adopt a second alternative.

[35]) It may be shown that no other choice of s and s' leads to a better value of $s+s'$.

(ii) We take the pair of numbers $s = 15$, $s' = 4$; it happens that we can, by a special device, suggested to us by Mr. A. E. Ingham, prove that

$$\nu_s(n) < A_\varepsilon n^{1-\lambda+\varepsilon}$$

for $s = 15$, and this proves to be sufficient for our purpose.

Let $\tau_t(\xi)$ be the number of numbers m for which (i) $16^t | m$, (ii) $0 < m \leqq \xi$, (iii) m is not representable as a sum of 15 fourth powers. If now $\frac{1}{2}n < m \leqq n$ and $16 \nmid m$, we have[35a])

$$\varrho(m) > A m^{\frac{11}{4}} > A n^{\frac{11}{4}} = A n^{sa-1};$$

and, if also m is non-representable,

$$|\sigma(m)| > A n^{sa-1}.$$

Hence by (6.17)[36])

$$\left(\tau_0(n) - \tau_0\left(\frac{1}{2}n\right)\right) n^{2sa-2} \leqq A \sum_{\frac{1}{2}n}^{n} (\sigma(m))^2 < A_\varepsilon n^{2sa-1-\lambda+\varepsilon},$$

$$\tau_0(n) - \tau_0\left(\frac{1}{2}n\right) < A_\varepsilon n^{1-\lambda+\varepsilon},$$

whence

$$\tau_0(n) < A_\varepsilon n^{1-\lambda+\varepsilon}. \tag{7.44}$$

Now a number $m = 16^t m'$ is representable as a sum of 15 fourth powers if and only if m' is so representable. Hence

$$\tau_t(\xi) = \tau_0(\xi \cdot 16^{-t})$$

and so

$$\nu_{4,15}(n) = \sum_{t=0}^{\infty} \tau_t(n) = \sum_{t=0}^{\infty} \tau_0(n \cdot 16^{-t}) < A_\varepsilon n^{1-\lambda+\varepsilon},$$

by (7.44). Now

$$\lambda = \operatorname{Min}\left(\frac{(15-2)(15-8)}{8 \cdot 15}, \frac{15-10}{16}, \frac{1}{2}\right) = \frac{5}{16},$$

and

$$\lambda = \frac{5}{16} > \frac{9}{32} = \frac{1}{2}\left(\frac{3}{4}\right)^2 = (1-2a)(1-a)^{s'-2}$$

if $s' = 4$. Our former argument therefore applies to show that

$$G(k) \leqq 15 + 4 = 19;$$

and the proof of Theorem **4** is completed.

[35a]) This is the special case of (6.31) in which $t = 0$.

[36]) Valid since $15 > 8 = 2k$.

8. Proof of Theorems 5—9.

8.1. In this section we use the Farey dissection of order $N_2 = [n^{1-3\alpha}]$, the major and minor arcs being defined by $q \leqq n^\alpha$ and $q > n^\alpha$, and α being a number of type c, which will be chosen small enough for our purposes. It is plain that, if α is small enough, every major arc $\mathfrak{M}_2$ is part of an $\mathfrak{M}_1$ and *a fortiori* of an $\mathfrak{M}$.

We return to the analysis of 6.1. We take (6.11), replacing $\mathfrak{M}_1$, $\mathfrak{m}_1$ by $\mathfrak{M}_2$, $\mathfrak{m}_2$, and we have to show that each of the terms J_1, J_2, J_3, J_4, J_5 is less than $Cn^{2sa-1-c}$. An examination of the argument shows that, except in respect of J_3, it requires very little alteration.

(i) The term J_5 is disposed of by Lemma 16, and no further discussion is necessary.

(ii) J_2 and J_4 are disposed of by (6.13) and (6.15), each of which depends on Lemma 19 (and so on Lemmas 17 and 18). The result of Lemma 17 stands with N_2 in place of N_1; and (5.28) is replaced by

$$\sum_\xi \int |G^2| \, |d\theta| < C\nu^4 N_2^{2sa-1} \log(N_2+2)$$
$$< Cn^{(1-3\alpha)(2sa-1)+4\beta} \log n.$$

This is plainly of the form required if β is chosen sufficiently small.

The other sums which appear in (5.35) and (5.36)[37] present no difficulty. The first sum on the right hand side of (5.35) is disposed of by the trivial Lemma 13; and the second, and likewise the corresponding sum in (5.36), by Lemma 15, which is still available. We require only that $q > \nu$ on every $\mathfrak{m}_2$, and this will certainly be so if $\beta < \alpha$.

(iii) J_1 is disposed of by (6.121) or (6.122). Here we use the latter inequality and observe that it is true *a fortiori* for arcs $\mathfrak{M}_2$.

Thus each of the terms J_1, J_2, J_4, J_5 is of the form required. This could naturally have been proved more simply, had we been working from the beginning in terms of the dissection we are using now.

8.2. It remains to prove that

$$J_3 = \sum \int_{\mathfrak{m}_2} |f|^{2s} \, |d\theta| < Cn^{2sa-1-c}. \tag{8.21}$$

It is here only that we use Hypothesis K.

If $\beta < \alpha$, as we have supposed, and x is on an $\mathfrak{m}_2$, then either (i) it is on an $\mathfrak{m}_1$ or (ii) it is on an $\mathfrak{M}_1$ for which $q > \nu$. In case (i)

$$|f| < B_\varepsilon n^{a\varkappa+\varepsilon} < Cn^{a-c} \tag{8.22}$$

[37]) With $\mathfrak{M}_2$, $\mathfrak{m}_2$ for $\mathfrak{M}_1$, $\mathfrak{m}_1$.

on fixing ε; and in case (ii)

$$(8.23)\qquad |f| \leqq |\varphi| + |f-\varphi| < A\left|\frac{S_{p,q}}{q}\right| |y|^{-a} + B_\varepsilon n^{a\kappa+\varepsilon}$$
$$< Bq^{-a}n^a + B_\varepsilon n^{a\kappa+\varepsilon} < Cn^{a-c},$$

on fixing ε and β. Hence

$$(8.24)\qquad J_3 < Cn^{2(s-k)(a-c)}\int_0^{2\pi}|f|^{2k}d\psi < Cn^{2sa-2-c}(1+\Sigma(r_k(m))^2|x|^{2m})$$
$$< C_\varepsilon n^{2sa-2-c}(1-|x|)^{-1-\varepsilon} < C_\varepsilon n^{2sa-1-c+\varepsilon} < Cn^{2sa-1-c},$$

on fixing ε once more.

This completes the proof of Theorem **5**. Theorems **6** and **7** require no further proof, the first following from **5** as **2** followed from **1**, and **7** being merely a restatement.

8.3. In proving Theorem **8**, we revert to the method of P. N. 1. We have

$$(8.31)\qquad r_s(n) = \frac{1}{2\pi i}\int\frac{f^s}{x^{n+1}}dx$$
$$= \sum\frac{1}{2\pi i}\int_{\mathfrak{M}_2}\frac{f^s}{x^{n+1}}dx + \sum\frac{1}{2\pi i}\int_{\mathfrak{m}_2}\frac{f^s}{x^{n+1}}dx = S_1 + S_2,$$

$$(8.32)\qquad S_1 = \sum\frac{1}{2\pi i}\int_{\mathfrak{M}_2}\frac{f^s-\varphi^s}{x^{n+1}}dx + \sum\frac{1}{2\pi i}\int_{\mathfrak{M}_2}\frac{\varphi^s - F_{p,q}}{x^{n+1}}dx$$
$$+ \sum\frac{1}{2\pi i}\int_{\mathfrak{M}_2}\frac{F_{p,q}}{x^{n+1}}dx = S_3 + S_4 + S_5,$$

say. We prove first that

$$(8.33)\qquad |S_j| < Cn^{sa-1-c} \qquad (j = 2, 3, 4).$$

Taking S_2 first, we have

$$|S_2| < A\Sigma\int_{\mathfrak{m}_2}|f|^s|d\theta| < Cn^{(s-2k)(a-c)}\Sigma\int_{\mathfrak{m}_2}|f|^{2k}|d\theta|$$

by (8.22) and (8.23). Arguing as in § 8.2, we find

$$|S_2| < C_\varepsilon n^{sa-1-c+\varepsilon} < Cn^{sa-1-c}.$$

This disposes of S_2. That S_4 is of the form required is obvious; it is in fact less than C (as in Lemma 13). As regards S_3, we have

$$|f^s - \varphi^s| < A_s|f-\varphi|^s + A_s|f-\varphi||\varphi|^{s-1}$$
$$< C_\varepsilon n^{sa\kappa+\varepsilon} + C_\varepsilon n^{a\kappa+\varepsilon}|y|^{-a(s-1)}$$

(since every $\mathfrak{M}_2$ is part of an $\mathfrak{M}_1$). The first term here contributes less than

$$C_\varepsilon n^{sa\varkappa+\varepsilon} \sum \int_{\mathfrak{M}_2} |d\theta| < C_\varepsilon n^{sa\varkappa+\varepsilon} . A \sum_{q \leqq n^\alpha} \sum_p \frac{1}{q n^{1-3\alpha}}$$
$$< C_\varepsilon n^{sa\varkappa+4\alpha-1+\varepsilon} < C n^{sa-1-c},$$

when ε and α are chosen appropriately. And the second term contributes less than

$$C_\varepsilon n^{a\varkappa+\varepsilon} \sum \int_{\mathfrak{M}_2} |y|^{-a(s-1)} |d\theta| < C_\varepsilon n^{a\varkappa+\varepsilon} \cdot n^{a(s-1)-1} \cdot \sum_{q \leqq n^\alpha} \sum_p 1$$
$$< C_\varepsilon n^{sa-1-a(1-\varkappa)+2\alpha+\varepsilon} < C n^{sa-1-c}$$

This completes the proof of (8.33).

8.4. The proof of Theorem 8 is thus reduced to a proof that

$$(8.41) \qquad S_5 = \sum \frac{1}{2\pi i} \int_{\mathfrak{M}_2} \frac{F_{p,q}}{x^{n+1}} dx = \varrho_s(n) + O(n^{sa-1-c}).$$

We write

$$(8.42) \quad S_5 = \sum \frac{1}{2\pi i} \int_\Gamma \frac{F_{p,q}}{x^{n+1}} dx - \sum \frac{1}{2\pi i} \int_\eta \frac{F_{p,q}}{x^{n+1}} dx = S_6 + S_7,$$

η being, as in P. N. 1, the arc of Γ complementary to ξ (or $\mathfrak{M}_2$). On η, $|\theta| > \frac{A}{q N_2}$, and so

$$|F_{p,q}| < C \left| g(|x| e^{-i\theta}) \right| < C |\theta|^{-sa},$$

by Lemma 12. Hence

$$\int_\eta |F_{p,q}| \, |d\theta| < C \int_{\frac{A}{qN_2}}^\infty \theta^{-sa} d\theta < C (q N_2)^{sa-1},$$

$$(8.43) \qquad |S_7| < A \sum \int_\eta |F_{p,q}| \, |d\theta| < C N_2^{sa-1} \sum_{q \leqq n^\alpha} \sum_p q^{sa-1}$$
$$< C N_2^{sa-1} n^{(sa+1)\alpha} < C n^{sa-1-2\alpha(sa-2)} < C n^{sa-1-c},$$

since $sa > 2$ and α is a c.

Finally

$$S_6 = \mathfrak{C} n^{sa-1} \sum_{p, q \leqq n^\alpha} \left(\frac{S_{p,q}}{q}\right)^s e_q(-np) = n^{sa-1} (\mathfrak{S}(n) - \mathfrak{S}_2(n))$$

(the ν of $\mathfrak{S}_2(n)$ being now n^α). Hence, by Lemma 3 (3.313), (8.42), and (8.43),

$$S_5 = n^{sa-1} \mathfrak{S}(n) + O(n^{sa-1-c}) + O(n^{sa-1-\alpha(sa-2)}) = \varrho_s(n) + O(n^{sa-1-c});$$

which completes the proof.

8.5. Theorem **9** is an obvious corollary of Theorem **8**. We conclude the memoir by showing that when, as is usually the case[38]), $\Gamma(k) \leqq k+1$, Theorem **9** may be deduced in an elementary manner from Theorem **6**.

We know that

$$\sum_1^n r_{k,k}(m) \sim Bn.$$

Hence, if Hypothesis K is true, there are more than $B_\varepsilon n^{1-\varepsilon}$ numbers less than n and representable by k k-th powers[39]). If n_1 is such a number less than n, and n is not representable by $2k+1$ powers, then none of the numbers $n-n_1$ is representable by $k+1$. That this should happen for an infinity of values of n would plainly contradict Theorem **6**.

[38]) See P. N. 4, 184, for an analysis of the exceptional cases.
[39]) In fact, in the notation of § 7, $\alpha_s = 1$ $(s \geqq k)$.

(Eingegangen am 18. Juni 1924.)

CORRECTIONS

In the last displayed formula but two on p. 32, read $\nu_{s,\,l-l'}(n)$ on the right.
p. 10. In the second line of the proof of Lemma 2, read $s = 4$ in place of $k = 4$.

SOME PROBLEMS OF "PARTITIO NUMERORUM" (VIII): THE NUMBER $\Gamma(k)$ IN WARING'S PROBLEM

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1. *Introduction.*

1.1. In this memoir we continue our researches concerning the number $\Gamma(k)$, some of whose properties were developed in the fourth and sixth memoirs of the series†.

In § 2 we give a formal proof of what we have stated before‡ concerning the arithmetical interpretation of $\Gamma(k)$, viz. that $\Gamma(k)$ is (except when $k=4$) the least number s such that every arithmetical progression contains an infinity of numbers which are sums of at most s positive k-th powers.

In §§ 3–5 we are concerned with upper bounds for $\Gamma(k)$. Since each of $k+1$ and $\Gamma(k)$ is a lower bound for $G(k)$, it is of interest to determine completely all cases in which $\Gamma(k) > k+1$. Our earlier researches left little doubt which these cases are, but an exact proof that $\Gamma(k) \leqslant k+1$

† G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum': (IV): The singular series in Waring's Problem and the value of the number $G(k)$", *Math. Zeitschrift*, 12 (1922), 161–188: and (VI): "Further researches in Waring's Problem", *ibid.*, 23 (1925), 1–37. We refer to these memoirs as P.N. 4, P.N. 6. Some errors in P.N. 4 are corrected in P.N. 6, but the main results of P.N. 6 will not be required here.

We assume that the reader is acquainted with the ideas and notation of P.N. 4, but, for the sake of clearness, we summarise the principal results, so far as they are relevant to our present purpose, in §§ 1.2—1.3.

We may add that "P.N. 7", which is still unpublished, contains an application of our methods to the problem of the order of magnitude of the difference between successive primes. We prove (subject to our generalized form of the Riemann hypothesis) that

$$\varliminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log n} \leqslant \tfrac{2}{3}.$$

‡ P.N. 4, 188; P.N. 6, 7.

1928, 4 (with J. E. Littlewood) *Proceedings of the London Mathematical Society*, (2) 28, 518–42.

in all others was lacking; and this we now supply, the key to the proof being contained in our present Theorem 3 (§ 4).

In the cases which were formerly the main obstacle, however, we now find that we can prove $\Gamma(k) \leqslant k$ as easily as $\Gamma(k) \leqslant k+1$. We are naturally led to attempt to determine all cases in which $\Gamma(k) > k$, and this, we find, involves the solution of a further problem, to which the key is Theorem 2 (§ 3). This theorem, like Theorem 3, is difficult; but the trouble involved in proving it is not wasted, partly because the theorem has a certain intrinsic interest, and partly because the lemmas on which it depends are important in the actual calculation of $\Gamma(k)$ for particular values of k.

In § 5 we prove Theorem 4, which shows that $\Gamma(k) \leqslant k$ except in five standard cases, in each of which $\Gamma(k)$ is determined explicitly. To these five cases we add a sixth, in which we can prove that $\Gamma(k) = k$.

Finally, § 6 is concerned with questions of numerical calculation.

1.2. It is convenient to insert here a brief summary of the notations and conclusions of P.N. 4, so far as these are directly relevant. It is assumed there that $k > 2$, and we shall suppose this here also, except where the contrary is explicitly indicated.

The singular series is

$$S = S(k, s, n) = \sum_{q=1}^{\infty} A_q,$$

where $$A_1 = 1, \quad A_q = A_q(k, s, n) = \sum_p \left(\frac{S_{p,q}}{q}\right)^s e^{-2np\pi i/q} \quad (q > 1),$$

$S_{q,q}$ being a generalized Gaussian sum and p running through values less than q and prime to q. Here $A_{qq'} = A_q A_{q'}$ if $(q, q') = 1$, so that

$$S = \chi_2 \chi_3 \chi_5 \ldots = \prod_{\varpi} \chi_{\varpi}\dagger,$$

where ϖ is a prime and

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + \ldots = \sum_{\lambda} A_{\varpi^{\lambda}}.$$

The series and product are absolutely convergent for $s \geqslant 4$‡ (except for $k = 2$). Given k and ϖ, we define θ and ϕ by

$$\varpi^{\theta} \| k\S, \quad \phi = \theta + 1 \ (\varpi > 2), \quad \phi = \theta + 2 \ (\varpi = 2),$$

† In P.N. 4 necessities of printing compelled us to write ϖ as π.

‡ P.N. 4, 176 (Theorem 3).

§ $x \mid y$ means "x is a divisor of y"; $x^a \| y$ means "x^a is the highest power of x which divides y".

and we sometimes write ρ for ϖ^{ϕ}. We have $\phi < k$, except† when $k = 4$, $\varpi = 2$.

We denote by

$$N(\varpi^{\lambda}, \nu) = N(\varpi^{\lambda}, \nu, s)$$

the number of primitive solutions of

$$x_1^k + x_2^k + \ldots + x_s^k \equiv \nu \pmod{\varpi^{\lambda}},$$

i.e. the number of solutions for which $0 \leqslant x_r < \varpi^{\lambda}$ $(r \leqslant s)$ and not every x_r is divisible by ϖ. If $(\varpi^k)^{\beta} | n$‡ then§

$$\chi_{\varpi} = B\varpi^{\phi(1-s)} N(\rho, 0) + \varpi^{\beta(k-s)+\phi(1-s)} N\left(\rho, \frac{n}{\varpi^{\beta k}}\right),$$

where $B = 0$ if $\beta = 0$ and $B > 0$ if $\beta > 0$. Thus in all cases $\chi_{\varpi} \geqslant 0$.

We denote by $\gamma'_{\varpi} = \gamma'_{\varpi}(k)$ the least number m such that

$$\chi_{\varpi} > 0$$

for $s \geqslant m$ and all n, and by $\gamma_{\varpi} = \gamma_{\varpi}(k)$ the least number m such that

$$\chi_{\varpi} \geqslant h_{\varpi} = h(k, s, \varpi) > 0$$

for $s \geqslant m$ and all n. Then $\gamma'_{\varpi} = \gamma_{\varpi}$ except when $k = 4$, $\varpi = 2$ (when $\gamma'_2 = 15$ and $\gamma_2 = 16$)||. Our number $\Gamma(k)$ is defined by

$$\Gamma(k) = \operatorname*{Max}_{\varpi} \gamma_{\varpi}.$$

In order that $\chi_{\varpi} \geqslant h_{\varpi} > 0$ for all n, *i.e.* that $s \geqslant \gamma_{\varpi}$, it is sufficient that $\chi_{\varpi} \geqslant h_{\varpi} > 0$ for all sufficiently large n. The necessary and sufficient condition for this is that

$$N(\rho, n) > 0$$

for all n¶, in which case $N(\varpi^{\lambda}, n) > 0$ for $\lambda \geqslant \phi$††.

1.3. Much of our analysis depends on a particular division into classes of the residues to modulus ρ. We suppose that $\varpi > 2$, so that

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† P.N. 4, 178.

‡ Not $\varpi^{\beta k} | n$.

§ P.N. 4, 166, Theorem 2. We do not need here the actual value of B when $\beta > 0$.

|| P.N. 4, 178 (Lemma 19).

¶ P.N. 4, 178 (Lemma 19).

†† This is a particular consequence of P.N. 4, Lemma 5.

$\phi = \theta+1$. We write†

$$k = \varpi^{\theta} k^* = \varpi^{\theta} \epsilon k_0,$$

where $$\epsilon = (k^*, \varpi-1) = (\varpi^{-\theta} k, \varpi-1);$$

and $$d = \frac{\varpi-1}{\epsilon},$$

so that $(k_0, d) = 1$‡. Finally we write

$$c = c_{\varpi} = c_{\varpi}(k) = \frac{\varpi^{\phi}-1}{\varpi-1}\epsilon+1 = \frac{\varpi^{\phi}-1}{d}+1,$$

and we divide the residues into c classes $C_0, C_1, \ldots, C_{c-1}$ as follows§.

If G is a primitive root of ρ, and n_0 is prime to ρ, we have

$$n_0 \equiv G^{m_0\psi_0+e} \pmod{\rho},$$

where $$\psi_0 = \varpi^{\phi-1}\epsilon,$$

m_0 has one of the values $0, 1, \ldots, d-1$, and e one of the values $0, 1, \ldots, \psi_0-1$. The d values of n_0 which have a common e we class together in the class

$$C_e^0 \quad (e = 0, 1, \ldots, \psi_0-1),$$

and we denote a typical member of the class by a_e^0. These ψ_0 classes contain all residues prime to ρ.

Next, if n_i is a residue for which $\varpi^i | n_i$, where $0 < i < \phi$, we can write

$$n_i \equiv \varpi^i N_i \equiv \varpi^i G^{m_i\psi_i+e} \pmod{\rho},$$

where $$\psi_i = \varpi^{\phi-i-1}\epsilon,$$

m_i runs from 0 to $d-1$, and e from 0 to ψ_i-1. We thus define ψ new classes

$$C_e^i \quad (e = 0, 1, \ldots, \psi_i-1)$$

of members a_e^i. Finally, the residue 0 is the sole member of a class C_0^{ϕ}. The total number of classes is c, and we may denote them, in the

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† We did not use the symbol k^* in P.N. 4.

‡ To fix ideas, we give the examples:

$k = 34$, $\varpi = 3$, $\theta = 0$, $\phi = 1$, $\rho = 3$, $k^* = 34$, $\epsilon = 2$, $d = 1$, $k_0 = 17$;

$k = 34$, $\varpi = 17$, $\theta = 1$, $\phi = 2$, $\rho = 289$, $k^* = 2$, $\epsilon = 2$, $d = 8$, $k_0 = 1$;

$k = 34$, $\varpi = 103$, $\theta = 0$, $\phi = 1$, $\rho = 103$, $k^* = 34$, $\epsilon = 34$, $d = 3$, $k_0 = 1$;

$$34 = 3^0 . 2 . 17 = 17^1 . 2 . 1 = 103^0 . 34 . 1.$$

§ A misprint, C_c for C_{c-1}, runs through §§ 5.4—5.6 of P.N. 4.

order in which they have been defined, by $C_0, C_1, \ldots, C_{c-1}$, and typical members by $a_0, a_1, \ldots, a_{c-1}$.

The class C_0, which is particularly important, consists of the residues of k-th powers prime to ϖ. In particular, 1 is an a_0. Any two a_0's are incongruent (mod ρ), and therefore incongruent (mod ϖ). The product of an a_0 and an a_r is an a_r, and $a_0 a_r$, where a_r is given, may be identified with any a_r by choice of a_0†.

One additional remark will be useful. Throughout P.N. 4 we supposed that $k > 2$. But our analysis of the classes does not depend on this assumption, and is equally valid when $k = 1$ or $k = 2$. Thus when $k = 1$ we have $\theta = 0$, $\phi = 1$, $\epsilon = 1$, $d = \varpi - 1$, $c = 2$. There are two classes, C_0 consisting of all residues save 0, and C_1 consisting of 0 alone. It is, in fact, obvious that all residues save 0 are representable by a single first power prime to ϖ, and 0 by two.

1.4. We shall require, besides the ideas summarized in §§ 1.2—1.3, a number of propositions which we state here as lemmas. These lemmas are either contained substantially in P.N. 4, or are easy developments of propositions proved there.

LEMMA 1. *If $d > 1$ we have*

$$\Sigma a_0 \equiv 0 \pmod{\rho},$$

the summation extending over every residue of C_0.

For
$$\Sigma a_0 \equiv \sum_{m_0=0}^{d-1} G^{m_0\psi_0} = \frac{G^{d\psi_0}-1}{G^{\psi_0}-1} = \frac{G^{\varpi^{\phi-1}(\varpi-1)}}{G^{\psi_0}-1} \equiv 0,$$

since G^{ψ_0} is an a_0 different from 1 and therefore‡ incongruent to 1 (mod ϖ).

† For all this see P.N. 4, 180–181. The point of the incongruence of the a_0's is not made explicitly. To establish it we observe that if G is a primitive root (mod ρ) it is a primitive root (mod ϖ). Hence from

$$G^{m_0\psi_0} \equiv G^{m'\psi_0} \pmod{\varpi}$$

we deduce successively

$$(m_0 - m_0')\,\psi_0 \equiv 0 \pmod{\varpi - 1}, \quad (m_0 - m_0')\,\epsilon \equiv 0 \pmod{\varpi - 1},$$
$$m_0 - m_0' \equiv 0 \pmod{d},$$
$$m_0\psi_0 \equiv m_0'\psi_0 \quad \{\text{mod } \psi_0 d = \varpi^{\phi-1}(\varpi - 1)\},$$
$$G^{m_0\psi_0} \equiv G^{m_0'\psi_0} \pmod{\varpi^{\phi}}.$$

‡ See § 1.3.

LEMMA 2. *If $d > 1$ and a is an a_0 other than 1, then*

$$1+a+a^2+\ldots+a^{d-1} \equiv 0 \pmod{\rho}.$$

For $a \equiv x^k \not\equiv 1$, and

$$\sum_{r=0}^{d-1} a^r \equiv \frac{x^{dk}-1}{x^k-1}, \quad x^{dk} = x^{\varpi^\theta(\varpi-1)k_0} \equiv 1.$$

LEMMA 3. *Every residue* (mod ρ), *except* 0, *is representable* (mod ρ) *as the sum of at most $c-1$ a_0's, and every residue by c. This is true for $k \geqslant 2$.*

When $k > 2$ this is contained in P.N. 4, Lemma 21, and the result is valid for $k = 2$ (see the end of § 1 . 3).

LEMMA 4. *If $1 \leqslant c' < c$ and $\mu(c')$ is the number of classes representable by at most c' a_0's, then*

$$(1.41) \qquad \mu(c'+1) \geqslant \min\{\mu(c')+1, c\}.\dagger$$

If (1 . 41) is false for c', then

$$\mu(c'+1) = \mu(c') < c.$$

Suppose that C^* is a typical class of the $\mu(c')$ classes, and that C_r is a C^*. If a_r is any member of C_r, a_r+1 must belong to a C^*, since no new classes are representable by $c'+1$ a_0's.

Similarly $(a_r+1)+1 = a_r+2$, a_r+3, ..., and so all residues, belong to some C^*, which contradicts $\mu(c') < c$.

LEMMA 5. *If $d_1 | d$ and $d_1 > 1$, then the residue 0 is representable* (mod ρ) *by d_1 a_0's.*

This is proved in P.N. 4, in the course of the proof of Lemma 21.

LEMMA 6. *If $\varpi > 2$ and $\mathbf{k} = k/k_0$, then*

$$\gamma_\varpi(k) \leqslant 3 \qquad (\mathbf{k} \leqslant 2),$$
$$\gamma_\varpi(k) = \gamma_\varpi(\mathbf{k}) \quad (\mathbf{k} > 2).$$

(i) Suppose, first, that $\mathbf{k} = \varpi^\theta \epsilon > 2$. Then ϕ, ϵ, d, and ψ_0 are the same for k and $\mathbf{k}$, and the classes C_0 for k and $\mathbf{k}$, are identical since each consists of the residues

$$G^{m_0\psi_0} \quad (m_0 = 0, 1, \ldots, d-1);$$

which proves the second result. (The proof is valid for all $\mathbf{k}$.)

† Compare P.N. 4, 183, where we prove that

$$\nu(c'+1) \geqslant \min\{\nu(c')+1, c-1\},$$

$\nu(c')$ being the number of representable classes *exclusive of* C_{c-1} (the class consisting of 0 only).

(ii) Next suppose that $\mathbf{k} \leqslant 2$. Then plainly $\theta = 0$, $\phi = 1$, and we have to prove that any residue (mod ϖ) is representable by at most three $\mathbf{k}$-th powers. We have pointed out already (in § 1 . 4) that, when $\mathbf{k} = 1$, two are sufficient. When $\mathbf{k} = 2$ we must have $\theta = 0$, $\phi = 1$, $\epsilon = 2$, and $c = 3$, so that our conclusion follows from Lemma 3.

LEMMA 7. *If*

(i) $\varpi = 2, \quad \theta = 0,$ *then* $\gamma_2 = 2$;

(ii) $\varpi = 2, \quad \theta > 0,$ *then* $\gamma_2 = 2^{\theta+2}$;

(iii) $\varpi > 2, \quad \epsilon = \varpi - 1,$ *then* $\gamma_\varpi = \varpi^{\theta+1}$;

(iv) $\varpi > 2, \quad \epsilon = \frac{1}{2}(\varpi - 1),$ *then* $\gamma_\varpi = \frac{1}{2}(\varpi^{\theta+1} - 1)$, *except in the case* $\varpi = 3$, $\theta = 0$ (*when* $\gamma_\varpi = 2$).

See P.N. 4, Lemma 22. The exceptional case in (iv) is there omitted by inadvertence. It arises from the failure of $2 \leqslant c - 1$, implicitly assumed, and true in general.

LEMMA 8. *If*

(i) $k = 2^\theta, \quad \theta > 1,$ *then* $\Gamma(k) = 2^{\theta+2}$;

(ii) $k = 2^\theta 3, \quad \theta > 1,$ *then* $\Gamma(k) = 2^{\theta+2}$:

(iii) $k = \varpi^\theta(\varpi - 1) \quad \varpi > 2, \quad \theta > 0,$ *then* $\Gamma(k) = \varpi^{\theta+1}$;

(iv) $k = \frac{1}{2}\varpi^\theta(\varpi - 1), \quad \varpi > 2, \quad \theta > 0,$ *then* $\Gamma(k) = \frac{1}{2}(\varpi^{\theta+1} - 1)$.

See P.N. 4, Theorems 6, 7, 9, 10.

LEMMA 9. *If* $\delta = (\varpi - 1, k)$ *and*

$$S_{p,\varpi} = S_{p,\varpi,k} \sum_{h=0}^{\varpi-1} e^{2phk\pi i/\varpi} = \sum_{h=0}^{\varpi-1} e_\varpi(ph^k),$$

then

$$|S_{p,\varpi}| \leqslant (\delta - 1)\sqrt{\varpi}.$$

See P.N.4, Lemma 13. If $\delta = 1$ the sum vanishes.

LEMMA 10†. *If* $\chi_\kappa(p)$ *is a character* (mod ϖ), κ' *a* κ *for which* χ_κ^k *is identical with the principal character* χ_1, *and*

$$\tau_{\bar{\kappa}} = \sum_{l=0}^{\varpi-1} \bar{\chi}_\kappa(l)\, e_\varpi(l),$$

† The χ_κ will not be confused with the χ_ϖ of § 1, which do not occur again.

then (1) *no* τ *exceeds* $\sqrt{\varpi}$ *in absolute value, and* (2)

$$S_{p,\varpi} = \sum_{\kappa'} \tau_{\kappa'} \chi_{\kappa'}(p)$$

when $\delta > 1$.

These propositions also are included in the statement or proof of P.N. 4, Lemma 13. The number of the κ' is $\epsilon - 1$ (P.N. IV, Lemma 7).

2. *Arithmetical progressions.*

2.1. In this section we prove

THEOREM 1. *The necessary and sufficient condition that every arithmetical progression should contain an infinity of numbers which are sums of s positive k-th powers is*

$$s \geqslant \Gamma(k),$$

except when $k = 4$, *in which case it is* $s \geqslant 15$.

The materials for the proof are contained in P.N. 4. We suppose, first, that $k \neq 4$.

(i) Suppose that $s \geqslant \Gamma(k)$. Since $s \geqslant \gamma_\varpi$, the congruence

$$\sum_1^s x_r^k \equiv n$$

is soluble to any modulus ϖ^λ (for $\lambda \geqslant \phi$ and therefore for all λ), and therefore to any modulus q. If $x_1, x_2, \ldots, x_s$ is a solution, then

$$\sum_1^s (mq + x_r)^k \equiv n \pmod{q}$$

for every m, so that the progression $Nq+n$ contains an infinity of numbers of the form required.

(ii) Suppose that every arithmetical progression contains numbers representable by s k-th powers (and that $k \neq 4$). Then for any ϖ, n there are numbers $\varpi^k m + n$ representable by s k-th powers, and the congruence

$$n \equiv \sum x_r^k \pmod{\varpi^k}$$

is soluble. For each of the values $n = 1, 2, \ldots, \varpi^\phi$ the solution is necessarily a primitive one, since otherwise n would be divisible by ϖ^k, and so by $\varpi^{\phi+1}$. *A fortiori* there is a primitive solution of the congruence

$$n \equiv \sum x_r^k \pmod{\varpi^\phi}$$

for each of these values of n, and since now n ranges through all residues

of the modulus we have $N(\rho, n) > 0$, and so $s \geqslant \gamma'_{\varpi} = \gamma_{\varpi}$. Since this is true for all ϖ, we have $s \geqslant \Gamma(k)$.

The argument in (i) fails for $k = 4$ because γ_{ϖ} and γ_{ϖ} differ when $\varpi = 2$. We have, in fact, $\gamma_2 = 16$, $\gamma'_2 = 15$, while

$$\gamma_{\varpi} = \gamma'_{\varpi} < 15$$

for every other ϖ. It is γ'_{ϖ} which is relevant in the argument (i) here, and every progression is permissible when $s \geqslant 15$. On the other hand, consider the progression $16m+15$. No number of this is representable by fewer than 15 fourth powers, and $s \geqslant 15$ is necessary.

3. *A theorem concerning the case* $\theta = 0$.

3.1. Our main theorem in this section is

THEOREM 2. *If* $k > 2$, $\theta = 0$, $d > 1$, *then*

$$\gamma_{\varpi} \leqslant k.$$

The principal difficulty lies in the representation of the residue 0 (mod ϖ), and the essential lemma is

LEMMA 11. *If* $\varpi > 2$, $\theta = 0$, *then the residue* 0 *has a representation by at most* s a_0*'s whenever*

$$s > 2, \quad \varpi \geqslant (\epsilon-1)^{2(s-1)/(s-2)}.$$

If $\epsilon = 1$, $d = \varpi - 1$, $c = 2$, and 0 is representable by 2 a_0's. If $\epsilon = 2$, $d = \frac{1}{2}(\varpi-1)$, $c = 3$, and it is representable by 3. These cases (in which the main inequality is nugatory) are therefore trivial, and we may suppose that $\epsilon > 2$.

Let $N+1$ be the number of solutions of

$$X = \sum_{r=1}^{s} x_r^k \equiv 0 \pmod{\varpi} \quad (0 \leqslant x_r < \varpi).$$

One solution is given by $x_r \equiv 0$ for all r, and so what we have to prove is that $N > 0$. Now $\sum_p e_{\varpi}(pX)$ has the value $\varpi - 1$ if $X \equiv 0$ and the value -1 if $X \not\equiv 0$. Hence

$$\begin{aligned} \sum_p S^s_{p,\varpi} &= \sum_p \sum_{x_r=0}^{\varpi-1} e_{\varpi}(pX) \\ &= \sum_{x_r} \sum_p e_{\varpi}(pX) = \sum_{X\equiv 0} \sum_p e_{\varpi}(pX) + \sum_{X \not\equiv 0} \sum_p e_{\varpi}(pX) \\ &= (N+1)(\varpi-1) - (\varpi^s - N - 1) = (N+1)\varpi - \varpi^s, \end{aligned}$$

$$(N+1)\varpi \geqslant \varpi^s - \sum_p |S_{p,\varpi}|^s. \tag{3.11}$$

Now
$$\delta = (\varpi - 1, k) = \epsilon,$$
and so, by Lemma 9,
$$(3.12) \qquad |S_{p,\varpi}| \leqslant (\epsilon - 1)\sqrt{\varpi}.$$
Also
$$\sum_p |S_{p,\varpi}|^2 = \sum_p \sum_{x,y=0}^{\varpi-1} e_\varpi\{p(x^k - y^k)\}$$
$$= \Bigl(\sum_{x^k \equiv y^k} + \sum_{x^k \not\equiv y^k}\Bigr) \sum_p e_\varpi\{p(x^k - y^k)\}$$
$$= \sum_{x^k \equiv y^k} (\varpi - 1) + \sum_{x^k \not\equiv y^k} (-1).$$
Given $x \not\equiv 0$, there are just ϵ y's for which $y^k \equiv x^k$. Hence
$$(3.13) \qquad \sum |S_{p,\varpi}|^2 = \{1 + \epsilon(\varpi - 1)\}(\varpi - 1) - \{\varpi^2 - 1 - \epsilon(\varpi - 1)\}$$
$$= (\epsilon - 1)\varpi(\varpi - 1).$$
From (3.12) and (3.13) we deduce
$$(3.14) \qquad \sum |S_{p,\varpi}|^s \leqslant \{(\epsilon - 1)\sqrt{\varpi}\}^{s-2} \sum |S_{p,\varpi}|^2 = (\epsilon - 1)^{s-1} \varpi^{\frac{1}{2}s}(\varpi - 1).$$
It then follows from (3.11) that $N > 0$ whenever
$$\varpi^s - \varpi > (\epsilon - 1)^{s-1} \varpi^{\frac{1}{2}s}(\varpi - 1),$$
and *a fortiori* when $\varpi^{s-1} \geqslant (\epsilon - 1)^{s-1} \varpi^{\frac{1}{2}s}$, the inequality of the lemma.

3.2. There is a companion result for residues other than 0. This is not required in the proof of any of our main theorems, but it is essential in the calculation of special values of $\Gamma(k)$, and has a certain independent interest.

LEMMA 12. *Suppose that* $\theta = 0$ *and* $n \not\equiv 0 \pmod{\varpi}$. *Then* n *has a representation by at most* s a_0'*s whenever*
$$s > 1, \quad \varpi > (\epsilon - 1)^{2s/(s-1)}.$$
As in Lemma 11, we may suppose $\epsilon > 2$. We have to prove that $N > 0$, N being now the number of solutions of
$$X \equiv n \pmod{\varpi}, \quad 0 \leqslant x_r < \varpi.$$
Now
$$(3.21) \qquad \sum_p S_{p,\varpi}^s e_\varpi(-np) = \sum_p \sum_{x_r=0}^{\varpi-1} e_\varpi\{p(X - n)\}$$
$$= \sum_{X \equiv n} \sum_p e_\varpi\{p(X - n)\} + \sum_{X \not\equiv n} \sum_p e_\varpi\{p(X - n)\}$$
$$= N(\varpi - 1) - (\varpi^s - N) = N\varpi - \varpi^s.$$

But, by Lemma 10,

$$S_{p,\varpi} = \Sigma \tau_{\bar{\kappa}'} \chi_{\kappa'}(p).$$

Hence

$$N\varpi - \varpi^s = \sum_p \Sigma^* \mathbf{T}\mathbf{X}(p)\, e_\varpi(-np) = \Sigma^* \mathbf{T} \sum_p \mathbf{X}(p)\, e_\varpi(-np),$$

where $\mathbf{T}$ is the product of s τ's, $\mathbf{X}(p)$ the product of s χ's (and so a character), and the number of terms in Σ^* is $(\epsilon-1)^s$. The inner sum is -1 if $\mathbf{X}(p)$ is the principal character, since $n \not\equiv 0$, and it is the product of a χ and a τ otherwise; so that, by Lemma 10, its absolute value does not exceed $\sqrt{\varpi}$. Hence

$$|N\varpi - \varpi^s| \leqslant (\epsilon-1)^s \varpi^{\frac{1}{2}(s+1)},$$

$$N\varpi \geqslant \varpi^s - (\epsilon-1)^s \varpi^{\frac{1}{2}(s+1)},$$

and $N > 0$ if $\varpi^s > (\epsilon-1)^s \varpi^{\frac{1}{2}(s+1)}$, the inequality of the lemma.

Since $(s-1)^2 > s(s-2)$, we have as a corollary

Lemma 13. *The result of Lemma 11 is true also for residues other than 0.*

We shall also require the trivial

Lemma 14. *If x is integral and not less than 4, then*

$$(x-1)^{2(x-1)/(x-2)} < x(2x-1).$$

3.3. *Proof of Theorem* 2. Since $\varpi \geqslant d+1 > 2$, we have $\phi = 1$, $c = \epsilon+1$. We begin by showing that we can make certain simplifying assumptions.

First, we may suppose $\epsilon > 2$. For $k = \epsilon k_0$, $\mathbf{k} = k/k_0 = \epsilon$. Hence, if $\epsilon \leqslant 2$, we have, by Lemma 6,

$$\gamma_\varpi(k) \leqslant 3 \leqslant k.$$

Next, we may suppose $k_0 = 1$, so that $k = \epsilon$. For suppose the theorem proved in this case. Then, since $\mathbf{k} = \epsilon > 2$, we have, by Lemma 6,

$$\gamma_\varpi(k) = \gamma_\varpi(\mathbf{k}) \leqslant \mathbf{k} = \epsilon \leqslant k.$$

Next, it follows from Lemma 3 that every residue (mod ϖ), except perhaps 0, is representable by at most $c-1 = \epsilon = k$ a_0's, and we have only to show that 0 is so representable.

Finally, it follows from Lemma 5 that 0 is so representable whenever d is even, and, in particular, whenever ϵ is odd; since we may then take $d_1 = 2$. We may therefore suppose that ϵ is even and d is odd.

To sum up, if the theorem is false for some k and ϖ, then a k and ϖ (not necessarily the same) exist for which

(A) $\varpi > 2,\ \theta = 0,\ k_0 = 1,\ k = \epsilon,\ c = \epsilon+1$;

(B) *ϵ is even, $\epsilon \geqslant 4$, d is odd*;

(C) *the residue* 0 *is not representable* (mod ϖ) *by fewer than* $\epsilon+1$ a_0's.

From these assumptions we deduce successively (D), (E), (F), (G) below. Since (G) contradicts (D), the theorem is then proved.

3.41. (D) $\varpi < \epsilon(2\epsilon-1)$, $d < 2\epsilon-1$†.

Take $s = \epsilon$ in Lemma 11. The result contradicts (C) unless $\varpi < (\epsilon-1)^{2(\epsilon-1)/(\epsilon-2)}$. Since $\epsilon \geqslant 4$, by (B), Lemma 14 gives $\varpi < \epsilon(2\epsilon-1)$; and then

$$d = (\varpi-1)/\epsilon < 2\epsilon-1.$$

3.42. (E) *There is an a_0, say a_0^*, such that a_0^*+1 is also an a_0.*

First, the number of classes representable by two or fewer a_0's is two at most; for otherwise we should have, by Lemma 4,

$$\mu(c-1) \geqslant \min\{\mu(2)+c-3,\ c\} = c,$$

and all residues would be representable by $c-1 = \epsilon$ a_0's, contrary to (C).

Next, if no a_0^* exists, no a_0+1 is an a_0, and, in particular, $1+1=2$ is not an a_0, so that every a_0+1 belongs to the same class as 2. It follows that $a_0+1 \equiv 2a_0'$ for every a_0 and some a_0'. Incongruent a_0's correspond to incongruent a_0''s and a_0' runs through the whole of C_0 with a_0. Hence, by Lemma 1,

$$\Sigma(a_0+1) \equiv 2\Sigma a_0, \quad d \equiv \Sigma a_0 \equiv 0;$$

and this is impossible because $d \mid \varpi-1$.

3.43. (F) $a_0^* \not\equiv 1$ (mod ϖ).

If $a_0^* \equiv 1$, 2 is an a_0, by (E); and so $1, 2, 2^2, 2^3, \ldots$ are all a_0's. Now any number up to $2^{\epsilon+1}-2$ can be represented in the binary scale by at most ϵ non-zero digits; and, after (D),

$$\varpi < \epsilon(2\epsilon-1) < 2^{\epsilon+1}-2$$

(since $\epsilon \geqslant 4$). Hence

$$\varpi = 2^{n_1}+2^{n_2}+\ldots+2^{n_r} \quad (0 \leqslant n_r < n_{r-1} < \ldots < n_1,\ r \leqslant \epsilon),$$

and $\varpi \equiv 0$ is representable by at most ϵ a_0's, contrary to (C).

† To avoid any misunderstanding, we repeat that (D)–(G) are not true theorems, but deductions from hypotheses which prove untenable.

3.43. (G) $d > 2\epsilon - 1$.

Take $a = a_0^*$ in Lemma 2. Then a and $1+a$ are a_0's, so that $a^m(1+a)$ is an a_0 for every m. Since d is odd, Lemma 2 gives

$$0 \equiv 1+a+a^2+\ldots+a^{d-1} = (1+a)+(1+a)\,a^2+\ldots+(1+a)\,a^{d-3}+a^{d-1}.$$

This is a representation of 0 by $\frac{1}{2}(d-3)+2 = \frac{1}{2}(d+1)$ a_0's. Hence, by (C),

$$\tfrac{1}{2}(d+1) > \epsilon, \quad d > 2\epsilon - 1.$$

Finally, (G) contradicts (D). Hence (A), (B), (C) are not simultaneously possible; and this proves the theorem.

4. *A theorem concerning the case $\theta > 0$.*

4.1. THEOREM 3. *If $\theta > 0$, $d > 2$, then*

$$\gamma_\varpi \leqslant k.$$

The main difficulties here arise with residues (mod ρ) other than 0.

We denote by $N[m]$, where $m > 0$, any number that is a sum of m k-th powers (positive or zero). The symbol is not a one-valued function of m; m (together with k) determines a certain class of integers, and the symbol denotes an undetermined member of this class. An equation in N's implies that any number that is of the form on the left of the equation is also of the form on the right.

4.2. LEMMA 15. $N[m]N[n] = N[mn]$.

For
$$N[m]N[n] = \sum_{i=1}^{m} x_i^k \sum_{j=1}^{n} y_j^k = \sum_{i,j} (x_i y_j)^k.$$

LEMMA 16. *For any n*

$$n \equiv N[\epsilon] \pmod{\varpi}.$$

This is trivial if $n \equiv 0 \pmod{\varpi}$, since 0 is the sum of ϵ 0^{k}'s. If $n \not\equiv 0 \pmod{\varpi}$, let $k' = \epsilon$, and let c', a_0' denote the number c and a typical a_0 for the index k'. Then, by Lemma 3, n can be expressed as a sum of at most $c'-1 = \epsilon$ a_0''s. Now an a_0 taken (mod ϖ) is plainly an a_0'; moreover, since the a_0's are incongruent (mod ϖ) (§ 1.3), and since the number d is the same for k and ϵ (for the prime ϖ), it follows that the set of a_0' is identical with the set of a_0 considered to mod ϖ. Hence n is representable (mod ϖ) by not more than ϵ a_0's.

4.3. We come now to the lemma containing the crucial step in the proof.

LEMMA 17†. *If $d > 2$ then there is a u, not a multiple of ϖ, such that*

$$u\varpi \equiv N[\varpi-1] \pmod{\varpi^2}.$$

Since $d \geqslant 3$, there exist three k-th powers, 1^k, α^k, β^k prime to ϖ and incongruent $(\mathrm{mod}\ \varpi)$. We define two positive integers a, b by

$$a \equiv \alpha^k \pmod{\varpi^2} \quad (0 < a < \varpi^2),$$

$$b \equiv \beta^k \pmod{\varpi^2} \quad (0 < b < \varpi^2).$$

Since neither a nor b is congruent to 0 or to 1 $(\mathrm{mod}\ \varpi)$, there exist integers μ, ν such that

$$(\mu-1)\varpi+2 \leqslant a \leqslant \mu\varpi-1 \quad (0 < \mu \leqslant \varpi),$$

$$(\nu-1)\varpi+2 \leqslant b \leqslant \nu\varpi-1 \quad (0 < \nu \leqslant \varpi).$$

Then $$\mu\varpi = a+m \equiv \alpha^k+m \pmod{\varpi^2},$$

where $$1 \leqslant m \leqslant \varpi-2.$$

Thus $\mu\varpi$ is representable $(\mathrm{mod}\ \varpi^2)$ as a sum of $1+m \leqslant \varpi-1$ k-th powers, and similarly for $\nu\varpi$. The result of the lemma therefore holds (with $u=\mu$ or $u=\nu$) unless $\mu=\nu=\varpi$. Now in this case, since a and b cannot both be ϖ^2-1, one at least, say a, must satisfy

$$(\varpi-1)\varpi+2 \leqslant a \leqslant \varpi^2-2,$$

so that $\varpi^2 = a+m$, where $2 \leqslant m \leqslant \varpi-2$. Then, h denoting *any* positive integer,

$$-\varpi = -hm+(hm-\varpi) \equiv h\alpha^k+(hm-\varpi) \pmod{\varpi^2}.$$

In this we take $h = h_m$, the least integer not less than ϖ/m. Then $-\varpi$ is expressible $(\mathrm{mod}\ \varpi^2)$ as a sum of

$$s_m = h_m+(h_m m-\varpi)$$

k-th powers, and the result of the lemma (with $u=-1$) will certainly hold if it is true that $s_m \leqslant \varpi-1$ for every m in $2 \leqslant m \leqslant \varpi-2$. Now

† In our original version the form of this lemma was slightly different. The present form, which leads rather more simply to Theorem 3, was suggested to us by Mr. Ingham. We are indebted to him also for the proof given here of the lemma; this represents a great gain in simplicity over the original. We have finally to thank Mr. Ingham also for detecting a number of minor errors in the paper as a whole.

$m(h_m-1) \leqslant \varpi-1$. Hence

$$s_m \leqslant (m+1)\left(1+\frac{\varpi-1}{m}\right)-\varpi = m+\frac{\varpi-1}{m}.$$

If now $\frac{1}{2}(\varpi-1) < m \leqslant \varpi-2$, then $s_m < \varpi-2+2 = \varpi$. If on the other hand $2 \leqslant m \leqslant \frac{1}{2}(\varpi-1)$, then $s_m \leqslant \frac{1}{2}(\varpi-1)+\frac{1}{2}(\varpi-1) = \varpi-1$. In either case $s_m \leqslant \varpi-1$, and the proof of the lemma is completed.

Lemma 18. *For any integer n*

$$n\varpi \equiv N[\epsilon(\varpi-1)] \pmod{\varpi^2}.$$

We choose v so that $uv \equiv n \pmod{\varpi}$, u being the u of Lemma 17. Then

$$n\varpi \equiv v\,.\,u\varpi \equiv N[\epsilon]\,.\,u\varpi \pmod{\varpi^2} \qquad \text{(Lemma 16)}$$

$$\equiv N[\epsilon]\,N[\varpi-1] \equiv N[\epsilon(\varpi-1)] \pmod{\varpi^2}$$

(Lemmas 17, 15)

4.4. *Proof of Theorem* 3. For any n we have

$$\begin{aligned}
n &= N[\epsilon]+n_1\varpi \quad \text{(Lemma 16)}\\
&= N[\epsilon]+N[\epsilon(\varpi-1)]+n_2\varpi^2 \quad \text{(Lemma 18)}\\
&= N[\epsilon]+N[\epsilon(\varpi-1)]+\varpi N[\epsilon(\varpi-1)]+n_3\varpi^3 \quad \text{(Lemma 18)}\\
&= \dots\dots\dots\\
&= N[\epsilon]+N[\epsilon(\varpi-1)]+\varpi N[\epsilon(\varpi-1)]+\dots+\varpi^{\theta-1}N[\epsilon(\varpi-1)]+n_{\theta+1}\varpi^{\theta+1}\\
&\equiv N[\epsilon+\epsilon(\varpi-1)+\epsilon(\varpi^2-\varpi)+\dots+\epsilon(\varpi^{\theta}-\varpi^{\theta-1})] \pmod{\rho}\\
&\equiv N[\epsilon\varpi_\theta] \pmod{\rho}.
\end{aligned}$$

If $n \not\equiv 0 \pmod{\rho}$ the $\epsilon\varpi^\theta$ k-th powers by which n is thus represented $(\text{mod}\ \rho)$ cannot all be divisible by ϖ, since $\phi \leqslant k$. Hence any $n \not\equiv 0$ is representable $(\text{mod}\ \rho)$ as a sum of at most $\epsilon\varpi^\theta$ a_0's. The same thing is true of the residue 0, by Lemma 5 (since $1 < d < \varpi^\theta\epsilon$). Hence

$$\gamma^{\varpi} \leqslant \varpi^\theta\epsilon \leqslant k.$$

5. *Determination of all cases in which* $\Gamma(k) > k$

5.1. We use π, like ϖ, to denote an odd prime; there is no danger of confusion with the common notation†. We distinguish six special

† π, ϑ are given by k; ϖ is a prime which may assume any value when k is given, and θ is determined by k and ϖ.

classes of values of k, viz. :

I. $k = 2^{\vartheta}$ $(\vartheta > 1)$.

II. $k = 2^{\vartheta}.3$ $(\vartheta > 1)$.

III. $k = \pi^{\vartheta}(\pi-1)$ $(\vartheta > 0)$.

IV. $k = \frac{1}{2}\pi^{\vartheta}(\pi-1)$ $(\vartheta > 0)$.

V. $k = \pi-1$, *k not belonging to any preceding class.*

VI. $k = \frac{1}{2}(\pi-1)$, *k not belonging to any preceding class.*

It is easily verified that these classes are mutually exclusive.

THEOREM 4. *The values of $\Gamma(k)$ when k belongs to a special class are given by the formulae* :

I. $\Gamma(k) = \gamma_2 = 2^{\vartheta+2} = 4k > k+1$.

II. $\Gamma(k) = \gamma_2 = 2^{\vartheta+2} = \frac{4}{3}k > k+1$.

III. $\Gamma(k) = \gamma_\pi = \pi^{\vartheta+1} > k+1$.

IV. $\Gamma(k) = \gamma_\pi = \frac{1}{2}(\pi^{\vartheta+1}-1) > k+1$, *except in the particular case* $k = 3 = \frac{1}{2}\,.\,3\ (3-1)$, *when* $\Gamma(k) = k+1$.

V. $\Gamma(k) = \gamma_\pi = \pi = k+1$.

VI. $\Gamma(k) = \gamma_\pi = \frac{1}{2}(\pi-1) = k$.

When k does not belong to a special class, then

$$\Gamma(k) \leqslant k.$$

The values of $\Gamma(k)$ in cases I–IV are given in Lemma 8. We suppose then that k does not belong to any of these classes.

Let ϖ be any prime. We shall prove that $\gamma_\varpi \leqslant k$, except when k belongs to class V and $\varpi = \pi$. There are four cases to consider :

(i) $\varpi = 2$, $\theta = 0$. Here $\gamma_\varpi = \gamma_2 = 2 < k$, by Lemma 7.

(ii) $\varpi = 2$, $\theta > 0$. Here $\epsilon = 1$, and k_0 is odd and greater than 3, since otherwise k would belong to class I, class II, or class III. Hence, by Lemma 7,

$$\gamma_\varpi = \gamma_2 = 2^{\theta+2} = 4k/k_0 < k.$$

(iii) $\varpi > 2$, $\theta = 0$. If $k_0 > 1$, then

$$\gamma_\varpi \leqslant c = \epsilon+1 \leqslant \epsilon k_0 = k,$$

by Lemma 3. If $k_0 = 1$, $d > 1$, then $\gamma_\varpi \leqslant k$ by Theorem 2. If $k_0 = 1$, $d = 1$, then $\epsilon = \varpi-1$, $k = \varpi-1$; and this is case V, with $\pi = \varpi$.

(iv) $\varpi > 2$, $\theta > 0$. If $k_0 > 1$, then, by Lemma 3,

$$\gamma_\varpi \leqslant c = \frac{\varpi^{\theta+1}-1}{\varpi-1}\epsilon+1 < 2\varpi^\theta\epsilon+1 \leqslant \varpi^\theta\epsilon k_0+1 = k+1,$$

and so $\gamma_\varpi \leqslant k$. If $k_0 = 1$, then $d > 2$, since $d = 1$ or 2 makes k belong to class III or class IV. Hence $\gamma_\varpi \leqslant k$, by Theorem 3.

It remains only to observe that $\gamma_\pi = \pi = k+1$ in case V, and $\gamma_\pi = \frac{1}{2}(\pi-1) = k$ in case VI, by Lemma 7. With this the proof is completed.

6. *The numerical calculation of* $\Gamma(k)$.

6.1. Table I gives the values of $\Gamma(k)$ for $k \leqslant 36$, the special class, if any, to which k belongs, and (under the heading ϖ) the least ϖ (in case more than one exists) for which $\gamma_\varpi = \Gamma(k)$.

The value of $\Gamma(k)$ may be found for any k by a finite process of calculation. A routine method may be set out as follows.

We observe first that $\Gamma(k) \geqslant 3$. Since $\Gamma(kk') \geqslant \Gamma(k)$ and $\Gamma(2^\vartheta) = 2^{\vartheta+2} > 3$ when $\vartheta \geqslant 2$, it is necessary to prove this only when k is an odd prime π. In this case there are $\pi-1$ π-th powers (mod π^2), at most $(\pi-1)^2+\pi-1 < \pi^2$ numbers are representable by *one or two* of them, and so $\gamma_\pi(\pi) \geqslant 3$.

We cannot prove that $\Gamma(k) \geqslant 4$, though no case of $\Gamma(k) = 3$ is known. Lemmas 11 and 13 show that $\gamma_\varpi \leqslant 3$ provided that

$$\varpi > (k-1)^{2(3-1)/(3-2)} = (k-1)^4 \dagger.$$

The number γ_2 is given for any k by Lemma 7, and, in particular, we know that for an even $k > 3$ we have $\gamma_2 \geqslant 8$, $\Gamma(k) \geqslant 8$. If now no (odd) ϖ exists for which $\gamma_\varpi > \text{Max}(\gamma_2, 3)$, then $\Gamma(k) = \text{Max}(\gamma_2, 3)$. If, however, such ϖ's exist, they must satisfy $\varpi \leqslant (k-1)^4$, they can be found by a finite search, and $\Gamma(k)$ is equal to the greatest of the γ_ϖ corresponding.

A prime ϖ giving in this way $\Gamma(k) = \gamma_\varpi > \text{Max}(\gamma_2, 3)$ must further satisfy either (i) $\varpi \mid k$, or (ii) $\varpi \equiv 1 \pmod{\delta}$, where $\delta \mid k$ and δ is not unity or a positive power of 2. For if such a ϖ satisfies neither (i) nor (ii) we must have $\theta = 0$ and $\epsilon = 2^a$, where $a \geqslant 0$, in which case

$$\gamma_\varpi \leqslant c = 2^a+1 \leqslant 2^{a+1} \leqslant \gamma_2.$$

In case (ii) ϖ belongs to the progression $m\delta+1$; it is fairly certain in practice that it will be the *least* prime in its progression.

An examination of the prime divisors of k and of the least primes in the progressions $m\delta+1$ generally suggests fairly strongly some numerical

† So that the condition $\theta = 0$ for the applicability of Lemmas 11 and 13 is satisfied.

value s for $\Gamma(k)$. We have then to prove, first, that $\gamma_\varpi = s$ for the selected ϖ, and then that $\gamma_\varpi \leqslant s$ both for every prime divisor of k and for every ϖ, in each of the progressions, for which

$$\varpi \leqslant (\epsilon-1)^{2(s-1)/(s-2)}.$$

If k is odd, however, we need consider only, in the progressions, primes satisfying the more favourable inequality

$$\varpi \leqslant (\epsilon-1)^{2s/(s-1)}.$$

For in this case the a_0's are equal and opposite in pairs, and 0 is representable by two of them, so that we may use Lemma 12 instead of Lemma 13.

It is easy to compute $\Gamma(k)$ with *practical* certainty for values of k considerably larger than 36; but the necessity of examining primes of the order k^4 makes a complete proof impracticable for those k whose $\Gamma(k)$ happens to be small.

6.2. *An example*: $k = 34$. We illustrate the process by proving that $\Gamma(34) = 10$. Here $\gamma_2 = 8$, by Lemma 7. It is improbable that $\Gamma(k)$ is so small as this, and the most likely values for the critical ϖ are 17 and 103, the smallest prime in the progression $17m+1$, the only progression that has to be considered. Since $103 < 17^2 = 289$, we begin by examining $\varpi = 103$.

We denote by ν_r a typical number, other than 0, representable (mod 103) by just r a_0's; ν_1 is an a_0. Since $d = 3$, 0 is representable by three a_0's. Calculation gives

ν_1 1, 46, 56.

ν_2 2, 9, 47, 57, 92, 102.

ν_3 3, 10, 35, 45, 48, 55, 58, 65, 93.

ν_4 4, 8, 11, 18, 36, 49, 59, 66, 81, 91, 94, 101.

ν_5 5, 12, 19, 24, 34, 37, 44, 50, 54, 60, 64, 67, 74, 82, 90, 95.

ν_6 6, 7, 13, 17, 20, 25, 27, 38, 51, 61, 68, 70, 75, 80, 83 96 100.

ν_7 14, 21, 23, 26, 28, 33, 39, 43, 52, 53, 62, 63, 69, 71, 73, 76, 84, 97.

ν_8 15, 16, 22, 29, 40, 72, 77, 79, 85, 89, 98, 99.

ν_9 30, 32, 41, 42, 78, 86.

ν_{10} 31, 87, 88.

These numbers exhaust the residues: hence $\gamma_{103} = 10$.

It remains to verify that $\gamma_\varpi \leqslant 10$ for the remaining primes $34m+1$, and that $\gamma_{17} \leqslant 10$. Taking $\varpi = 34m+1$, and $s = 10$ in Lemmas 11 and 12, we have to verify that any residue (mod ϖ), other than 0, is

representable by ten a_0's when

$$(6.21) \qquad \varpi \leqslant (34-1)^{2.10/(10-1)} < 2380,$$

and that 0 is similarly representable in the further range

$$(6.22) \qquad 2381 \leqslant \varpi \leqslant (34-1)^{2(10-1)/(10-2)} < 2614.$$

The values of ϖ satisfying (6 . 22) are 2381, 2551. For these d is 70 and 75, both divisible by $d_1 = 5$. Hence in these cases, by Lemma 5, 0 is representable by five a_0's.

The ϖ's of the range (6 . 21) are 137, 239, 307, 409, 443, 613, 647, 919, 953, 1021, 1123, 1259, 1327, 1361, 1429, 1531, 1667, 1871, 1973, 2143, 2347.

We have to show that any residue to any of these moduli is representable by at most ten 34-th powers, and calculation shows that this is possible with a good deal to spare. With the higher moduli it is not necessary to calculate more than about ten a_0's.

Finally, to prove that $\gamma_{17} \leqslant 10$, we have to show that all residues (mod 289) are similarly representable; and here again there is something to spare.

6 . 3. It is natural to expect that $\Gamma(k)$ tends to infinity with k. But so far are we from being able to prove this that (as we stated in § 6 . 1) we cannot prove even

$$\underline{\lim}\, \Gamma(k) \geqslant 4\,;$$

and these problems seem to be extremely difficult. Thus to prove $\Gamma(k) \geqslant 4$ for all, or all large, k, or, what is the same thing, to prove $\Gamma(\pi) \geqslant 4$ for all, or all large, π, it is necessary to prove, for each π, that either $\gamma_\pi(\pi) \geqslant 4$ or $\gamma_\varpi(\pi) \geqslant 4$ for some prime $\varpi = 2m\pi+1$. The first alternative is improbable for a given π (since the number of a_0's is $d = \pi - 1$). The second would be established if we could show that, for any large π, a prime $\varpi = 2m\pi+1$ exists with $d = 2m < \varpi^{\frac{1}{3}} - 1$†, which is roughly the same as $2m < \pi^{\frac{1}{2}}$. Such a theorem, if true, must be very deep, and it is not easy to find other ways of attacking the problem.

6 . 4. This being so, it is worth while to carry our numerical computations beyond the limits of Table 1, contenting ourselves with lower bounds for $\Gamma(k)$ when the exact calculation becomes impracticable. In what follows we suppose that $\varpi > 2$, and we use the following considerations to assign a lower bound for γ_ϖ (where we cannot appeal to any more precise result).

† For if $d + d^2 + d^3 < \varpi$ there must exist some residue not representable by one, two, or three a_0's.

We use $N_s(d)$ to denote either the number of distinct residues (mod ρ), other than 0, which are representable by s or fewer a_0's, *or any upper bound of this number*. Here d has its usual meaning. If $N_s(d)$ were the exact number of residues, the notation would be inappropriate, since it would depend on k otherwise than through d; but it will appear that, by taking an appropriate upper bound, we can make it depend on d and s only. Similarly we use $\nu_s(d)$ to denote an upper bound for the number of residues representable by s a_0's and not by less†. We may plainly take

$$(6.41) \qquad N_s(d) = \sum_{r=1}^{s} \nu_r(d).$$

Let $P_r(s)$ be the number of partitions of s into just r of 1, 2, 3, .., repetitions being allowed and order counting. Then

$$(6.42) \qquad P_r(s) = \binom{s-1}{r-1} \quad (r \leqslant s), \quad P_r(s) = 0 \quad (r > s)‡.$$

Let

$$(6.43) \qquad Q_r(s) = \sum_{t=1}^{s} P_r(t) = \sum_{t=r}^{s} P_r(t) = \binom{s}{r}.$$

Finally, when $d > 1$, let $D_r(d)$ be an upper bound for the number of combinations of d a_0's just r at a time, order not counting, repetitions not being allowed, and any combination which contains a set of a_0's whose sum is congruent to 0 (mod ρ) being rejected§. Evidently $D_r(d) = 0$ when $r > k$; and this is true also for $r = d$, since $\Sigma a_0 \equiv 0$, by Lemma 1. And

$$(6.44) \qquad D_r(d) \leqslant \binom{d}{r} \quad (r < d).$$

A little consideration shows that the number of residues which are representable by just s a_0's, and in which just r different a_0's appear, is at most $P_r(s)\, D_r(d)$. Hence we may take

$$\nu_s(d) \leqslant \sum_{r=1}^{d-1} P_r(s)\, D_r(d),$$

$$(6.45) \quad N_s(d) \leqslant \sum_{r=1}^{d-1} Q_r(s)\, D_r(d) = \sum_{r=1}^{d-1} \binom{s}{r} D_r(d) \leqslant \sum_{r=1}^{d-1} \binom{s}{r}\binom{d}{r}.$$

It is evident that, if $N_s(d) < \varpi^\phi - 1$, then $\gamma_\varpi \geqslant s+1$. Hence (6.45) may be used to find a lower bound for γ_ϖ. If d is *even*, however, we

† Thus if -1 is an a_0, $a_0+1+(-1)$ represents a_0 by three a_0's, but this representation is to be rejected.

‡ $P_r(s)$ is the coefficient of x^s in $(x+x^2+x^3+\dots)^r$.

§ Without this provision the number of combinations would of course be $\binom{d}{r}$

can do better than (6 . 45). For, if $a = 2d_1$, the a_0's are of the type

$$\pm a_1, \pm a_2, \ldots, \pm a_{d_1},$$

and, in estimating $D(d)$, we can reject any combination in which a_r and $-a_r$ both occur. This leads to the inequalities

$$(6.46) \quad D_r(d) \leqslant \frac{d(d-2)\ldots(d-2r+2)}{r!} \quad (0 < r \leqslant d_1), \quad D_r(d) = 0 \ (r > d_1).$$

$$(6.47) \qquad N_s(d) \leqslant \sum_{r=1}^{d_1} \binom{s}{r} \frac{d(d-2)\ldots(d-2r+2)}{r!}.$$

We use (6 . 45) when $d = 3$, and (6 . 46) when d is even and greater than 3. It so happens that we do not find it necessary to consider odd values of d greater than 3.

The values of $N_s(d)$, given by (6 . 45) and (6 . 46), are set out in Table 2. If one of them is less than ρ, then $\gamma_{\varpi} \geqslant s+1$.

6 . 5. Table 3 gives (1) the values of $\Gamma(k)$ when k belongs to a special class, (2) lower bounds for $\Gamma(k)$ in other cases, for $36 < k \leqslant 200$. The special values are starred; in all other cases we know, of course, that $\Gamma(k) \leqslant k$.

The lower bounds are found in four ways.

(i) In the first place

$$\Gamma(k) \geqslant \operatorname*{Max}_{\delta \mid k} \Gamma(\delta).$$

When the lower bound is found in this way, we write $k > \delta$ (*e.g.* $45 > 9$) in the k-column, δ being the relevant divisor of k.

(ii) If ϖ is the least prime $3k+1$ or $dk+1 = 2d_1k+1$, we can obtain a lower bound for γ_{ϖ}, and so for $\Gamma(k)$, by the use of Table 2 and the argument of § 6 . 4. In this case the relevant value of d is entered in the second column. It so happens that this d is, in fact, the least d, odd or even, for which $dk+1$ is prime, except in the single case $k = 62$, when $d = 5$ and $d = 6$ both give primes, and $d = 6$ gives the better lower bound. In a larger table, of course, it might be necessary to consider odd d greater than 3.

(iii) In one case, $k = 197$, when $d = 18$, $\varpi = 3547$, we have, in order to show that $\Gamma(197) > 4$, gone beyond the principles of § 6 . 4 and resorted to calculation†. The tables then show that, for $k \leqslant 200$, $\Gamma(k) > 3$, and that, except for 2, 3, 7, and 19, $\Gamma(k) > 4$.

† For which we are indebted to Mr. F. G. Maunsell. The a_0's occur in equal and opposite pairs, half of them being

1, 291, 447, 1162, 1163, 1177, 1468, 1548, 1552.

The residue 7 requires five of these.

(iv) In two cases, $k = 38$ and $k = 62$, the best result is derived from the fact that $\gamma_2 = 8$ (Lemma 7). Here we have written $38 = 2 \,.\, 19$ and $62 = 2 \,.\, 31$ in the k-column.

When the actual lower bound is given by (i) or (iv), we have still entered in the second column the appropriate d, or "$\geqslant b$", where b is a lower bound of it, so that it can be verified that (ii) does not give a better result Further refinements of the principles employed in (ii) appear to be practically ineffective.

6.6. We have explored the possibilities $\Gamma(k) = 3$, $\Gamma(k) = 4$ in the range $2 < k \leqslant 3000$. Our results are that $\Gamma(k) > 3$ in all cases; and $\Gamma(k) > 4$, except for $k = 2, 3, 7, 19$, for which it is 4, and possibly (but very improbably) $k = 1163, 1637, 1861, 1997, 2053$.

We need only consider odd k, since $\gamma_2 \geqslant 8$ for even k. In the second it is easy to see that we can confine ourselves to proving that only these exceptions occur among *prime* $k < 3000$. For this result, together with the inequality $\Gamma(kk') \geqslant \Gamma(k)$, shows that $\Gamma(k) > 3$ for odd composite $k \leqslant 3000$ (indeed, $k \leqslant 200^2$), and $\Gamma(k) > 4$, unless k is divisible by *two* of 3, 7, 19. Now all products of two of these numbers, except $19^2 = 361$, occur in Table 3 with a $\Gamma(k) > 4$. In the case $k = 361$ we consider $\gamma_{19}(361)$, when $\varpi^\phi = 19^3$, $d = 18$, and the argument of § 6.4 shows that $\gamma_{19} > 4$. It follows that we need consider only *prime* values of k.

Considering now only primes below 3000, we calculate for each k the least d (necessarily even) for which $dk+1$ is prime. We find

$d \leqslant 12$ for $200 < k < 227$;

$d \leqslant 16$ for $227 \leqslant k < 540$, except for $k = 227$ $(d = 24)$ and $k = 457$ $(d = 30)$;

$d \leqslant 22$ for $540 < k < 1000$;

$d \leqslant 26$ for $1000 < k < 1302$, except for $k = 1163$ $(d = 32)$;

$d \leqslant 30$ for $1302 < k < 2000$, except for $k = 1637$ $(d = 38)$, $k = 1861$ $(d = 40)$, and $k = 1997$ $(d = 44)$;

$d \leqslant 34$ for $2000 < k < 2500$, except for $k = 2053$ $(d = 46)$;

$d \leqslant 36$ for $2500 < k < 3000$.

The sufficient condition for $\Gamma(k) > 3$ is

$$N_3(d) < \varpi - 1 = dk,$$

which is

$$d^2 + 3d + 8 < 6k; \tag{6.61}$$

and that for $\Gamma(k) > 4$ is

$$d^3+4d^2+20d+32 < 24k. \tag{6.62}$$

The inequality (6.61) is satisfied in all cases with a good deal to spare, and the limit 3000 could be extended considerably without excessive labour. The inequality (6.62) is satisfied by the upper bounds of d set out for the various ranges of k above. Thus $\Gamma(k) > 4$, except in the exceptional cases, which require detailed calculation. Mr. Maunsell has carried this out for $k = 227$ and $k = 457$†, but beyond this point the work becomes very heavy.

TABLE 1.

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\Gamma(k)$	4	16	5	9	4	32	13	12	11	16	6	14	15	64	6	27	4
Class	IV	I	VI	III	—	I	IV	IV	VI	II	—	VI	VI	I	—	III	—
ϖ	3	2	11	3	7	2	3	5	23	2	53	29	31	2	103	3	229

k	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$\Gamma(k)$	25	24	23	23	32	10	26	40	29	29	31‡	5	128	33	10	35	36
Class	III	IV	V	VI	II	—	VI	IV	V	VI	V	—	I	VI	—	VI	V
ϖ	5	7	23	47	2	101	53	3	29	59	31	311	2	67	103	71	73

TABLE 2: $N_s(d)$.

$s \backslash d$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	30	45	62	84	108	135	165	198	234	273	315	360	408	459	513	570	630	693
4	40	60	82	112	144	180	220	264	312	364	420	480	544	612	684	760	840	
6	120	210	336	504	740	990	1320	2052										
8	320	680	1288															
10	680	1682																
12	1288																	

† For $k = 227$, $\varpi = 3449$, the a_0's are

1, 78, 489, 543, 603, 621, 635, 1238, 1474, 1475, 1518, 2007,

and the same numbers with negative sign. The residue 21 requires 5.

For $k = 457$, $\varpi = 13711$, the a_0's are

1, 702, 792, 1060, 1394, 2186, 2425, 3127, 3148, 3149, 3442, 3726, 5107, 6167, 6543,

and the same numbers with negative sign. The residue 52 requires 5.

‡ There is a misprint, 30 for 31, in the value of $\Gamma(30)$ given in the table of P.N. 4.

TABLE 3.

k	d	Γ	k	d	Γ
37	4	$\geqslant 9$	$^*81 = \frac{1}{2}3^4(3-1)$	—	121
$38 = 2.19$	6	$\geqslant 8$	$^*82 = 83-1$	1	83
$^*39 = 79-1$	2	39	$^*83 = \frac{1}{2}(167-1)$	2	83
$^*40 = 41-1$	1	41	$84 > 42$	$\geqslant 3$	$\geqslant 49$
$^*41 = \frac{1}{2}(83-1)$	2	41	$85 > 17$	$\geqslant 12$	$\geqslant 6$
$^*42 = 7(7-1)$	—	49	$^*86 = \frac{1}{2}(173-1)$	2	86
$^*43 = \frac{1}{2}(87-1)$	2	43	$87 > 29$	$\geqslant 3$	$\geqslant 29$
$^*44 = \frac{1}{2}(89-1)$	2	44	$^*88 = 89-1$	1	89
$45 > 9$	4	$\geqslant 13$	$^*89 = \frac{1}{2}(179-1)$	2	89
$^*46 = 47-1$	1	47	$^*90 = \frac{1}{2}(181-1)$	2	90
47	6	$\geqslant 6$	$91 > 13$	6	$\geqslant 8$
$^*48 = 2^4.3$	—	64	$92 > 23$	$\geqslant 3$	$\geqslant 23$
49	4	$\geqslant 10$	93	4	$\geqslant 14$
$^*50 = \frac{1}{2}(101-1)$	2	50	94	3	$\geqslant 14$
$^*51 = \frac{1}{2}(103-1)$	2	51	$^*95 = \frac{1}{2}(191-1)$	2	95
$^*52 = 53-1$	1	53	$^*96 = 3.2^5$	—	128
$^*53 = \frac{1}{2}(107-1)$	2	53	97	4	$\geqslant 14$
$^*54 = 3^3(3-1)$	—	81	$^*98 = \frac{1}{2}(197-1)$	2	98
$^*55 = \frac{1}{2}11(11-1)$	—	60	$^*99 = \frac{1}{2}(199-1)$	2	99
$^*56 = \frac{1}{2}(113-1)$	2	56	$^*100 = 5^2(5-1)$	—	125
57	4	$\geqslant 11$	101	6	$\geqslant 8$
$^*58 = \frac{1}{2}(117-1)$	1	59	$^*102 = 103-1$	1	103
$^*59 = \frac{1}{2}(119-1)$	2	59	103	6	$\geqslant 8$
$^*60 = 61-1$	1	61	$^*104 = \frac{1}{2}(209-1)$	2	104
61	6	$\geqslant 7$	$^*105 = \frac{1}{2}(211-1)$	2	105
$62 = 2.31$	6	$\geqslant 8$	$^*106 = 107-1$	1	107
$^*63 = \frac{1}{2}(127-1)$	2	63	107	6	$\geqslant 8$
$^*64 = 2^6$	—	256	$108 > 54$	$\geqslant 4$	$\geqslant 81$
$^*65 = \frac{1}{2}(131-1)$	2	65	109	10	$\geqslant 5$
$66 > 11$	3	$\geqslant 11$	$^*110 = 11(11-1)$	—	121
67	4	$\geqslant 12$	$^*111 = \frac{1}{2}(223-1)$	2	111
$^*68 = \frac{1}{2}(137-1)$	2	68	$^*112 = 113-1$	1	113
$^*69 = \frac{1}{2}(139-1)$	2	69	$^*113 = \frac{1}{2}(227-1)$	2	113
$^*70 = \frac{1}{2}(141-1)$	1	71	$^*114 = \frac{1}{2}(229-1)$	2	114
71	8	$\geqslant 5$	$115 > 23$	$\geqslant 4$	$\geqslant 23$
$^*72 = 73-1$	1	73	$^*116 = \frac{1}{2}(233-1)$	2	116
73	4	$\geqslant 12$	$117 > 9$	$\geqslant 6$	$\geqslant 13$
$^*74 = \frac{1}{2}(149-1)$	2	74	$118 > 59$	$\geqslant 3$	$\geqslant 59$
$^*75 = \frac{1}{2}(151-1)$	2	75	$^*119 = \frac{1}{2}(239-1)$	2	119
76	3	$\geqslant 12$	$^*120 = \frac{1}{2}(241-1)$	2	120
$77 > 11$	6	$\geqslant 11$	$121 > 11$	$\geqslant 6$	$\geqslant 11$
$^*78 = \frac{1}{2}13(13-1)$	—	84	122	3	$\geqslant 16$
79	4	$\geqslant 13$	$123 > 41$	$\geqslant 4$	$\geqslant 41$
$^*80 = \frac{1}{2}(161-1)$	2	80	$124 > 4$	3	$\geqslant 16$

TABLE 3—(*continued*).

k	d	Γ	k	d	Γ
$*125 = \frac{1}{2}(251-1)$	2	125	163	4	$\geqslant 18$
$*126 = 127-1$	1	127	$164 > 41$	$\geqslant 3$	$\geqslant 41$
127	4	$\geqslant 16$	$*165 = \frac{1}{2}(331-1)$	2	165
$*128 = 2^7$	—	512	$*166 = 167-1$	1	167
$129 > 43$	$\geqslant 4$	$\geqslant 43$	167	14	$\geqslant 5$
$*130 = 131-1$	1	131	$*168 = \frac{1}{2}(337-1)$	2	168
$*131 = \frac{1}{2}(263-1)$	2	131	169	4	$\geqslant 18$
$132 > 33$	3	$\geqslant 33$	$170 > 10$	6	$\geqslant 12$
133	12	$\geqslant 5$	$171 > 9$	$\geqslant 6$	$\geqslant 13$
$*134 = \frac{1}{2}(269-1)$	2	134	$*172 = 173-1$	1	173
$*135 = \frac{1}{2}(271-1)$	2	135	$*173 = \frac{1}{2}(347-1)$	2	173
$*136 = \frac{1}{2}17(17-1)$	—	144	$*174 = \frac{1}{2}(349-1)$	2	174
137	6	$\geqslant 9$	$175 > 35$	$\geqslant 4$	$\geqslant 35$
$*138 = 139-1$	1	139	$*176 = 177-1$	1	177
139	4	$\geqslant 17$	$177 > 59$	4	$\geqslant 59$
$*140 = \frac{1}{2}(281-1)$	2	140	$*178 = 179-1$	1	179
$*141 = \frac{1}{2}(283-1)$	2	141	$*179 = \frac{1}{2}(359-1)$	2	179
142	4	$\geqslant 17$	$*180 = 181-1$	1	181
$143 > 11$	6	$\geqslant 11$	181	6	$\geqslant 10$
$144 > 72$	$\geqslant 3$	$\geqslant 73$	182	3	$\geqslant 19$
$145 > 29$	$\geqslant 4$	$\geqslant 29$	$*183 = \frac{1}{2}(367-1)$	2	183
$*146 = \frac{1}{2}(293-1)$	2	146	$184 > 46$	$\geqslant 4$	$\geqslant 47$
$*147 = \frac{1}{2}7^2(7-1)$	—	171	$185 > 37$	$\geqslant 8$	$\geqslant 9$
$*148 = 149-1$	1	149	$*186 = \frac{1}{2}(373-1)$	2	186
149	8	$\geqslant 6$	$187 > 11$	$\geqslant 6$	$\geqslant 11$
$*150 = 151-1$	1	151	$188 > 4$	$\geqslant 5$	$\geqslant 16$
151	6	$\geqslant 9$	$*189 = \frac{1}{2}(379-1)$	2	189
$152 > 8$	$\geqslant 3$	$\geqslant 32$	$*190 = 191-1$	1	191
$*153 = \frac{1}{2}(307-1)$	2	153	$*191 = 383-1$	2	191
154	3	$\geqslant 18$	$*192 = 3.2^6$	—	256
$*155 = \frac{1}{2}(311-1)$	2	155	193	4	$\geqslant 20$
$*156 = 13(13-1)$	—	169	$*194 = \frac{1}{2}(389-1)$	2	194
157	6	$\geqslant 9$	$195 > 39$	$\geqslant 4$	$\geqslant 39$
$*158 = \frac{1}{2}(317-1)$	2	158	$196 > 28$	$\geqslant 3$	$\geqslant 29$
$*159 = \frac{1}{2}(319-1)$	2	159	197	18	$\geqslant 5$
$160 > 32$	$\geqslant 3$	$\geqslant 128$	$*198 = 199-1$	1	199
161	6	$\geqslant 9$	199	4	$\geqslant 20$
$*162 = 163-1$	1	163	$*200 = \frac{1}{2}(401-1)$	2	200

(c) Goldbach's Problem

INTRODUCTION TO PAPERS ON GOLDBACH'S PROBLEM

The main papers in this section are P.N. III (1922, 3) and P.N. V (1924, 6). The other three papers are historical reviews and a preliminary announcement.

In P.N. III and P.N. V Hardy and Littlewood deal with Goldbach's Problem on the assumption that the following hypothesis is true.

Hypothesis R. There exists a real number $\Theta < 3/4$ such that all zeros of all L-series $L(s, \chi)$ formed with Dirichlet characters lie in the half-plane $\sigma \leqslant \Theta$.

Hypothesis R is weaker than the original Riemann hypothesis, but is asserted for a larger class of functions. It remains unproved today. Throughout P.N. III and P.N. V Hypothesis R is taken for granted.

Let $N_3(n)$ be the number of representations of the odd positive integer n as a sum of three primes. Let

$$C_3 = \prod_{\varpi} (1+(\varpi-1)^{-3}),$$

where ϖ runs through all odd primes. The authors prove that

$$N_3(n) \sim C_3 n^2 (\log n)^{-3} \prod_p \frac{(p-1)(p-2)}{p^2-3p+3},$$

where p runs through all odd prime divisors of n. In particular it follows that $N_3(n) > 0$ for large n.

The proof runs on lines similar to those of the preceding sections, though new difficulties cropped up and had to be overcome. The generating function is the third power of

$$f(x) = \sum_{\varpi} (\log \varpi) x^{\varpi},$$

where ϖ runs through all odd primes.

The introduction of the factor $\log \varpi$ is a familiar device to make the use of the prime number theorem for arithmetical progressions easier. The Farey dissection works as before; there are no minor arcs. If

$$x = e^{-Y+2\pi i p/q},$$

it follows from Mellin's formula that, on a Farey arc around p/q,

$$x^{\varpi} = e^{2\pi i p \varpi/q} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)\varpi^{-s}\, ds,$$

and apart from negligible terms, $f(x)$ can be expressed as a linear combination of integrals

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s) \frac{-L'(s, \chi)}{L(s, \chi)}\, ds,$$

where χ runs through all characters (primitive or not) mod q. The integrand is regular

for $\sigma > \Theta$, except for a pole at $s = 1$ when χ is the principal character, and is meromorphic for all s.

Hence Cauchy's theorem can be applied so as to shift the path of integration to the line $\sigma = -\frac{1}{4}$. The residue at $s = 1$ gives the principal terms in the approximation for $f(x)$; the contributions resulting from the zeros of $L(s, \chi)$ give simple poles in the strip $0 \leqslant \sigma \leqslant \Theta$. A careful estimate of their contribution to the value of the integral gives an approximation for $f(x)$ on the Farey arc. Then the older techniques can be applied; the singular series, though troublesome, does not present serious difficulties. And the asymptotic formula is proved.

In the rest of the paper, the authors apply their method heuristically to a large number of problems which are exceedingly difficult. Of their large number of conjectures only one has so far been proved, namely by Linnik, who showed in his book *The dispersion method for binary additive problems* (Leningrad, 1961) that every large integer is a sum of two squares and a prime. The same result had been proved earlier by C. Hooley (*Acta Math.* 97 (1957), 189–210) under the assumption of the generalized Riemann hypothesis.

In P.N. V (1924, 6)† the authors assume that Hypothesis R holds with $\Theta = \frac{1}{2}$, and deduce that there are at most $O(n^{\frac{1}{2}+\epsilon})$ even positive integers not exceeding n which are not the sum of 2 odd primes. The paper stands to P.N. III in the same relation as P.N. VI to the preceding papers on Waring's problem.

Since the time of Hardy and Littlewood much progress has been made in Goldbach's problem.

L. Schnirelmann (*Iswestija Donskowo Polytechn. Inst.* (Novotscherkask) 14 (1930), 3–28) proved the existence of a constant γ such that every integer > 1 is the sum of at most γ primes. I. M. Vinogradov (*Rec. Math. Moscou*, (2) 2 (1937), 179–95) proved that every large odd number is the sum of three odd primes, that is, he proved the Hardy–Littlewood theorem without the use of Hypothesis R. His proof is based on a clever introduction of an exponential sum, essentially transferring the sieve method to exponential level. U. V. Linnik (*Rec. Math.* [*Math. Sbornik*], N.S. 19 (61) (1946), 3–8) proved the result again. His proof reverts to the original Hardy–Littlewood pattern, but instead of using Hypothesis R he obtains an estimate for the total number of zeros, in a particular region, of all the L-functions to a given modulus.

For expositions of modern work on Goldbach's problem, the reader is referred to:

T. Estermann, *Introduction to modern prime number theory* (Cambridge Tract No. 41), Cambridge, 1952.

L. K. Hua, *Additive Primzahltheorie*, Leipzig, 1959.

K. Prachar, *Primzahlverteilung*, Berlin–Göttingen–Heidelberg, 1957.

I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers* (trans. K. F. Roth and A. Davenport), London, 1954.

H. H.

† An abstract appeared in *Proc. London Math. Soc.* (2) 22 (1924), xi.

Note on Messrs Shah and Wilson's paper entitled: 'On an empirical formula connected with Goldbach's Theorem'. By G. H. HARDY, M.A., Trinity College, and J. E. LITTLEWOOD, M.A., Trinity College.

[*Received* 22 January 1919: *read* 3 February 1919.]

1. The formulae discussed by Messrs Shah and Wilson were obtained in the course of a series of researches which have occupied us at various times during the last two years. A full account of our method will appear in due course elsewhere*; but it seems worth while to give here some indication of the genesis of these particular formulae, and others of the same character. We have added a few words about various questions which are suggested by Shah and Wilson's discussion.

The genesis of the formulae.

2. Let

$$f(x) = \Sigma \Lambda(n)\, x^n = \Sigma \Lambda(n)\, e^{-ny} = F(y)$$

and

$$f_\kappa(x) = F_\kappa(y) = \Sigma \chi_\kappa(n)\, \Lambda(n)\, e^{-ny},$$

where $\Lambda(n)$ is equal to $\log p$ when n is a prime p, or a power of p, and to zero otherwise, and $\chi_\kappa(n)$ is one of Dirichlet's 'characters to modulus q'†. Also let

$$x = \mathbf{x} e^{2p\pi i/q},$$

where p is positive, less than q, and prime to q; and suppose that $\mathbf{x}$ tends to unity by positive values.

It is known that

$$\sum_1^n \chi_\kappa(\nu)\, \Lambda(\nu) = o(n),$$

unless χ_κ is the 'principal' character χ_1, in which case

$$\sum_1^n \chi_\kappa(\nu)\, \Lambda(\nu) \sim \sum_1^n \Lambda(\nu) \sim n.$$

It follows that

$$(2{\cdot}1) \qquad f_1(\mathbf{x}) \sim \frac{1}{1-\mathbf{x}}$$

and

$$(2{\cdot}2) \qquad f_\kappa(\mathbf{x}) = o\left(\frac{1}{1-\mathbf{x}}\right) \quad (\kappa > 1).$$

* An outline of one of its most important applications is contained in a paper entitled 'A new solution of Waring's Problem', which will be published shortly in the *Quarterly Journal of Mathematics*.

† See Landau, *Handbuch*, pp. 391 *et seq.*

1919, 1 (with J. E. Littlewood) *Proceedings of the Cambridge Philosophical Society*, 19, 245–54.

Now

(2·3) $$f(x) = \Sigma\Lambda(n)\mathbf{x}^n e^{2np\pi i/q} = \sum_{j=1}^{q} e^{2jp\pi i/q} \sum_{n\equiv j} \Lambda(n)\mathbf{x}^n.$$

If j is prime to q, we have*

(2·4) $$\sum_{n\equiv j} \Lambda(n)\mathbf{x}^n = \frac{1}{\phi(q)} \sum_{\kappa=1}^{\phi(q)} \bar{\chi}_\kappa(j) f_\kappa(\mathbf{x}),$$

where $\bar{\chi}_\kappa$ is the character conjugate to χ_κ, and $\phi(q)$ is the number of numbers less than and prime to q. It follows from (2·1) and (2·2) that

(2·5) $$\sum_{n\equiv j} \Lambda(n)\mathbf{x}^n \sim \frac{\bar{\chi}_1(j)}{\phi(q)} \frac{1}{1-\mathbf{x}} = \frac{1}{\phi(q)} \frac{1}{1-\mathbf{x}}.$$

If on the other hand j is not prime to q, the formula (2·4) is untrue, as its right-hand side is zero. But in this case $\Lambda(n) = 0$ unless n is a power of q, so that

(2·6) $$\sum_{n\equiv j} \Lambda(n)\mathbf{x}^n = o\left(\frac{1}{1-\mathbf{x}}\right).$$

From (2·3), (2·5), and (2·6) it follows that

(2·7) $$f(x) \sim \frac{A_q}{1-\mathbf{x}},$$

where

(2·71) $$A_q = \frac{1}{\phi(q)} \sum_j e^{2jp\pi i/q} = \frac{1}{\phi(q)} \sum_j e^{2j\pi i/q},$$

the summation extending over all values of j less than and prime to q. The sum which appears in (2·71) has been evaluated by Jensen and Ramanujan†, and its value is $\mu(q)$, the well-known arithmetical function of q which is equal to zero unless q is a product $p_1 p_2 \dots p_\rho$ of different primes, and then equal to $(-1)^\rho$. Thus

(2·8) $$f(x) \sim \frac{\mu(q)}{\phi(q)} \frac{1}{1-\mathbf{x}} \ddagger.$$

3. The sum

(3·1) $$\omega(n) = \sum_{m+m'=n} \Lambda(m)\Lambda(m'),$$

* Landau, *l.c.*, p. 421.

† J. L. W. V. Jensen, 'Et nyt Udtryk for den talteoretiske Funktion $\sum_1^m \mu(n) = M(m)$', *Saertryk af Beretning om den 3 Skandinaviske Matematiker-Kongres*, Kristiania, 1915; S. Ramanujan, 'On certain trigonometrical sums and their applications in the theory of numbers', *Trans. Camb. Phil. Soc.*, vol. 22, 1918, pp. 259–276.

‡ If $\mu(q)$ is zero, this formula is to be interpreted as meaning

$$f(x) = o\left(\frac{1}{1-\mathbf{x}}\right).$$

which appears on the left-hand side of Shah and Wilson's equation (2), is the coefficient of x^n in the expansion of $\{f(x)\}^2$. And

$$\{f(x)\}^2 \sim \left\{\frac{\mu(q)}{\phi(q)}\right\}^2 \frac{1}{(1-x)^2} = \left\{\frac{\mu(q)}{\phi(q)}\right\}^2 \Sigma n x^n e^{-2np\pi i/q},$$

when $x \to e^{2p\pi i/q}$ along a radius vector. Our general method accordingly suggests to us to take

$$\Omega(n) = n\Sigma\left\{\frac{\mu(q)}{\phi(q)}\right\}^2 e^{-2np\pi i/q},$$

where the summation extends over $q = 1, 2, 3, \ldots$ and all values of p less than and prime to q, as an approximation to $\omega(n)$. Using Ramanujan's notation, this sum may be written

$$\Omega(n) = n\Sigma\left\{\frac{\mu(q)}{\phi(q)}\right\}^2 c_q(n). \tag{3·2}$$

The series (3·2) can be summed in finite terms. We have

$$c_q(n) = \Sigma\delta\mu\left(\frac{q}{\delta}\right), \tag{3·3}$$

the summation extending over all common divisors δ of q and n*; and it is easily verified, either by means of this formula or by means of the definition of $c_q(n)$ as a trigonometrical sum, that

$$c_{qq'}(n) = c_q(n)\, c_{q'}(n)$$

whenever q and q' are prime to one another. We may therefore write

$$\Omega(n) = n\Sigma A_q = n\Pi\chi_{\varpi},$$

where the product extends over all primes ϖ, and

$$\chi_{\varpi} = 1 + A_{\varpi} + A_{\varpi^2} + A_{\varpi^3} + \ldots = 1 + A_{\varpi},$$

since A_q contains the factor $\mu(q)$ and A_{ϖ^2}, A_{ϖ^3}, ... are accordingly zero.

If n is not divisible by ϖ, we have $c_{\varpi}(n) = \mu(\varpi) = -1$ and

$$A_{\varpi} = -\frac{1}{\{\phi(\varpi)\}^2} = -\frac{1}{(\varpi-1)^2};$$

while if n is divisible by ϖ we have

$$c_{\varpi}(n) = \mu(\varpi) + \varpi\mu(1) = \varpi - 1,$$

$$A_{\varpi} = \frac{1}{\varpi - 1}.$$

Hence

$$\Omega(n) = n\Pi'\left(1 + \frac{1}{\varpi-1}\right)\Pi''\left\{1 - \frac{1}{(\varpi-1)^2}\right\},$$

* Ramanujan, *l.c.*, p. 260.

where Π' applies to primes which divide n and Π'' to primes which do not.

It is evident that $\Omega(n)$ is zero if n is odd. On the other hand, if n is even, we have

$$\begin{aligned}\Omega(n) &= 2n\Pi\left\{1-\frac{1}{(\varpi-1)^2}\right\}\Pi\left[\left(1+\frac{1}{\mathbf{p}-1}\right)\Big/\left\{1-\frac{1}{(\mathbf{p}-1)^2}\right\}\right]\\ &= 2n\Pi\left\{1-\frac{1}{(\varpi-1)^2}\right\}\Pi\left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),\end{aligned}$$

where ϖ now runs through all odd primes and $\mathbf{p}$ through odd prime divisors of n.

The formula $$\omega(n) \sim \Omega(n)$$

is formula (2) of Shah and Wilson's paper*.

The incorrectness of Sylvester's formula.

4. It is easy to prove that *if any formula of the type*

$$(4{\cdot}1) \qquad \omega(n) \sim C\Omega(n)$$

be true, then C must be unity. In other words, our formula is the only formula of this type which can possibly be correct. This may be shown as follows.

Let

$$(4{\cdot}2) \qquad f(s) = \Sigma\frac{\Omega(n)}{n^s},$$

where n runs through all even values; and let $s-1=t$. The series is absolutely convergent if $s>2$, $t>1$. Replacing $\Omega(n)$ by its expression in terms of the prime divisors of n, and splitting up $f(s)$ into factors in the ordinary manner, we obtain

$$f(s) = \frac{2^{1-t}A}{1-2^{-t}}\Pi\left(1+\frac{\varpi-1}{\varpi-2}\frac{\varpi^{-t}}{1-\varpi^{-t}}\right) = \frac{2^{1-t}A\chi(t)}{1-2^{-t}},$$

say, where A is the same constant as in Shah and Wilson's paper, and ϖ runs through all odd primes.

Let

$$\psi(t) = \Pi\left(1+\frac{\varpi^{-t}}{1-\varpi^{-t}}\right) = \Pi\left(\frac{1}{1-\varpi^{-t}}\right) = (1-2^{-t})\zeta(t),$$

and suppose that $t\rightarrow 1$. Then

$$\begin{aligned}\frac{\chi(t)}{\psi(t)} &= \Pi\left\{\left(1+\frac{\varpi-1}{\varpi-2}\frac{\varpi^{-t}}{1-\varpi^{-t}}\right)\Big/\left(1+\frac{\varpi^{-t}}{1-\varpi^{-t}}\right)\right\}\\ &\rightarrow \Pi\left\{\left(1+\frac{1}{\varpi-2}\right)\Big/\left(1+\frac{1}{\varpi-1}\right)\right\}\\ &= \Pi\left\{\frac{(\varpi-1)^2}{\varpi(\varpi-2)}\right\} = \Pi\left\{\frac{(\varpi-1)^2}{(\varpi-1)^2-1}\right\} = \frac{1}{A};\end{aligned}$$

* When $\Omega(n)=0$, the formula is to be interpreted as meaning $\omega(n)=o(n)$.

and so

$$(4{\cdot}3)\qquad f(s) \sim 2A\chi(t) \sim 2(1-2^{-t})\,\zeta(t) \sim \frac{1}{t-1} = \frac{1}{s-2}.$$

This is a consequence of our hypothesis: the corresponding consequence of the hypothesis (4·1) would be

$$(4{\cdot}31)\qquad f(s) \sim \frac{C}{s-2}.$$

On the other hand, it is easy to prove* that

$$(4{\cdot}4)\qquad \omega(1) + \omega(2) + \ldots + \omega(n) \sim \tfrac{1}{2}n^2;$$

and from this to deduce that

$$\phi(s) = \Sigma \frac{\omega(n)}{n^s} \sim \frac{1}{s-2}$$

when $s \to 2$. This equation is inconsistent with (4·1) and (4·31), unless $C = 1$.

It follows that Sylvester's suggested formula is definitely erroneous.

It is more difficult to make a definite statement about the formula given by Brun. The formula to which his argument naturally leads is Shah and Wilson's formula (12); and this formula, like Sylvester's, is erroneous. But in fact Brun never enunciates this formula explicitly. What he does is rather to advance reasons for supposing that *some* formula of the type (4·1) is true, and to determine C on the ground of empirical evidence†. The result to which he is led is equivalent to that obtained by taking $C = 1{\cdot}5985/1{\cdot}3203 = 1{\cdot}2107$ ‡. The reason for so substantial a discrepancy is in effect that explained in the last section of Shah and Wilson's paper.

Further results.

5. The method of § 2 leads to a whole series of results concerning the number of decompositions of n into 3, 4, or any number of primes. The results suggested by it are as follows. Suppose

* Since $$\Sigma\Lambda(n)\,x^n \sim \frac{1}{1-x}$$
as $x \to 1$, we have $$\Sigma\omega(n)\,x^n = \{\Sigma\Lambda(n)\,x^n\}^2 \sim \frac{1}{(1-x)^2};$$
and the desired result follows from Theorem 8 of a paper published by us in 1912 ('Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive', *Proc. London Math. Soc.*, ser. 2, vol. 13, pp. 174–192). This, though the shortest, is by no means the simplest proof.

The formula (4·4) is substantially equivalent to Landau's formula (10) in Shah and Wilson's paper.

† Evidence connected not with Goldbach's theorem itself but with a closely related problem concerning pairs of primes differing by 2. See § 7.

‡ 1·5985 is Brun's constant, while 1·3203 is $2A$.

that $\nu_r(n)$ is the number of expressions of n as the sum of r primes Then if r is *odd* we have

(5·11) $$\nu_r(n) = o(n^{r-1})$$

if n is even, and

(5·12) $$\nu_r(n) \sim \frac{2B}{(r-1)!} n^{r-1} \Pi \left\{ \frac{(\mathbf{p}-1)^r - (\mathbf{p}-1)}{(\mathbf{p}-1)^r + 1} \right\}$$

if n is odd, $\mathbf{p}$ being an odd prime divisor of n, and

(5·13) $$B = \Pi \left\{ 1 + \frac{1}{(\varpi - 1)^r} \right\},$$

where ϖ runs through all odd primes. On the other hand, if r is *even*, we have

(5·21) $$\nu_r(n) = o(n^{r-1})$$

if n is odd, and

(5·22) $$\nu_r(n) \sim \frac{2C}{(r-1)!} n^{r-1} \Pi \left\{ \frac{(\mathbf{p}-1)^r + (\mathbf{p}-1)}{(\mathbf{p}-1)^r - 1} \right\},$$

where

(5·23) $$C = \Pi \left\{ 1 - \frac{1}{(\varpi - 1)^r} \right\},$$

if n is even. The last formula reduces to (1) of Shah and Wilson's paper when $r = 2$.

We have not been able to find a rigorous proof, independent of all unproved hypotheses, of any of these formulae. But we are able to connect them in a most interesting manner with the famous 'Riemann hypothesis' concerning the zeros of Riemann's function $\zeta(s)$. The Riemann hypothesis may be stated as follows: *$\zeta(s)$ has no zeros whose real part is greater than $\frac{1}{2}$.* If this be so, it follows easily that all the zeros of $\zeta(s)$, other than the trivial zeros $s = -2$, $s = -4, \ldots$, lie on the line $\sigma = \mathbf{R}(s) = \frac{1}{2}$. It is natural to extend this hypothesis as follows: *no one of the functions defined, when $\sigma > 1$, by the series*

$$L(s) = \Sigma \frac{\chi_\kappa(n)}{n^s},$$

possesses zeros whose real part is greater than $\frac{1}{2}$. We may call this the *extended Riemann hypothesis.* This being so, what we can prove is this, that *if the extended Riemann hypothesis is true, then the formulae* (5·11)—(5·23) *are true for all values of r greater than* 4.

The reasons for supposing the extended hypothesis true are of the same nature as those for supposing the hypothesis itself true. It should be observed, however, that it is necessary, before we generalise the hypothesis, to modify the form in which it is usually stated; for it is not proved (as it is for $\zeta(s)$ itself) that $L(s)$ can have no real zero between $\frac{1}{2}$ and 1.

6. A modification of our method enables us to attack a closely related problem, that of the existence of pairs of primes differing by a constant even number k.

We have

$$\Sigma \Lambda(n) \Lambda(n+k) r^{2n+k} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 e^{-ki\theta} d\theta,$$

where $f(x)$ is the same function as in § 1, and r is positive and less than unity. We divide the range of integration into a number of small arcs, correlated in an appropriate manner with a certain number of the points $e^{2p\pi i/q}$, and approximate to $|f(re^{i\theta})|^2$ on each arc by means of the formula (2·8). The result thus suggested is that

$$\Sigma \Lambda(n) \Lambda(n+k) r^{2n} \sim \frac{2A}{1-r^2} \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),$$

where A has the same meaning as in § 2 and $\mathbf{p}$ is an odd prime divisor of k. From this it would follow that

$$(6\cdot1) \qquad \sum_{\nu<n} \Lambda(\nu) \Lambda(\nu+k) \sim 2An \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right);$$

and that, if $N_k(n)$ is the number of prime pairs less than n, whose difference is k, then

$$(6\cdot2) \qquad N_k(n) \sim \frac{2An}{(\log n)^2} \Pi \left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right).$$

This formula is of exactly the same form as (1), except that $\mathbf{p}$ is now a factor of k and not of n. In particular we should have

$$(6\cdot3) \qquad N_2(n) \sim \frac{2An}{(\log n)^2},$$

and

$$(6\cdot4) \qquad N_6(n) \sim \frac{4An}{(\log n)^2}.$$

We should therefore conclude that there are about two pairs of primes differing by 6 to every pair differing by 2. This conclusion is easily verified. In fact the numbers of pairs differing by 2, below the limits*

100, 500, 1000, 2000, 3000, 4000, 5000,

are

9, 24, 35, 61, 81, 103, 125;

while the numbers of pairs differing by 6 are

16, 47, 73, 125, 168, 201, 241.

* To be precise, the numbers of pairs (p, p') such that $p' = p+2$ and p' does not exceed the limit in question.

The numbers of pairs differing by 4, which should be roughly the same as those of pairs differing by 2, are

9, 26, 41, 63, 86, 107, 121.

7. Brun, in his note already referred to, recognises the correspondence between the problem of §§ 2—4 and that of the prime-pairs differing by 2, and realises the identity of the constants involved in the formulae; but does not allude to the more general problem of prime-pairs differing by k. He does not determine the fundamental constant A, attempting only to approximate to it empirically by means of a count of prime-pairs differing by 2 and less than 100000, made by Glaisher in 1878*. The value of the constant thus obtained is, as was pointed out in § 4, seriously in error. The truth is that when we pass from (6·1), which, when $k=2$, takes the form

$$\sum_{\nu<n} \Lambda(\nu)\Lambda(\nu+2) \sim 2An,$$

to (6·3), the formula which presents itself most naturally is not (6·3) but

$$(7\cdot1) \qquad N_2(n) \sim 2A\int^n \frac{dx}{(\log x)^2}.$$

This formula is of course, in the long run, equivalent to (6·3). But

$$\int^n \frac{dx}{(\log x)^2} = \frac{n}{(\log n)^2}\left(1+\frac{2\,!}{\log n}+\frac{3\,!}{(\log n)^2}+\dots\right)^{\dagger};$$

and the second factor on the right-hand side is, for $n=100000$, far from negligible. Thus (6·3) may be expected, for such values of n, to give results considerably too small.

If we take the lower limit of integration in (7·1) to be 2, we find that the value of the right-hand side for $n=100000$ is, to the nearest integer, 1249, whereas the actual value of $N_2(n)$ is, according to Glaisher, 1224‡. The ratio is 1·02, and the agreement seems to be as good as can reasonably be expected.

The calculation of prime-pairs has been carried further by Mrs Streatfeild, whose results are exhibited in the following table:

* J. W. L. Glaisher, 'An enumeration of prime-pairs', *Messenger of Mathematics*, vol. 8, 1878, pp. 28–33. The number of pairs below 100000 is 1225.

† The series is naturally divergent, and must be closed, after a finite number of terms, with an error term of lower order than the last term retained.

‡ Glaisher reckons 1 as a prime and (1, 3) as a prime-pair, making 1225 in all.

n	$N_2(n)$	$2A\int_2^n \frac{dx}{(\log x)^2}$	Ratio
100,000	1224	1249	1·020
200,000	2159	2180	1·010
300,000	2992	3035	1·014
400,000	3801	3846	1·012
500,000	4562	4625	1·014
600,000	5328	5381	1·010

8. In a later paper* Brun gives a more general formula relating to prime-pairs (p, p') such that $p = ap' + 2$. This formula also involves an undetermined constant k. It is worth pointing out that our method is equally applicable to this and to still more general problems. Suppose, in the first place, that $\nu(n)$ is the number of expressions of n in the form

$$n = ap + bp',$$

where p and p' are primes†. We may suppose without loss of generality that a and b have no common factor.

The results suggested by our method are as follows. If n has any factor in common with a and b, then

$$\nu(n) = o\left\{\frac{n}{(\log n)^2}\right\};$$

and this is true even when n is prime to both a and b, unless *one* of n, a, b is even‡. But if n, a and b are coprime, and one of them even, then

$$\nu(n) \sim \frac{2A}{ab}\frac{n}{(\log n)^2}\Pi\left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),$$

where A is the constant of § 2, and the product is now extended over all odd primes which divide n or a or b.

* 'Sur les nombres premiers de la forme $ap + b$', *Archiv for Mathematik*, vol. 24, 1917, no. 14.

† We might naturally include powers of primes.

‡ These results are trivial. If n and a have a common factor, it divides bp', and is therefore necessarily p', which can thus assume but a finite number of values. If n, a, b are all odd, either p or p' must necessarily be 2.

Similarly, suppose $N(n)$ to be the number of pairs of solutions of the equation

$$ap' - bp = k$$

such that $p' < n$. It is supposed that a and b have no common factor. Then

$$N(n) = o\left\{\frac{n}{(\log n)^2}\right\}$$

unless k is prime to both a and b, and one of the three is even. If these conditions are satisfied

$$N(n) \sim \frac{2A}{a}\frac{n}{(\log n)^2}\Pi\left(\frac{\mathbf{p}-1}{\mathbf{p}-2}\right),$$

where $\mathbf{p}$ is now an odd prime factor of k, a, or b.

Goldbach's Theorem.

By G. H. Hardy[1]).

The famous problem of which I propose to speak tonight is probably as difficult as any of the unsolved problems of mathematics, and my lecture cannot be entirely easy. I must be content if I can give you some rough notion of the nature and of the history of the problem, and of the general ideas which have guided the mathematicians of the past and of the present in their efforts to find a solution.

Every even number is the sum of two primes: this is 'Goldbach's Theorem'. I may perhaps begin by the trivial observation that, if 'prime' is to mean what it means in modern mathematics, the theorem is obviously false. It fails for 2, which is a prime, but not the sum of two. We do not nowadays call 1 a prime, for, if we do, the factorisation of a number into primes is not unique. We must therefore insert the words 'greater than 2' in the enunciation of the theorem.

The theorem is stated in Goldbach's correspondense with Euler in the year 1742. It would seem that he had been anticipated by Descartes[2]). It was conjectured independently by Waring a little later, and it is, I believe, in Waring's Meditationes algebraicae (1770) that the conjecture appears first in print. Each of these authors appears to have observed that, if every even number (greater than 2) is the sum of two primes, then every number (greater than 5) is

[1]) A lecture to the Mathematical Society of Copenhagen on 6. October 1921.

[2]) In these matters of history I am content to follow Prof. *L. E. Dickson*'s History of the theory of numbers (Washington, 1919, vol. 1. pp. 421 et seq.). Dickson however attributes to Descartes the assertion that 'every even number is a sum of 1, 2, or 3 primes', and here the word 'even' should surely be deleted.

the sum of three. You will see later why this apparently trivial remark should be interesting to me.

The numerical evidence for the truth of the theorem is overwhelming. It has been verified up to 1000 by Cantor (1894—1895), to 2000 by Aubry (1896—1903), to 10,000 by Haussner (1896); and further numerical data, concerning special numbers or numbers of specified forms, have been accumulated by Ripert (1903), by Cunningham (1906), and by Shah and Wilson (1919). But most of the modern computations have been directed towards a more ambitious end, that of determining or verifying some asymptotic formula for the number of decompositions into primes of a given even number n.

I denote by $\nu(n)$ the number of ways in which $n = 2N$ can be expressed as a sum of two primes, or the number of solutions of the equation

$$n = 2N = p + p'. \tag{1}$$

According to Goldbach's Theorem, $\nu(n) > 0$ if $n > 2$; and a very hasty survey of the evidence is enough to make two things clear. In the first place, a great deal more is true than is asserted by the theorem. Not merely is $\nu(n)$ positive, but it is large when n is large. Secondly, while $\nu(n)$ tends to infinity with n, it does not do so in any very regular manner. The magnitude of $\nu(n)$ does not depend merely on the magnitude of n, but also on its arithmetic form.

It is important to observe that these conclusions are in complete agreement with the a priori judgement of common sense. We naturally argue thus. If $m < n$, and n is large, the chance that m is prime is approximately $1 : \log n$. If then we write n in every possible way in the form $n = m + m'$, the chance that both m and m' are prime is approximately $1 : (\log n)^2$. We should therefore expect the order of magnitude of $\nu(n)$ to be

$$\frac{n}{(\log n)^2}. \tag{2}$$

On the other hand, the arithmetical form of n is plainly also relevant. In the first place, it is obviously relevant whether n is odd or even: if n is odd there is no representation, unless $n = p + 2$, and then only one. It is not quite so obviously important to consider whether n is a multiple of 3.

We have however to exclude all cases in which either m or m' is a multiple of 3. If $3\,|\,n$[1]), the two sets of cases thus excluded are the same, while if $3 \nmid n$ the effect of the double exclusion is cumulative. W should therefore expect divisibility by 3 to increase $\nu(n)$; and a similar argument applies to any other prime, so that $\nu(n)$ should be largest when n is composed of a large number of different small prime factors. All these rough expectations prove to be in complete accordance with the facts.

Sylvester (1871) was the first mathematician to suggest an asymptotic formula for $\nu(n)$. Sylvester's rule is stated in words, and when translated into symbols is as follows:

$$\nu(n) \sim 2\pi(n) \prod \left(\frac{p-2}{p-1}\right), \tag{3}$$

where $\pi(n)$ is the number of primes not exceeding n, and the product extends over all primes p for which $3 \leqq p \leqq n$ and $p \nmid n$. We know, though Sylvester did not, that

$$\pi(n) \sim \frac{n}{\log n}. \tag{4}$$

This is the famous 'Primzahlsatz', and we can use it to simplify (3). We also require a theorem of Mertens (1874), to the effect that

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-C}}{\log x}, \tag{5}$$

where C is Euler's constant. This theorem, which lies much less deep than (4), shows that

$$\prod_{p \leqq \sqrt{n}} \left(\frac{p-2}{p-1}\right) = \prod_{p \leqq \sqrt{n}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \leqq \sqrt{n}} \left(1 - \frac{1}{p}\right) \sim \frac{2Ae^{-C}}{\log n},$$

where

$$A = \prod \left(1 - \frac{1}{(p-1)^2}\right), \tag{6}$$

[1]) Following Landau, I write '$3\,|\,n$' for 'n is divisible by 3', and '$3 \nmid n$' to express the contrary.

the product extending now over all odd primes. We thus obtain the formule

$$\nu(n) \sim \frac{4Ae^{-C}n}{(\log n)^2} \prod_{p \mid n} \left(\frac{p-1}{p-2}\right), \tag{7}$$

where p is an odd prime divisor of n, and this we may reasonably call 'Sylvester's formula'.

Sylvester's formula is certainly wrong. It is correct, apparently, in its most obviously interesting parts, in its crude order of magnitude, and in the irregularly oscillating factor which depends upon the arithmetic structure of n. It is the constant factor $4Ae^{-C}$ that cannot be correct. It is interesting to pause for a moment to consider how such a negative result can be established.

There is no mystery at all about the average value of $\nu(n)$. It is quite easy to prove [1] that

$$\frac{\nu(2) + \nu(4) + \cdots + \nu(n)}{n} \sim \frac{n}{2(\log n)^2}. \tag{8}$$

If now we assume any asymptotic formula for $\nu(n)$, we can test it by its compatibility with (8). We may assume a formula of Sylvester's type, but with an unspecified constant factor B; and we find, as the test of compatibility, that $B = 2A$. Thus the only possible formula of this type is

$$\nu(n) = \frac{2An}{(\log n)^2} \prod_{p \mid n} \left(\frac{p-1}{p-2}\right). \tag{9}$$

Sylvester's formula must be wrong, if not in principle, then by a constant multiplier $2e^{-C} = 1 \cdot 123 \cdots$ [2]). The formula (9), on the other hand, seems almost certainly correct.

Sylvester's contribution to the problem passed unnoticed for nearly 50 years. I turn now to the writers, Merlin, Brun, and Stäckel, who have attacked it recently, and particularly to Brun. The work of Mr Littlewood and myself belongs to a different circle of ideas, and I shall speak of it later.

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1) The actual theorem is due to Landau (1900).

2) There is considerably more difficulty in testing the discrepancy by comparison with the facts. Shah and Wilson have given a very clear and interesting discussion of this point.

The writings of these three mathematicians have a striking common characteristic: they use elementary methods only. You may ask me what an 'elementary' method is, and I must explain precisely what I understand by this expression. I do not mean an easy or a trivial method; an elementary method may be quite desperately ingenious and subtle. I am using the word in a definite and technical sense, and in this I am only following the common usage of arithmeticians. I mean, by an elementary method, a method which makes no use of the notion of an analytic function. And the question that I wish to put to you is this: is it reasonable, in the present state of mathematical knowledge, to hope to obtain an elementary proof of Goldbach's theorem?

If I reply to this question in the negative, as I must and shall, if I say that I am compelled to regard all such efforts as foredoomed to failure, I trust that you will not misunderstand me. I cannot believe that the methods of Merlin and Brun are sufficiently powerful or sufficiently profound to lead to a solution of the problem. But I am very far from meaning that I regard their work as devoid of interest and value. There is much in Brun's work in particular that seems to me very beautiful, and some of his theorems ought, I think, to find their way into every book on the theory of numbers.

We have however to take account both of the history and the logical structure of our subject. Let us turn back then for a moment to its central theorem, the 'Primzahlsatz' or 'prime number theorem' expressed by the equation (4). It seems plain that this must be at any rate an easier theorem than Goldbach's theorem. No elementary proof is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's Zeta-function[1]) has no zeros on a certain line[2]) A proof of such a theorem, not fundamentally dependent upon the ideas of the theory of functions, seems to me extraordinarily unlikely. It is rash to assert that a mathematical theorem cannot be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think

[1]) $\zeta(s) = \zeta(\sigma + it)$.
[2]) $\sigma = 1$.

that some theorems, as we say, 'lie deep', and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten.

You are probably familiar with the general idea of the 'sieve' or 'crible' of Eratosthenes. We write down all the integers

$$\left.\begin{array}{c} \underline{1},\ 2,\ 3,\ \underline{4},\ 5,\ \underline{\underline{6}},\ 7,\ \underline{8},\ \underline{9},\ \underline{\underline{10}},\ 11,\ \underline{\underline{12}},\ 13,\ \underline{\underline{14}},\ \underline{\underline{15}}, \\ \underline{16},\ 17,\ \underline{\underline{18}},\ 19,\ \underline{\underline{20}},\ \underline{\underline{21}},\ \underline{\underline{22}},\ \cdots,\ x \end{array}\right\} \quad (8)$$

up to x. We erase (or underline), first 1; then every even number after 2; then multiples of 3, except 3 itself; and so on, repeating the process for every p. The process comes to an end when p exceeds $\sqrt{x}$, for then only primes are left.

Suppose now that $p_1, p_2, \cdots, p_r$ are the first r primes, that x is large and r *fixed*, and that we use the sieve for these primes only. The number of numbers left is approximately

$$x\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right);$$

it is easy to see that the error in this enumeration is at most 2^r+r. If $\pi_r(x)$ is the number of numbers, not exceeding x, and prime or not divisible by any of these r primes, then

$$\pi_r(x) \sim x\prod_1^r\left(1-\frac{1}{p_\nu}\right). \tag{9}$$

We have supposed so far that r is fixed. The result would be much more interesting if we could suppose that r is a function of x. Let us assume provisionally that this is legitimate, and take r to be the largest prime not exceeding $\sqrt{x}$. We obtain

$$\pi(x) = \pi_r(x) \sim x\prod_{p\leqq\sqrt{x}}\left(1-\frac{1}{p}\right) \sim \frac{2e^{-C}x}{\log x}, \tag{10}$$

by Mertens' formula (5). This formula is false (so that our assumption was illegitimate), and it is significant that it is

wrong in just the same way as Sylvester's formula (7). Our failure may help to deepen our scepticism as to the results to be anticipated from the use of the principle of the sieve.

The 'sieve' used by Merlin and Brun for Goldbach's theorem (le crible de Merlin) is of a slightly more complex kind. We form the table

$$\left.\begin{array}{llllllll} 1, & 2, & 3, & 4, & 5, & 6, & 7, \cdots, n-1 & (m), \\ n-1, & n-2, & n-3, & n-4, & n-5, & n-6, & n-7, \cdots, \quad 1 & (m'). \end{array}\right\} \tag{11}$$

Here n is an even number, and the table, read vertically, shows all possible decompositions $n = m + m'$ of n into positive integers. We now perform the process of Eratosthenes on both rows of the table, starting from the right of the lower row, and operating with the first r primes. We consider a decomposition $m + m'$ to be erased when either of its constituents is erased.

There are two cases to be considered for each prime p. If $p \mid n$, the erasures in the second row fall immediately below the corresponding erasures in the first, and the number of decompositions erased is approximately $n : p$. If $p \nmid n$, the erasures never correspond, and their number is approximately $2n : p$. If then $\nu_r(n)$ is the number of decompositions into numbers m, m' which are prime or not divisible by any of the first r primes, we have

$$\nu_r(n) \sim n \prod_{p \mid n,\, p \leqq p_r} \left(1 - \frac{1}{p}\right) \prod_{p \nmid n,\, p \leqq p_r} \left(1 - \frac{2}{p}\right). \tag{12}$$

This formula is correct so long as r is fixed.

Let us assume once more that this formula is correct when p_r is about $\sqrt{n}$. We obtain

$$\nu(n) = \nu_r(n) \sim \tfrac{1}{2} n \prod_{p \mid n} \frac{p-1}{p-2} \prod_{p \leqq \sqrt{n}} \left(1 - \frac{2}{p}\right),$$

where the value 2 of p is now excluded. But, if $3 \leqq p \leqq \sqrt{n}$, we have

$$\prod\left(1 - \frac{2}{p}\right) = \prod\left(1 - \frac{1}{(p-1)^2}\right) \prod\left(1 - \frac{1}{p}\right)^2 \sim \frac{16Ae^{-2C}}{(\log n)^2},$$

by Mertens' formula (5). We thus obtain

$$\nu(n) \sim \frac{8Ae^{-2C}n}{(\log n)^2} \prod_{p \mid n} \left(\frac{p-1}{p-2}\right); \tag{13}$$

and this is the formula to which Brun's argument naturally leads[1]). The formula, like Sylvester's, is wrong, and by a factor $4e^{-2C} = 1 \cdot 263 \cdots$. It will be observed that Sylvester's formula (7) is the geometric mean between (9) and (13).

I have explained that I do not believe that a proof of Goldbach's theorem is likely to be found by methods such as these. But it is certainly possible to prove s o m e t h i n g in this sort of way, and what Brun has proved seems to me very interesting indeed. He has proved, for example that e v e r y l a r g e e v e n n u m b e r n c a n b e e x p r e s s e d i n t h e f o r m $m + m'$, w h e r e m a n d m' a r e n u m b e r s c o m p o s e d o f a t m o s t 9 o d d p r i m e f a c t o r s, a n d t h a t t h e n u m b e r o f s u c h d e c o m p o s i t i o n s i s o f o r d e r $n : (\log n)^2$ a t l e a s t. His method of proof is elementary enough, but a little complicated, and the idea which underlies it can probably be explained most clearly by reference to a simpler problem.

The number of numbers not exceeding x, and prime or not divisible by any of the primes $p_1, p_2, \cdots, p_r$, is (if $p_r < x$)

$$N = r + [x] - \sum \left[\frac{x}{p_\lambda}\right] + \sum \left[\frac{x}{p_\lambda p_\mu}\right] - \sum \left[\frac{x}{p_\lambda p_\mu p_\nu}\right] + \cdots, \tag{14}$$

where $[x]$ is the largest integer in x. This can be shown at once by the method of Eratosthenes, and it is an imediate deduction that

$$N > x\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right) - A 2^r, \tag{15}$$

where A is a constant. No very interesting consequences can be drawn from this; but Brun has shown that, if we are content to allow the first term on the right of (15) to be multiplied by a constant less than 1, we can materially reduce the order of the second, considering as a function of r. More precisely, he proves that

[1]) The formulae (13) is not enunciated explicitly by Brun. His procedure was rather to develop his argument until it leads to a formula o f t h i s t y p e, and to attempt to determine the constant on other grounds. The determination of the only possible constant by averaging was effected independently by Stäckel, and by Mr. Littlewood and myself.

$$N > 0.3\,x\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)-p_r^5. \qquad (16)$$

Suppose now that p_r is about $x^{\frac{1}{6}}$. Then the first term is (by Mertens' theorem) of order $x:\log x$, while the second is of order $x^{\frac{5}{6}}$. Thus N is at least of order $x:\log x$. If a number does not exceed x, and all of its prime factors are greater than $x^{\frac{1}{6}}$, the number of its prime factors is 5 at most. We are led to the theorem that the number of numbers less than x, and with five prime factors at most, is at least of the order $x:\log x$.

The theorem is trivial, for Tschebyschef proved (and by purely elementary methods) that the number of primes less than x is of this order of magnitude. I have chosen this trivial theorem, however, merely as a simple illustration, and Brun has proved much more. In the first place, he has extended the argument to the numbers of an arbitrary arithmetical progression $mk + l$[1]). This theorem also is of course not new, for we know, from the work of de la Vallée Poussin, that any such progression contains the prescribed amount of primes. The method, on the other hand, is in this case most interesting, for there is no elementary proof of any of the theorems concerning the primes of an arithmetical progression. What is still more important is that Brun has been able to treat the 'crible de Merlin' in the same kind of way, and to deduce the very striking and beautiful theorem that I enunciated a moment ago; a theorem, I may add, which Mr. Littlewood and I are unable to prove analytically, with the much more powerful machinery at our command.

It seems clear however that, if we are to attack the main problem with any prospect of success, the methods of the theory of functions are indispensable; and I shall now attempt to explain the method by which Mr. Littlewood and I have attacked it. The general lines of the argument are obvious and inevitable enough; the difficulties lie in carrying it through to its conclusion.

We write

$$\psi(x) = x^2 + x^3 + x^5 + \cdots = \Sigma x^p, \qquad (17)$$

$$(\psi(x))^r = \Sigma \nu_r(n)\, x^n, \qquad (18)$$

[1]) The uninteresting case in which k and l have a common factor being naturally excluded.

so that $\nu_r(n)$ is the number of representations of n as a sum of r primes[1]. We have

$$\nu_r(n) = \frac{1}{2\pi i}\int_C \frac{(\psi(x))^r}{x^{n+1}}\,dx, \tag{19}$$

by Cauchy's theorem, the path of integration C being the circle $|x| = R$, where $0 < R < 1$. We take

$$R = 1 - \frac{1}{n}. \tag{20}$$

We do not base our analysis, however, on the actual formulae that I have written. It is more convenient to use the formulae

$$f(x) = x^2 \log 2 + x^3 \log 3 + x^5 \log 5 + \cdots = \Sigma x^p \log p, \tag{21}$$

$$(f(x))^r = \Sigma N_r(n) x^n, \tag{22}$$

$$N_r(n) = \frac{1}{2\pi i}\int_C \frac{(f(x))^r}{x^{n+1}}\,dx. \tag{23}$$

Here

$$N_r(n) = \Sigma \log p_1 \log p_2 \cdots \log p_r, \tag{24}$$

where the summation applies to all sets $p_1, p_2, \cdots, p_r$ whose sum is n. Goldbach's theorem asserts that $\nu_2(n)$ (or, what is the same thing, that $N_2(n)$) is positive for $n = 4, 6, \cdots$. Our object is the more comprehensive one of finding an asymptotic formula for $\nu_r(n)$; and it is easy to show that, if we can find such a formula for $N_r(n)$, we can deduce one for $\nu_r(n)$ by division by $(\log n)^r$ [2].

Our fundamental idea is the same as that which has guided us in our work on Waring's problem. The unit circle is a barrier of singularities for $f(x)$; we may say, roughly, that $f(x)$ becomes large when x approaches the circle. There are however certain special points of the circle in whose neighbourhood $f(x)$ is largest, and whose contributions to the integral (23) are of dominating importance. These points are the 'rational points h, k', for which

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[1] $\nu_r(n)$ has no connection with the $\nu_r(n)$ of (12).

[2] If $0 < \delta < 1$, we have $(1 - \delta)\log n < \log p < \log n$ for nearly all the primes in question.

$$x = e^{\frac{2h\pi i}{k}} = e\left(\frac{h}{k}\right) = x_{h,k}, \quad \frac{h}{k} = \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \cdots$$

It is obvious, for example, that the point 0, 1, for which $x = 1$, must be most important of all; for $f(x)$ is a series with positive coefficients, and increases most rapidly when x is positive. In general, one may expect the importance of the point h, k to diminish as k increases; and in fact

$$f(x) \sim \frac{\mu(k)}{\varphi(k)}\left(\log \frac{1}{X}\right)^{-1}, \tag{25}$$

where $\mu(k)$ and $\varphi(k)$ are the well known arithmetical functions of Möbius and Euler[1]), if $x = Xx_{h,k}$ and $X \to 1$ by positive values.

Suppose that a formula of the type of (25) has been proved to hold in the neighbourhood of every point h, k. The natural procedure is then as follows. We divide the circle C into a large number of small arcs $\xi_{h,k}$, each associated with a particular point h, k. We substitute for $f(x)$, in the part of the integral (23) which is taken along $\xi_{h,k}$, the approximation derived from (25); we evaluate the resulting integrals; and we thus obtain an approximation for $N_r(n)$ in the form of an infinite series. This series, which we call the singular series, can happily be summed in finite form.

The analysis to which this process leads is intricate and difficult. It will be best that I should begin by stating what we can prove. In the first place we have, for reasons which will appear later, to assume the truth of an unproved hypothesis. The celebrated hypothesis of Riemann may be stated as follows: the real part of a zero of the Zeta-function does not exceed $\frac{1}{2}$. Our hypothesis is a natural extension of this. The theory of the distribution of primes depends, in general, on the theory of $\zeta(s)$. Their distribution in an arithmetical progression $mk + l$, where l is prime to k, depends upon a number of associated functions denoted generically by $L(s)$. Thus when $k = 4$ there are two such functions, defined respectively by

$$L_1(s) = 1^{-s} + 3^{-s} + 5^{-s} + \cdots, \quad L_2(s) = 1^{-s} - 3^{-s} + 5^{-s} \cdots. \tag{26}$$

[1]) $\mu(k) = (-1)^m$ if k is the product of m different primes, $\mu(k) = 0$ otherwise; $\varphi(k)$ is the number of numbers less than and prime to k.

The first is $(1-2^{-s})\zeta(s)$, but the second is a new transcendant. There are $\varphi(k)$ such functions associated with a given k.

The most natural generalisation of Riemann's hypothesis is

HYPOTHESIS R. The real part of a zero of $L(s)$ does not exceed $\frac{1}{2}$.

We do not actually need quite the full force of this hypothesis, but I shall be content to state it in its simplest and most striking form.

Our main theorem is then a follows: If hypothesis R is true, then every large[1]) odd number is the sum of three odd primes. The number $\nu_3(n)$ of representations is given asymptotically by the formula

$$\nu_3(n) \sim B \frac{n^2}{(\log n)^3} \prod \left(\frac{(p-1)(p-2)}{p^2-3p+3} \right), \tag{27}$$

where

$$B = \prod \left(1 + \frac{1}{(p-1)^3} \right). \tag{28}$$

The product in (27) extends over the odd prime divisors of n, and that in (28) over all odd primes.

The complete proof of this theorem, and of similar theorems concerning representations of numbers by four or any larger number of primes, will appear shortly in the Acta Mathematica. At the moment I shall attempt to explain to you only

(1) how the final formula arises,

(2) why it should be necessary to assume hypothesis R,

(3) why our method succeeds for three or more primes, but fails for two.

I shall not take these questions in the order in which I have stated them: I begin with the second. You will remember that it is necessary to approximate to the function $f(x)$ in the neighbourhood of the point h, k. Taking first the simplest case, suppose that $h=0$, $k=1$, $x_{h,k}=1$.

We write $x = e^{-y}$, and use the well known formula

$$e^{-y} = \frac{1}{2\pi i} \int y^{-s} \Gamma(s)\, ds \tag{29}$$

of Mellin. From this we deduce

[1]) That is, every such number from a certain point onwards.

$$f(x) = \Sigma \log p\, e^{-py} = \frac{1}{2\pi i}\int y^{-s}\Gamma(s) \sum \frac{\log p}{p^s}\, ds \qquad (30)$$

which is substantially (though not exactly)

$$-\frac{1}{2\pi i}\int y^{-s}\Gamma(s)\frac{\zeta'(s)}{\zeta(s)}\, ds. \qquad (31)$$

The integrals are taken along a line parallel to the imaginary axis and passing to the right of the point $s = 1$.

The subject of integration has poles when $s = 1$, when s is zero or a negative integer, and at all the complex zeros of $\zeta(s)$, which we denote generally by ρ. If we assume Riemann's hypothesis, every ρ has the real part $\frac{1}{2}$. It is natural to suppose that, if we move the path of integration further and further to the left, and apply Cauchy's theorem, we shall express $f(x)$ in the form of an infinite series, in which the most important terms will be the terms corresponding to the residues for 1 and ρ. If we denote these terms by

$$\frac{1}{y} + \sum \frac{A_\rho}{y_\rho}, \qquad (32)$$

then the term corresponding to ρ is of order $y^{-\frac{1}{2}}$ when y is small, and we may hope that the order of the series will be much the same. In this case we shall prove that $f(x) \sim y^{-1}$, which is the simplest case of (25), and that the order of magnitude of the error is not notably greater than that of $y^{-\frac{1}{2}}$. It is evident that our estimate of the error will depend essentially on our assumption.

Passing now to the general case, I denote by $\lambda(n)$ the arithmetical function of n which is equal to $\log n$ when n is prime and otherwise to zero. We have now

$$f(x) = f(Xx_{h,k}) = \Sigma X^n \lambda(n) e\left(\frac{nh}{k}\right). \qquad (33)$$

We write

$$n = mk + l \ (l = 1, 2, \cdots, k;\ m = 0, 1, 2, \cdots),$$

and we obtain

$$\left.\begin{aligned} f(x) &= \sum_{l,m} X^{mk+l}\lambda(mk+l)\, e\left(\frac{lh}{k}\right) \\ &= \sum_{l} X^l e\left(\frac{lh}{k}\right) \sum_{m} X^{mk}\lambda(mk+l). \end{aligned}\right\} \qquad (34)$$

Thus $f(x)$ is expressed as the sum of a finite number of functions, each of which plainly depends on the distribution of primes in an arithmetical progression $mk+l$. We are thus led to make an assumption, analogous to the hypothesis of Riemann, about all the functions $L(s)$ on which this distribution depends; in other words, we are led to hypothesis R. Unless we make some such assumption our difficulties, already considerable, will be terribly increased.

We therefore assume hypothesis R. We now write $X=e^{-Y}$ and express $f(x)$, by means of the formulae (33) and (34), in the form of an integral

$$\frac{1}{2\pi i}\int Y^{-s}\Gamma(s)Z(s)ds, \tag{35}$$

where $Z(s)$ is a linear combination of the logarithmic derivatives $L'(s):L(s)$ of the L-functions associated with modulus k. The dominant term of $f(x)$, in the neighbourhood of $x_{h,k}$, is the residue of the integrand for $s=1$, and a simple calculation shows that this is the function which appears on the right hand side of (25).

We now return to the integral (23), and consider the separate contributions of the arcs $\xi_{h,k}$. If we substitute for $f(x)$, on $\xi_{h,k}$, the approximation given by (25), and transform the integral by the substitutions

$$x=Xx_{h,k}=Xe\left(\frac{h}{k}\right),\quad X=e^{-Y}, \tag{36}$$

we obtain

$$\left(\frac{\mu(k)}{\varphi(k)}\right)^{r}e\left(-\frac{nh}{k}\right)\frac{1}{2\pi i}\int Y^{-r}e^{nY}dY. \tag{37}$$

The path of integration is now a segment of a straight line, parallel to the imaginary axis in the plane of Y, and passing to the right of the origin.

We can substitute for this segment the complete straight line of which it is part, the error involved in this approximation proving to be unimportant. The integral can then be evaluated in finite terms, and (37) assumes the form

$$\frac{n^{r-1}}{(r-1)!}\left(\frac{\mu(k)}{\varphi(k)}\right)^{r}e\left(-\frac{nh}{k}\right). \tag{38}$$

We have finally to sum with respect to h and k, and we obtain the formula

$$N_r(n) \sim \frac{n^{r-1}}{(r-1)!} S, \tag{39}$$

where

$$S = \sum_{h,k}' \left(\frac{\mu(k)}{\varphi(k)}\right)^r e\left(-\frac{nh}{k}\right) = \sum_k' \left(\frac{\mu(k)}{\varphi(k)}\right)^r c_k(n), \tag{40}$$

$$c_k(n) = \sum_h' e\left(-\frac{nk}{k}\right), \tag{41}$$

and h, in (41), is less than and prime to k.

The series S, the 'singular series', is the series on which the solution of all these problems depends. It can, as I said before, be summed in finite terms[1]. If r and n are of opposite parity, then $S = 0$. If they are of the same parity, then

$$S = 2C_r \prod_{p \mid n} \left(\frac{(p-1)^r + (-1)^r(p-1)}{(p-1)^r - (-1)^r}\right), \tag{42}$$

where

$$C_r = \prod \left(1 - \frac{(-1)^r}{(p-1)^r}\right). \tag{43}$$

If $r = 3$, we are led to (27), and if $r = 2$ to (9), the powers of $\log n$ which appear in these formulae being introduced in the passage from N to ν; and there are similar formulae for every value of r.

There is unfortunately a vital difference between the case $r = 2$, which corresponds to Goldbach's theorem, and all the rest. We have to fill in the skeleton which I have presented to you, and to transform it into an accurate proof; and in doing this we find ourselves compelled to suppose that $r > 2$. It only remains that I should explain to you shortly the reason for this regrettable limitation[2].

Our work depends upon a system of approximate formulae for $f(x)$, each valid near a particular point of the unit circle. It will be sufficient for my purpose to consider the point $x = 1$. If $x = e^{-y}$, and y is positive, our formula for $f(x)$ is of the form

[1]) The summation is a simple but entertaining exercise in elementary algebra, which I must be content to take for granted.

[2]) The explanation which follows must be taken merely as a first approximation to the truth.

$$f(x) = \frac{1}{y} + O\left(\frac{1}{y^{\frac{1}{2}+\varepsilon}}\right),$$

where ε may be any positive number. Thus $(f(x))^r$ consist of two parts, a term y^{-r}, which is exactly known, and an error term whose order is, so far as our analysis shows, not less than $O(y^{-\frac{1}{2}r})$; and each of these terms gives rise to a corresponding term in the final formula. The contribution of the dominant term can be found precisely, and is of order n^{r-1}. The contribution of the error term can only be estimated in a cruder manner, and the best that we can say about it is that it is of order $O(n^{\frac{1}{2}r})$. This error must, if our approximation is to succeed, be smaller than the dominant term; and this requires that $r - 1 > \frac{1}{2}r$ or $r > 2$. In fact I have done rather more than justice to our method. It would appear, from what I have stated, that we stand on the very margin of success. But there are further complications in the analysis, and we fail by a power $n^{\frac{1}{4}}$.

I implied, when I was discussing the methods of Merlin and Brun, that it was hardly reasonable to suppose that they could possibly succeed. You may naturally ask me what I think of the prospects of our own, and the question is one to which I find it rather difficult to reply. You must make me, for the moment, a present of hypothesis *R*. I presume that the hypothesis of Riemann will some day be proved. Hypothesis *R* will, I am sure, be proved within a week from then, and the proofs will be substantially the same. There is nothing whatever to suggest that, in these respects, one *L*-function behaves unlike another.

Apart from this I would reply that, the hypothesis once proved or granted, I see no particular reason why our method should not succeed. It seems to me adequate for the problem; the ideas which underlie it are not too easy and lie sufficiently deep. It fails in detail, and not in principle, even as it is; the failure is not a failure of the method, but of the analytical powers of Mr. Littlewood and myself. The method seems to embody the essential features of the problem, and leads, naturally and inevitably, to what is plainly the real result. I believe that, when the problem is solved, it will be solved in some such way as this.

SOME PROBLEMS OF 'PARTITIO NUMERORUM'; III: ON THE EXPRESSION OF A NUMBER AS A SUM OF PRIMES.

BY

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1. Introduction.

1. 1. It was asserted by GOLDBACH, in a letter to EULER dated 7 June, 1742, that *every even number 2m is the sum of two odd primes*, and this proposition has generally been described as 'Goldbach's Theorem'. There is no reasonable doubt that the theorem is correct, and that the number of representations is large when m is large; but all attempts to obtain a proof have been completely unsuccessful. Indeed it has never been shown that every number (or every *large* number, any number, that is to say, from a certain point onwards) is the sum of 10 primes, or of 1 000 000; and the problem was quite recently classified as among those 'beim gegenwärtigen Stande der Wissenschaft unangreifbar'.[1]

In this memoir we attack the problem with the aid of our new transcendental method in 'additiver Zahlentheorie'.[2] We do not solve it: we do not

[1] E. LANDAU, 'Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion', *Proceedings of the fifth International Congress of Mathematicians*, Cambridge, 1912, vol. 1, pp. 93—108 (p. 105). This address was reprinted in the *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 21 (1912), pp. 208—228.

[2] We give here a complete list of memoirs concerned with the various applications of this method.

G. H. HARDY.

1. 'Asymptotic formulae in combinatory analysis', *Comptes rendus du quatrième Congrès des mathematiciens Scandinaves à Stockholm*, 1916, pp. 45—53.

2. 'On the expression of a number as the sum of any number of squares, and in particular of five or seven', *Proceedings of the National Academy of Sciences*, vol. 4 (1918), pp. 189—193.

even prove that any number is the sum of 1 000 000 primes. In order to prove anything, we have to assume the truth of an unproved hypothesis, and, even on this hypothesis, we are unable to prove Goldbach's Theorem itself. We show, however, that the problem is not 'unangreifbar', and bring it into contact with the recognized methods of the Analytic Theory of Numbers.

3. 'Some famous problems of the Theory of Numbers, and in particular Waring's Problem' (Oxford, Clarendon Press, 1920, pp. 1—34).

4. 'On the representation of a number as the sum of any number of squares, and in particular of five', *Transactions of the American Mathematical Society*, vol. 21 (1920), pp. 255—284.

5. 'Note on Ramanujan's trigonometrical sum $c_q(n)$', *Proceedings of the Cambridge Philosophical Society*, vol. 20 (1921), pp. 263—271.

G. H. Hardy and J. E. Littlewood.

1. 'A new solution of Waring's Problem', *Quarterly Journal of pure and applied mathematics*, vol. 48 (1919), pp. 272—293.

2. 'Note on Messrs. Shah and Wilson's paper entitled: On an empirical formula connected with Goldbach's Theorem', *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 245—254.

3. 'Some problems of 'Partitio numerorum'; I: A new solution of Waring's Problem', *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen* (1920), pp. 33—54.

4. 'Some problems of 'Partitio numerorum'; II: Proof that any large number is the sum of at most 21 biquadrates', *Mathematische Zeitschrift*, vol. 9 (1921), pp. 14—27.

G. H. Hardy and S. Ramanujan.

1. 'Une formule asymptotique pour le nombre des partitions de *n*', *Comptes rendus de l'Académie des Sciences*, 2 Jan. 1917.

2. 'Asymptotic formulae in combinatory analysis', *Proceedings of the London Mathematical Society*, ser. 2, vol. 17 (1918), pp. 75—115.

3. 'On the coefficients in the expansions of certain modular functions', *Proceedings of the Royal Society of London* (A), vol. 95 (1918), pp. 144—155.

E. Landau.

1. 'Zur Hardy-Littlewood'schen Lösung des Waringschen Problems', *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen* (1921), pp. 88—92.

L. J. Mordell.

1. 'On the representations of numbers as the sum of an odd number of squares', *Transactions of the Cambridge Philosophical Society*, vol. 22 (1919), pp. 361—372.

A. Ostrowski.

1. 'Bemerkungen zur Hardy-Littlewood'schen Lösung des Waringschen Problems', *Mathematische Zeitschrift*, vol. 9 (1921), pp. 28—34.

S. Ramanujan.

1. 'On certain trigonometrical sums and their applications in the theory of numbers', *Transactions of the Cambridge Philosophical Society*, vol. 22 (1918), pp. 259—276.

N. M. Shah and B. M. Wilson.

1. 'On an empirical formula connected with Goldbach's Theorem', *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 238—244.

Our main result may be stated as follows: *if a certain hypothesis* (a natural generalisation of Riemann's hypothesis concerning the zeros of his Zeta-function) *is true, then every large odd number n is the sum of three odd primes; and the number of representations is given asymptotically by*

$$\bar{N}_3(n) \sim C_3 \frac{n^2}{(\log n)^3} \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p}-1)(\mathfrak{p}-2)}{\mathfrak{p}^2-3\mathfrak{p}+3}\right), \tag{1. 11}$$

where $\mathfrak{p}$ *runs through all odd prime divisors of* n, *and*

$$C_3 = \Pi\left(1+\frac{1}{(\varpi-1)^3}\right), \tag{1. 12}$$

the product extending over all odd primes ϖ.

Hypothesis R.

1. 2. We proceed to explain more closely the nature of our hypothesis. Suppose that q is a positive integer, and that

$$h = \varphi(q)$$

is the number of numbers less than q and prime to q. We denote by

$$\chi(n) = \chi_k(n) \qquad (k = 1, 2, \ldots, h)$$

one of the h Dirichlet's 'characters' to modulus q[1]: χ_1 is the 'principal' character.

By $\bar{\chi}$ we denote the complex number conjugate to χ: $\bar{\chi}$ is a character.

By $L(s, \chi)$ we denote the function defined for $\sigma > 1$ by

$$L(s) = L(\sigma + it) = L(s, \chi) = L(s, \chi_k) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Unless the contrary is stated the modulus is q. We write

$$\bar{L}(s) = L(s, \bar{\chi}).$$

By

$$\varrho = \beta + i\gamma$$

[1] Our notation, so far as the theory of L-functions is concerned, is that of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, book 2, pp. 391 *et seq.*, except that we use q for his k, k for his $\varkappa$, and ϖ for a typical prime instead of p. As regards the 'Farey dissection', we adhere to the notation of our papers *3* and *4*.

We do not profess to give a complete summary of the relevant parts of the theory of the L-functions; but our references to Landau should be sufficient to enable a reader to find for himself everything that is wanted.

we denote a typical zero of $L(s)$, those for which $\gamma = 0$, $\beta \leqq 0$ being excluded. We call these the *non-trivial* zeros. We write $N(T)$ for the number of ϱ'*s* of $L(s)$ for which $0 \leqq \gamma \leqq T$.

The natural extension of Riemann's hypothesis is

HYPOTHESIS R. Every ϱ has its real part less than or equal to* $\frac{1}{2}$.[1]

We shall not have to use the full force of this hypothesis. What we shall in fact assume is

HYPOTHESIS R. There is a number $\Theta < \frac{3}{4}$ *such that*

$$\beta \leqq \Theta$$

for every ϱ of every $L(s)$.

The assumption of this hypothesis is fundamental in all our work; *all the results of the memoir, so far as they are novel, depend upon it*[2]; and we shall not repeat it in stating the conditions of our theorems.

We suppose that Θ has its smallest possible value. In any case $\Theta \geqq \frac{1}{2}$. For, if ϱ is a complex zero of $L(s)$, $\bar{\varrho}$ is one of $\bar{L}(s)$. Hence $1 - \bar{\varrho}$ is one of $\bar{L}(1-s)$, and so, by the functional equation[3], one of $L(s)$.

Further notation and terminology.

1. 3. We use the following notation throughout the memoir.

A is a positive absolute constant wherever it occurs, but not the same constant at different occurrences. B is a positive constant depending on the single parameter r. O's refer to the limit process $n \to \infty$, the constants which they involve being of the type B, and o's are uniform in all parameters *except* r.

ϖ is a prime. $\mathfrak{p}$ (which will only occur in connection with n) is an odd prime divisor of n. p is an integer. If $q = 1$, $p = 0$; otherwise

$$0 < p < q, \quad (p, q) = 1.$$

(m, n) is the greatest common factor of m and n. By $m \mid n$ we mean that n is divisible by m; by $m \nmid n$ the contrary.

$\Lambda(n)$, $\mu(n)$ have the meanings customary in the Theory of Numbers. Thus $\Lambda(n)$ is $\log \varpi$ if $n = \varpi^m$ and zero otherwise: $\mu(n)$ is $(-1)^k$ if n is a product of

[1] The hypothesis must be stated in this way because

(a) it has not been proved that no $L(s)$ has real zeros between $\frac{1}{2}$ and 1,

(b) the L-functions associated with *imprimitive* (uneigentlich) characters have zeros on the line $\sigma = 0$.

[2] Naturally many of the results stated incidentally do not depend upon the hypothesis.

[3] Landau, p. 489. All references to 'Landau' are to his *Handbuch*, unless the contrary is stated.

k different prime factors, and zero otherwise. The fundamental function with which we are concerned is

$$f(x) = \sum_{\varpi} \log \varpi \; x^{\varpi}. \tag{1. 31}$$

To simplify our formulae we write

$$e(x) = e^{2\pi i x}, \quad e_q(x) = e\left(\frac{x}{q}\right).$$

Also

$$c_q(n) = \sum_{p} e_q(n p). \tag{1. 32}$$

If χ_k is primitive,

$$\tau_k = \tau(\chi_k) = \sum_{p} e_q(p)\, \chi_k(p) = \sum_{m=1}^{q} e_q(m)\, \chi_k(m).^{1} \tag{1. 33}$$

This sum has the absolute value[2] $\sqrt{q}$.

The Farey dissection.

1. 4. We denote by Γ the circle

$$|x| = e^{-H} = e^{-\frac{1}{n}}. \tag{1. 41}$$

We divide Γ into arcs $\xi_{p,q}$ which we call *Farey arcs*, in the following manner. We form the Farey's series of order

$$N = [\sqrt{n}], \tag{1. 42}$$

the first and last terms being $\frac{0}{1}$ and $\frac{1}{1}$. We suppose that $\frac{p}{q}$ is a term of the series, and $\frac{p'}{q'}$ and $\frac{p''}{q''}$ the adjacent terms to the left and right, and denote by $j_{p,q}$ $(q > 1)$ the intervals

$$\frac{p}{q} - \frac{1}{q(q+q')}, \quad \frac{p}{q} + \frac{1}{q(q+q'')};$$

by $j_{0,1}$ and $j_{1,1}$ the intervals $\left(0, \frac{1}{N+1}\right)$ and $\left(1 - \frac{1}{N+1}, 1\right)$. These intervals just

[1] $\chi_k(m) = 0$ if $(m, q) > 1$.

[2] Landau, p. 497.

fill up the interval $(0, 1)$, and the length of each of the parts into which $j_{p,q}$ is divided by $\frac{p}{q}$ is less than $\frac{1}{qN}$ and not less than $\frac{1}{2qN}$. If now the intervals $j_{p,q}$ are considered as intervals of variation of $\frac{\theta}{2\pi}$, where $\theta = \arg x$, and the two extreme intervals joined into one, we obtain the desired dissection of Γ into arcs $\xi_{p,q}$.[1]

When we are studying the arc $\xi_{p,q}$, we write

$$x = e^{\frac{2p\pi i}{q}} X = e_q(p) X = e_q(p) e^{-Y}, \tag{1.43}$$

$$Y = \eta + i\theta. \tag{1.44}$$

The whole of our work turns on the behaviour of $f(x)$ as $|x| \to 1$, $\eta \to 0$, and we shall suppose throughout that $0 < \eta \leqq \frac{1}{2}$. When x varies on $\xi_{p,q}$, X varies on a congruent arc $\zeta_{p,q}$, and

$$\theta = -\left(\arg x - \frac{2p\pi}{q}\right)$$

varies (in the inverse direction) over an interval $-\theta'_{p,q} \leqq \theta \leqq \theta_{p,q}$. Plainly $\theta_{p,q}$ and $\theta'_{p,q}$ are less than $\frac{2\pi}{qN}$ and not less than $\frac{\pi}{qN}$, so that

$$\bar{\theta}_{p,q} = \operatorname{Max}(\theta_{p,q}, \theta'_{p,q}) < \frac{A}{qN}.$$

In all cases $Y^{-s} = (\eta + i\theta)^{-s}$ has its principal value

$$\exp(-s \log(\eta + i\theta)),$$

wherein (since η is positive)

$$-\frac{1}{2}\pi < \Im \log(\eta + i\theta) < \frac{1}{2}\pi.$$

By $N_r(n)$ we denote the number of representations of n by a sum of r primes, attention being paid to order, and repetitions of the same prime being allowed, so that

$$\sum_{n=2}^{\infty} N_r(n) x^n = \left(\sum_{\varpi} x^{\varpi}\right)^r. \tag{1.45}$$

[1] The distinction between major and minor arcs, fundamental in our work on Waring's Problem, does not arise here.

By $\nu_r(n)$ we denote the sum

$$\nu_r(n) = \sum_{\varpi_1 + \varpi_2 + \cdots + \varpi_r = n} \log \varpi_1 \log \varpi_2 \ldots \log \varpi_r, \tag{1. 46}$$

so that

$$\sum_{n=2}^{\infty} \nu_r(n) x^n = (f(x))^r. \tag{1. 47}$$

Finally S_r is the *singular series*

$$S_r = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n). \tag{1. 48}$$

2. Preliminary lemmas.

2. 1. *Lemma 1. If* $\eta = \Re(Y) > 0$ *then*

$$f(x) = f_1(x) + f_2(x), \tag{2. 11}$$

where

$$f_1(x) = \sum_{(q,n)>1} \Lambda(n) x^n - \sum_{\varpi} \log \varpi (x^{\varpi^2} + x^{\varpi^3} + \cdots), \tag{2. 12}$$

$$f_2(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s} \Gamma(s) Z(s) \, ds, \tag{2. 13}$$

Y^{-s} *has its principal value,*

$$Z(s) = \sum_{k=1}^{h} C_k \frac{L'_k(s)}{L_k(s)}, \tag{2. 14}$$

C_k *depends only on* p, q *and* χ_k,

$$C_1 = -\frac{\mu(q)}{h} \tag{2. 15}$$

and

$$|C_k| \leq \frac{\sqrt{q}}{h}. \tag{2. 16}$$

We have

$$f_2(x) = f(x) - f_1(x) = \sum_{(q,n)=1} \Lambda(n)\, x^n$$

$$= \sum_{1 \leq j \leq q,\, (q,j)=1} e_q(pj) \sum_{l=0}^{\infty} \Lambda(lq+j)\, e^{-(lq+j)Y}$$

$$= \sum_j e_q(pj) \sum_l \Lambda(lq+j) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s}\, \Gamma(s)\, (lq+j)^{-s}\, ds,$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Y^{-s}\, \Gamma(s)\, Z(s)\, ds,$$

where

$$Z(s) = \sum_j e_q(pj) \sum_l \frac{\Lambda(lq+j)}{(lq+j)^s}.$$

Since $(q, j) = 1$, we have[1]

$$\sum_l \frac{\Lambda(lq+j)}{(lq+j)^s} = -\frac{1}{h} \sum_{k=1}^{h} \bar{\chi}_k(j) \frac{L'_k(s)}{L_k(s)};$$

and so

$$Z(s) = \sum_{k=1}^{h} C_k \frac{L'_k(s)}{L_k(s)},$$

where

$$C_k = -\frac{1}{h} \sum_{j=1}^{q} e_q(pj)\, \bar{\chi}_k(j).$$

Since $\bar{\chi}_k(j) = 0$ if $(q, j) > 1$, the condition $(q, j) = 1$ may be omitted or retained at our discretion.

Thus[2]

$$C_1 = -\frac{1}{h} \sum_{1 \leq j \leq q,\, (q,j)=1} e_q(pj)$$

$$= -\frac{1}{h} \sum_{1 \leq m \leq q,\, (q,m)=1} e_q(m) = -\frac{\mu(q)}{h}.$$

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[1] Landau, p. 421.

[2] Landau, pp. 572—573.

Again, if $k > 1$ we have[1]

$$C_k = -\frac{1}{h}\sum_{j=1}^{q} e_q(pj)\,\bar{\chi}_k(j) = -\frac{\chi_k(p)}{h}\sum_{m=1}^{q} e_q(m)\,\bar{\chi}_k(m).$$

If $\bar{\chi}_k$ is a primitive character,

$$\sum_{m=1}^{q} e_q(m)\,\bar{\chi}_k(m) = \tau(q, \bar{\chi}_k),$$

$$|\tau(q, \bar{\chi}_k)| = \sqrt{q},^{2}$$

$$|C_k| = \frac{\sqrt{q}}{h}.$$

If χ is imprimitive, it belongs to $Q = \frac{q}{d}$, where $d > 1$. Then $\bar{\chi}_k(m)$ has the period Q, and

$$\sum_{m=1}^{q} e_q(m)\,\chi_k(m) = \sum_{n=1}^{Q} e_q(n)\,\bar{\chi}_k(n)\sum_{l=0}^{d-1} e_q(lQ).$$

The inner sum is zero. Hence $C_k = 0$, and the proof of the lemma is completed.[3]

2. 2. *Lemma 2. We have*

$$|f_1(x)| < A(\log(q+1))^A \eta^{-\frac{1}{2}}. \tag{2. 21}$$

We have

$$f_1(x) = \sum_{(q,n)>1} \Lambda(n)x^n - \sum_{\varpi} \log\varpi(x^{\varpi^2} + x^{\varpi^3} + \cdots) = f_{1,1}(x) - f_{1,2}(x).$$

But

$$|f_{1,1}(x)| \leq \sum_{\varpi \mid q} \log\varpi \sum_{r=1}^{\infty} |x|^{\varpi^r}$$

$$< A\log(q+1)\log q\sum_{r=1}^{\infty}|x|^{2^r} < A(\log(q+1))^2\sum_{r=1}^{\infty} e^{-\eta 2^r}$$

$$< A(\log(q+1))^A \log\frac{1}{\eta} < A(\log(q+1))^A \eta^{-\frac{1}{2}}.$$

[1] Landau, p. 485. The result is stated there only for a primitive character, but the proof is valid also for an imprimitive character when $(p, q) = 1$.

[2] Landau, pp. 485, 489, 492.

[3] See the additional note at the end.

Also

$$\sum_{r \geqq 2,\, \varpi^r \leqq \xi} \log \varpi < A\sqrt{\xi},$$

and so

$$|f_{1,2}(x)| \leqq \sum_{r \geqq 2,\, \varpi} \log \varpi \, |x|^{\varpi^r} < A(1-|x|) \sum_n \sqrt{n}\,|x|^n$$

$$< A(1-|x|)^{-\frac{1}{2}} < A\eta^{-\frac{1}{2}}.$$

From these two results the lemma follows.

2. 3. *Lemma 3. We have*

$$\frac{L'(s)}{L(s)} = -\frac{\mathfrak{b}}{s-1} + \frac{\mathfrak{d}-\mathfrak{b}}{s} + b - \frac{1}{2}\psi\left(\frac{s+\mathfrak{a}}{2}\right) + \sum_\varrho \left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right), \tag{2. 31}$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

the $\mathfrak{a}$'s, $\mathfrak{b}$'s, $\mathfrak{d}$'s *and* b's *are constants depending upon* q *and* χ, $\mathfrak{a}$ *is* 0 *or* 1,

$$\mathfrak{b}_1 = 1, \qquad \mathfrak{b}_k = 0 \qquad (k > 1), \tag{2. 32}$$

and

$$0 \leqq \mathfrak{d} < A \log (q+1). \tag{2. 33}$$

All these results are classical except the last.[1]

The precise definition of $\mathfrak{d}$ is rather complicated and does not concern us. We need only observe that $\mathfrak{d}$ does not exceed the number of different primes that divide q,[2] and so satisfies (2. 33).

2. 41. *Lemma 4. If* $0 < \eta \leqq \frac{1}{2}$, *then*

$$f(x) = \frac{\mu(q)}{hY} + \sum_{k=1}^{h} C_k G_k + P, \tag{2. 411}$$

where

$$G_k = \sum_{\varrho_k} \Gamma(\varrho)\, Y^{-\varrho}, \tag{2. 412}$$

[1] Landau, pp. 509, 510, 519.

[2] Landau, p. 511 (footnote).

(2. 413) $$|P| < A\sqrt{q}\,(\log(q+1))^A\left(\frac{1}{h}\sum_{k=1}^{h}|b_k| + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}}\delta^{-\frac{1}{2}}\right),$$

(2. 414) $$\delta = \operatorname{arc\,tan}\frac{\eta}{|\theta|}.$$

We have, from (2. 13) and (2. 14),

(2. 415) $$f_2(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)Z(s)\,ds$$

$$= \sum_{k=1}^{h}\frac{C_k}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)\frac{L'_k(s)}{L_k(s)}\,ds = \sum_{k=1}^{h} C_k f_{2,k}(x),$$

say. But[1]

(2.416) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}\,ds = -\frac{\mathfrak{b}}{Y} + R + \sum_{\varrho}\Gamma(\varrho)Y^{-\varrho} + \frac{1}{2\pi i}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}\,ds,$$

where

$$R = \left\{Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}\right\}_0,$$

$\{f(s)\}_0$ denoting generally the residue of $f(s)$ for $s = 0$.

Now[2]

$$\frac{L'(s)}{L(s)} = \log\frac{\pi}{Q} + \sum_{\nu=1}^{\mathfrak{c}}\frac{\varepsilon_\nu \log\varpi_\nu}{\varpi_\nu^s - \varepsilon_\nu} + \sum_{\nu=1}^{\mathfrak{c}}\frac{\bar{\varepsilon}_\nu \log\varpi_\nu}{\varpi_\nu^{1-s} - \bar{\varepsilon}_\nu}$$

$$-\frac{1}{2}\psi\left(\frac{s+\mathfrak{a}}{2}\right) - \frac{1}{2}\psi\left(\frac{1-s+\mathfrak{a}}{2}\right) - \frac{\bar{L}'(1-s)}{\bar{L}(1-s)},$$

where Q is the divisor of q to which χ belongs, $\mathfrak{c}$ is the number of primes which divide q but not Q, $\varpi_1, \varpi_2, \ldots$ are the primes in question, and ε_ν is a root of unity. Hence, if $\sigma = -\frac{1}{4}$, we have

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[1] This application of Cauchy's Theorem may be justified on the lines of the classical proof of the 'explicit formulae' for $\psi(x)$ and $\pi(x)$: see Landau, pp. 333—368. In this case the proof is much easier, since $Y^{-s}\Gamma(s)$ tends to zero, when $|t| \to \infty$, like an exponential $e^{-a|t|}$. Compare pp. 134—135 of our memoir 'Contributions to the theory of the Riemann Zeta-function and the theory of the distribution of primes', *Acta Mathematica*, vol. 41 (1917), pp. 119—196.

[2] Landau, p. 517.

(2. 417)
$$\left|\frac{L'(s)}{L(s)}\right| < A \log q + A\mathfrak{d} \log q + A \log(|t|+2) + A$$
$$< A(\log(q+1))^A \log(|t|+2).$$

Again, if $s = -\frac{1}{4} + it$, $Y = \eta + i\theta$, we have

$$|Y^{-s}| = |Y|^{\frac{1}{4}} \exp\left(t \operatorname{arc\,tan} \frac{\theta}{\eta}\right),$$

$$|Y^{-s}\Gamma(s)| < A|Y|^{\frac{1}{4}}(|t|+2)^{-\frac{3}{4}} \exp\left(-\left(\frac{1}{2}\pi - \operatorname{arc\,tan}\frac{|\theta|}{\eta}\right)|t|\right),$$

$$< A|Y|^{\frac{1}{4}} \frac{|t|^{-\frac{1}{2}}}{\log(|t|+2)} e^{-\delta|t|};$$

and so

(2. 418)
$$\left|\frac{1}{2\pi i}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} Y^{-s}\Gamma(s)\frac{L'(s)}{L(s)}\,ds\right| < A(\log(q+1))^A |Y|^{\frac{1}{4}} \int_0^\infty t^{-\frac{1}{2}} e^{-\delta t}\,dt$$
$$< A(\log(q+1))^A |Y|^{\frac{1}{4}} \delta^{-\frac{1}{2}}.$$

2. 42. We now consider R. Since

$$\sum\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) = 0 \qquad (s=0),$$

we have

$$R = \{(\mathfrak{b}+b)\Gamma(s)\}_0 + \left\{\frac{\mathfrak{d}-\mathfrak{b}}{s} Y^{-s}\Gamma(s)\right\}_0 - \frac{1}{2}\left\{Y^{-s}\Gamma(s)\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right\}_0$$
$$= A_1(\mathfrak{b}+b) - (\mathfrak{d}-\mathfrak{b})(A_2 + A_3 \log Y) + C_1(\mathfrak{a}) + C_2(\mathfrak{a}) \log Y,$$

where each of the C's has one of two absolute constant values, according to the value of $\mathfrak{a}$. Since

$$0 \leq \mathfrak{b} \leq 1,\quad 0 \leq \mathfrak{d} < A \log(q+1),\quad |\log Y| < A \log\frac{1}{\eta} < A\eta^{-\frac{1}{2}},$$

we have

(2. 421)
$$|R| < A|b| + A\log(q+1)\,\eta^{-\frac{1}{2}}.$$

From (2. 415), (2. 416), (2. 418), (2. 421) and (2. 15) we deduce

$$f_{2,k}(x) = -\frac{\mathfrak{b}}{Y} + G_k + P_k,$$

$$|P_k| < A\,(\log(q+1))^A \left(|b| + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}} \delta^{-\frac{1}{2}}\right),$$

(2. 422) $$f_2(x) = +\frac{\mu(q)}{hY} + \sum_k C_k G_k + P,$$

(2. 423) $$|P| < A\sqrt{q}\,(\log(q+1))^A \left(\frac{1}{h}\sum_k |b_k| + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}}\delta^{-\frac{1}{2}}\right).$$

Combining (2. 422) and (2. 423) with (2. 11) and (2. 21), we obtain the result of Lemma 4.

2. 5. *Lemma* 5. *If* $q > 1$ *and* χ_k *is a primitive (and therefore non-principal*[1]*) character, then*

(2. 51) $$L(s) = \frac{a e^{bs}}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} \prod_{\varrho} \left(\left(1 - \frac{s}{\varrho}\right) e^{\frac{s}{\varrho}}\right),$$

where

$$a = a(q, \chi) = a_k,$$

(2. 521) $$|L(1)| = \pi q^{-\frac{1}{2}} |L(0)| \qquad (\mathfrak{a} = 1),$$

(2. 522) $$|L(1)| = 2 q^{-\frac{1}{2}} |L'(0)| \qquad (\mathfrak{a} = 0).$$

Further

(2. 53) $$1 - \Theta \leq \Re(\varrho) \leq \Theta,$$

and

(2. 54) $$\left|\frac{L'(1)}{L(1)}\right| < A\,(\log(q+1))^A.$$

This lemma is merely a collection of results which will be used in the proof of Lemmas 6 and 7. They are of very unequal depth. The formula (2. 51) is classical.[2] The two next are immediate deductions from the functional equation for $L(s)$.[3] The inequalities (2. 53) follow from the functional equation and the

[1] Landau, p. 480.
[2] Landau, p. 507.
[3] Landau, pp. 496, 497.

absence (for primitive χ) of factors $1-\varepsilon_\nu \varpi_\nu^{-s}$ from L. Finally (2. 54) is due to GRONWALL.[1]

2. 61. *Lemma 6. If $M(T)$ is the number of zeros ϱ of $L(s)$ for which*

$$0 \leq T \leq |\gamma| \leq T+1,$$

then

$$M(T) < A(\log(q+1))^A \log(T+2). \tag{2. 611}$$

The ϱ's of an imprimitive $L(s)$ are those of a certain primitive $L(s)$ corresponding to modulus Q, where $Q|q$, together with the zeros (other than $s=0$) of certain functions

$$E_\nu = 1 - \varepsilon_\nu \varpi_\nu^{-s},$$

where

$$|\varepsilon_\nu| = 1, \quad \varpi_\nu | q.$$

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[1] T. H. GRONWALL, 'Sur les séries de Dirichlet correspondant à des caractères complexes', *Rendiconti del Circolo Matematico di Palermo*, vol. 35 (1913), pp. 145—159. Gronwall proves that

$$\frac{1}{|L(1)|} < A \log q (\log\log q)^{\frac{3}{8}}$$

for every *complex* χ, and states that the same is true for *real* χ if hypothesis R (or a much less stringent hypothesis) is satisfied. LANDAU ('Über die Klassenzahl imaginär-quadratischer Zahlkörper', *Göttinger Nachrichten*, 1918, pp. 285—295 (p. 286, f. n. 2)) has, however, observed that, in the case of a real χ, Gronwall's argument leads only to the slightly less precise inequality

$$\frac{1}{|L(1)|} < A \log q \sqrt{\log\log q}.$$

Landau also gives a proof (due to HECKE) that

$$\frac{1}{|L(1)|} < A \log q$$

for the special character $\left(\frac{-q}{n}\right)$ associated with the fundamental discriminant $-q$.

The first results in this direction are due to Landau himself ('Über das Nichtverschwinden der Dirichletschen Reihen, welche komplexen Charakteren entsprechen', *Math. Annalen*, vol. 70 (1911), pp. 69—78). Landau there proves that

$$\frac{1}{|L(1)|} < A (\log q)^5$$

for complex χ.

It is easily proved (see p. 75 of Landau's last quoted memoir) that

$$|L'(1)| < A(\log q)^2,$$

so that any of these results gives us more than all that we require.

The number of ϖ_ν's is less than $A \log(q+1)$, and each E_ν has a set of zeros, on $\sigma = 0$, at equal distances

$$\frac{2\pi}{\log \varpi_\nu} > \frac{2\pi}{\log(q+1)}.$$

The contribution of these zeros to $M(T)$ is therefore less than $A(\log(q+1))^2$; and we need consider only a primitive (and therefore, if $q > 1$, non-principal) $L(s)$.

We observe:

(a) that $\mathfrak{a}$ is the same for $L(s)$ and $\bar{L}(s)$;

(b) that $L(s)$ and $\bar{L}(s)$ are conjugate for real s, so that the b corresponding to $\bar{L}(s)$ is $\bar{b}$, the conjugate of the b of $L(s)$;

(c) that the typical ϱ of $\bar{L}(s)$ may be taken to be either $\bar{\varrho}$ or (in virtue of the functional equation) $1 - \varrho$, so that

$$S = \sum \left(\frac{1}{\varrho} + \frac{1}{1-\varrho}\right) = \sum \left(\frac{1}{\varrho} + \frac{1}{\bar{\varrho}}\right)$$

is real.

Bearing these remarks in mind, suppose first that $\mathfrak{a} = 1$. We have then, from (2. 51) and (2. 521),

$$\frac{\pi^2}{q} = \left|\frac{L(1)\bar{L}(1)}{L(0)\bar{L}(0)}\right| = A\left|e^b \prod\left(\left(1-\frac{1}{\varrho}\right)e^{\frac{1}{\varrho}}\right) e^{\bar{b}} \prod\left(\left(1-\frac{1}{1-\varrho}\right)e^{\frac{1}{1-\varrho}}\right)\right|$$

$$= A\, e^{2\Re(b)+S},$$

since

$$\left(1-\frac{1}{\varrho}\right)\left(1-\frac{1}{1-\varrho}\right) = 1.$$

Thus

(2. 612) $$|2\Re(b) + S| < A \log(q+1).$$

On the other hand, if $\mathfrak{a} = 0$, we have, from (2. 51) and (2. 522),

$$\frac{4}{q} = \left|\frac{L(1)\bar{L}(1)}{L'(0)\bar{L}'(0)}\right| = A\left|e^b \prod\left(\left(1-\frac{1}{\varrho}\right)e^{-\frac{1}{\varrho}}\right) e^{\bar{b}} \prod\left(\left(1-\frac{1}{1-\varrho}\right)e^{\frac{1}{1-\varrho}}\right)\right|,$$

and (2. 612) follows as before.

2. 62. Again, by (2. 31)

(2. 621) $$\frac{L'(1)}{L(1)} = \mathfrak{b} + b - \frac{1}{2}\psi\left(\frac{1+\mathfrak{a}}{2}\right) + \sum\left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right),$$

for every non-principal character (whether primitive or not). In particular, when χ is primitive, we have, by (2. 621), (2. 54), and (2. 33),

$$(2.\ 622) \qquad |\Re(b)+S| = \left|\Re\frac{L'(1)}{L(1)} - \mathfrak{b} + \frac{1}{2}\psi\left(\frac{1+\mathfrak{a}}{2}\right)\right| < A\,(\log(q+1))^A.$$

Combining (2. 612) and (2. 622) we see that

$$(2.\ 623) \qquad S < A\,(\log(q+1))^A$$

and

$$(2.\ 624) \qquad |\Re(b)| < A\,(\log(q+1))^A.$$

2. 63. If now $q > 1$, and χ is primitive (so that $\mathfrak{b} = 0$), and $s = 2 + iT$, we have, by (2. 31), (2. 33), and (2. 624),

$$\begin{aligned}
0 &< \sum\left(\frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} + \frac{\beta}{\beta^2+\gamma^2}\right) = \Re\sum\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right) \\
&= \Re\frac{L'(s)}{L(s)} - \Re\left(\frac{\mathfrak{b}}{s}\right) - \Re(b) + \frac{1}{2}\Re\left(\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right) \\
&\leqq \left|\frac{L'(s)}{L(s)}\right| + \left|\frac{\mathfrak{b}}{s}\right| + |\Re(b)| + \left|\psi\left(\frac{s+\mathfrak{a}}{2}\right)\right| \\
&< A + A\log(q+1) + A\,(\log(q+1))^A + A\log(|T|+2) \\
&< A\,(\log(q+1))^A \log(|T|+2),
\end{aligned}$$

$$\sum_{|T-\gamma|\leqq 1}\frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} < A\,(\log(q+1))^A\log(|T|+2).$$

Every term on the left hand side is greater than A, and the number of terms is not less than $M(T)$. Hence we obtain the result of the lemma. We have excluded the case $q = 1$, when the result is of course classical.[1]

2. 71. *Lemma 7. We have*

$$(2.\ 711) \qquad |b| < Aq\,(\log(q+1))^A.$$

Suppose first that χ is non-principal. Then, by (2. 621) and (2. 54),

$$(2.\ 712) \qquad |b| < A\,(\log(q+1))^A + \left|\sum\left(\frac{1}{1-\varrho}+\frac{1}{\varrho}\right)\right|.$$

[1] Landau, p. 337.

We write

(2. 713)
$$\sum = \sum_1 + \sum_2,$$

where $\sum_1$ is extended over the zeros for which $1 - \Theta \leqq \Re(\varrho) \leqq \Theta$ and $\sum_2$ over those for which $\Re(\varrho) = 0$. Now $\sum_1 = S'$, where S' is the S corresponding to a primitive $L(s)$ for modulus Q, where $Q \mid q$. Hence, by (2. 623),

(2. 714)
$$\left|\sum_1\right| < A\,(\log(Q+1))^A < A\,(\log(q+1))^A.$$

Again, the ϱ's of $\sum_2$ are the zeros (other than $s = 0$) of

$$\prod_\nu \left(1 - \frac{\varepsilon_\nu}{\varpi_\nu^s}\right),$$

the ϖ_ν's being divisors of q and ε_ν an m-th root of unity, where $m = \varphi(Q) < q$[1]; so that the number of ϖ_ν's is less than $A \log q$ and

$$\varepsilon_\nu = e^{2\pi i \omega_\nu},$$

where either $\omega_\nu = 0$ or

$$\frac{1}{q} \leqq |\omega_\nu| \leqq \frac{1}{2}.$$

Let us denote by ϱ_ν a zero (other than $s = 0$) of $1 - \varepsilon_\nu \varpi_\nu^{-s}$, by ϱ'_ν a ϱ_ν for which $|\varrho_\nu| \leqq 1$, and by ϱ''_ν a ϱ_ν for which $|\varrho_\nu| > 1$. Then

(2. 715)
$$\left|\sum_2 \left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right)\right| \leqq \sum_\nu \left(\sum_{\varrho'_\nu} + \sum_{\varrho''_\nu}\right) \left|\frac{1}{1-\varrho} + \frac{1}{\varrho}\right|.$$

Any ϱ_ν is of the form

$$\varrho_\nu = \frac{2\pi i (m + \omega_\nu)}{\log \varpi_\nu},$$

where m is an integer. Hence the number of zeros ϱ'_ν is less than $A \log \varpi_\nu$ or than $A \log(q+1)$; and the absolute value of the corresponding term in our sum is less than

(2. 716)
$$\frac{A}{|\varrho|} < \frac{A \log \varpi_\nu}{|\omega_\nu|} < A q \log(q+1);$$

[1] For (Landau, p. 482) $\varepsilon_\nu = X(\varpi_\nu)$, where X is a character to modulus Q.

so that

(2. 717) $$\left|\sum_{\varrho'_\nu}\right| < Aq\,(\log (q+1))^2.$$

Also

(2. 718) $$\left|\sum_{\varrho''_\nu}\right| \leq \sum_{\varrho''_\nu}\left|\frac{1}{\varrho(1-\varrho)}\right| < \sum_{\varrho''_\nu}\frac{1}{|\varrho|^2}$$

$$< A\,(\log \varpi_\nu)^2 \sum_{m=1}^{\infty}\frac{1}{m^2} < A\,(\log (q+1))^2.$$

From (2. 715), (2. 717) and (2. 718) we deduce

(2. 719) $$\left|\sum\nolimits_2\right| < Aq\,(\log (q+1))^A;$$

and from (2. 713), (2. 714) and (2. 719) the result of the lemma.

2. 72. We have assumed that χ is not a principal character: For the principal character (mod. q) we have [1]

$$L_1(s) = \prod_{\varpi \mid q}\left(1 - \frac{1}{\varpi^s}\right)\zeta(s).$$

Since $\mathfrak{a} = 0$, $\mathfrak{b} = 1$, we have

$$\sum_{\varpi \mid q}\frac{\log \varpi}{\varpi^s - 1} + \frac{\zeta'(s)}{\zeta(s)} = \frac{L'_1(s)}{L_1(s)}$$

$$= \frac{\mathfrak{b}-1}{s} - \frac{1}{s-1} + b - \frac{1}{2}\psi\left(\frac{1}{2}s\right) + \sum\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right),^2$$

$$\sum_{\varpi \mid q}\frac{\log \varpi}{\varpi - 1} + \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) = \mathfrak{b} - 1 + b - \frac{1}{2}\psi\left(\frac{1}{2}\right) + \sum\left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right),$$

$$|b| < A \log (q+1) + \left|\sum\left(\frac{1}{1-\varrho} + \frac{1}{\varrho}\right)\right|.$$

This corresponds to (2. 712), and from this point the proof proceeds as before.

[1] Landau, p. 423.

[2] $\sum$ refers to the complex zeros of $L_1(s)$, not merely to those of $\zeta(s)$.

2. 81. *Lemma 8. If* $0 < \eta \leqq \frac{1}{2}$ *then*

(2. 811)
$$f(x) = \frac{\mu(q)}{hY} + \sum_{k=1}^{h} C_k G_k + P,$$

where

(2. 812)
$$G_k = \sum_{\varrho_k} \Gamma(\varrho)\, Y^{-\varrho},$$

(2. 813)
$$|P| < A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}} \delta^{-\frac{1}{2}}\right),$$

(2. 814)
$$\delta = \operatorname{arc\,tan} \frac{\eta}{|\theta|}.$$

This is an immediate corollary of Lemmas 4 and 7.

2. 82. *Lemma 9. If* $0 < \eta \leqq \frac{1}{2}$ *then*

(2. 821)
$$f(x) = \varphi + \Phi,$$

where

(2. 822)
$$\varphi = \frac{\mu(q)}{hY},$$

(2. 823)
$$|\Phi| < A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right)\right),$$

(2. 824)
$$\delta = \operatorname{arc\,tan} \frac{\eta}{|\theta|}.$$

We have

(2. 825)
$$|G_k| \leqq \sum\nolimits_1 |\Gamma(\varrho)\, Y^{-\varrho}| + \sum\nolimits_2 |\Gamma(\varrho)\, Y^{-\varrho}|,$$

where $\sum_1$ extends over ϱ_k's for which $|\gamma| \geqq 1$, $\sum_2$ over those for which $|\gamma| < 1$. In $\sum_1$ we have

$$|\Gamma(\varrho)\, Y^{-\varrho}| = |\Gamma(\beta + i\gamma)|\,|Y|^{-\beta} \exp\left(\gamma \operatorname{arc\,tan} \frac{\theta}{\eta}\right)$$

$$\leqq A|\gamma|^{\beta-\frac{1}{2}} |Y|^{-\beta} \exp\left(-\left(\frac{1}{2}\pi - \operatorname{arc\,tan} \frac{|\theta|}{\eta}\right)|\gamma|\right)$$

$$\leqq A|\gamma|^{\Theta-\frac{1}{2}} |Y|^{-\Theta} e^{-\delta|\gamma|}$$

(since $|Y| < A$ and, by hypothesis R, $\beta \leqq \Theta$). The number $M(T)$ of ϱ's for which $|\gamma|$ lies between T and $T+1$ $(T \geqq 0)$ is less than $A\,(\log(q+1))^A \log(T+2)$, by (2. 611). Hence

$$\sum\nolimits_1 |\gamma|^{\Theta-\frac{1}{2}} e^{-\delta|\gamma|} \leqq A\,(\log(q+1))^A \sum_{n=0}^{\infty} (n+1)^{\Theta-\frac{1}{2}} \log(n+2)\, e^{-\delta n}$$

$$< A\,(\log(q+1))^A\, \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta}+2\right),$$

(2. 826) $$\sum\nolimits_1 |\Gamma(\varrho)\, Y^{-\varrho}| < A\,(\log(q+1))^A |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta}+2\right).$$

2. 83. Again, once more by (2. 611), $\sum_2$ has at most $A\,(\log(q+1))^A$ terms. We write

(2. 831) $$\sum\nolimits_2 = \sum\nolimits_{2,1} + \sum\nolimits_{2,2},$$

$\sum_{2,1}$ applying to zeros for which $1-\Theta \leqq \beta \leqq \Theta$, and $\sum_{2,2}$ to those for which $\beta = 0$. Now, in $\sum_2$,

$$|Y^{-\varrho}| = |Y|^{-\beta} \exp\left(\gamma \operatorname{arc\,tan} \frac{\theta}{\eta}\right) < A\,|Y|^{-\beta};$$

and in $\sum_{2,1}$, $|\Gamma(\varrho)| < A$. Hence

(2. 832) $$\left|\sum\nolimits_{2,1}\right| < A\,|Y|^{-\beta} \sum\nolimits_{2,1} |\Gamma(\varrho)| < A\,|Y|^{-\Theta} \sum\nolimits_{2,1} 1 < A\,(\log(q+1))^A |Y|^{-\Theta}.$$

Again, in $\sum_{2,2}$, $|Y| < A$ and

$$\frac{1}{|\varrho|} < A\,q \log(q+1),$$

by (2. 716); so that

(2. 833) $$\left|\sum\nolimits_{2,2}\right| < A \sum\nolimits_{2,2} |\Gamma(\varrho)| = A \sum\nolimits_{2,2} \frac{|\Gamma(1+\varrho)|}{|\varrho|}$$

$$< A \sum\nolimits_{2,2} \frac{1}{|\varrho|} < A\,q\,(\log(q+1))^A.$$

From (2. 825), (2. 826), (2. 831), (2. 832), and (2. 833), we obtain

(2. 834) $$|G_k| < A\,(\log(q+1))^A \left(q + |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta}+2\right)\right) = H_k,$$

say; and from (2. 811), (2. 812), (2. 813), (2. 821), (2. 822) and (2. 834) we deduce

$$|\Phi| = \left|\sum_{k=1}^{h} C_k G_k + P\right|$$

$$< \sum_{k=1}^{h} |C_k G_k| + A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{\frac{1}{4}}\delta^{-\frac{1}{2}}\right)$$

$$< \frac{\sqrt{q}}{h}\sum_{k=1}^{h} H_k + A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{-\Theta}\delta^{-\Theta-\frac{1}{2}}\log\left(\frac{1}{\delta}+2\right)\right)$$

$$< A\sqrt{q}\,(\log(q+1))^A \left(q + \eta^{-\frac{1}{2}} + |Y|^{-\Theta}\delta^{-\Theta-\frac{1}{2}}\log\left(\frac{1}{\delta}+2\right)\right);$$

that is to say (2. 823).

2. 9. *Lemma 10. We have*

(2. 91) $$h = \varphi(q) > Aq\,(\log q)^{-A}.$$

We have in fact[1]

$$\varphi(q) > (1-\delta)\, e^{-C}\frac{q}{\log\log q} \qquad (q > q_0(\delta))$$

for every positive δ, C being Euler's constant.

3. Proof of the main theorems.

Approximation to $\nu_r(n)$ by the singular series.

3. 11. **Theorem A.** *If r is an integer, $r \geqq 3$, and*

(3. 111) $$(f(x))^r = \sum \nu_r(n)\, x^n,$$

so that

(3. 112) $$\nu_r(n) = \sum_{\varpi_1+\varpi_2+\cdots+\varpi_r = n} \log\varpi_1 \log\varpi_2 \cdots \log\varpi_r,$$

then

(3. 113) $$\nu_r(n) = \frac{n^{r-1}}{(r-1)!}\mathfrak{S}_r + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)}(\log n)^B\right) \sim \frac{n^{r-1}}{(r-1)!}\mathfrak{S}_r,$$

[1] Landau, p. 217.

where

$$S_r = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n). \tag{3. 114}$$

It is to be understood, here and in all that follows, that O's refer to the limit-process $n \to \infty$, and that their constants are functions of r alone.

If $n \geqq 2$, we have

$$\nu_r(n) = \frac{1}{2\pi i}\int (f(x))^r \frac{dx}{x^{n+1}}, \tag{3. 115}$$

the path of integration being the circle $|x| = e^{-H}$, where $H = \frac{1}{n}$, so that

$$1 - |x| = \frac{1}{n} + O\left(\frac{1}{n^2}\right) \infty \frac{1}{n}.$$

Using the Farey dissection of order $N = [\sqrt{n}]$, we have

$$\begin{aligned} \nu_r(n) &= \sum_{q=1}^{N} \sum_{p<q,\,(p,q)=1} \frac{1}{2\pi i}\int_{\xi_{p,q}} (f(x))^r \frac{dx}{x^{n+1}} \\ &= \sum e_q(-np) \frac{1}{2\pi i}\int_{\xi_{p,q}} (f(x))^r \frac{dX}{X^{n+1}} \\ &= \sum e_q(-np)\, j_{p,q}, \end{aligned} \tag{3. 116}$$

say. Now

$$\begin{aligned} |f^r - \varphi^r| &\leq |\Phi| (|f^{r-1}| + |f^{r-2}\varphi| + \cdots + |\varphi^{r-1}|) \\ &< B(|\Phi f^{r-1}| + |\Phi \varphi^{r-1}|). \end{aligned}$$

Also $|X^{-n}| = e^{nH} < A$. Hence

$$j_{p,q} = l_{p,q} + m_{p,q}, \tag{3. 117}$$

where

$$l_{p,q} = \frac{1}{2\pi i}\int_{\xi_{p,q}} \varphi^r \frac{dX}{X^{n+1}}, \tag{3. 118}$$

$$|m_{p,q}| = O\left(\int_{-\theta'_{p,q}}^{\theta_{p,q}} (|\Phi f^{r-1}| + |\Phi \varphi^{r-1}|)\, d\theta\right). \tag{3. 119}$$

3. 12. We have $\eta = H = \frac{1}{n}$ and $q \leqq \sqrt{n}$, and so, by (2. 823),

$$|\Phi| < A n^{\frac{3}{4}} (\log n)^A + A (\log n)^A \sqrt{q} \, |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right), \tag{3. 121}$$

where $\delta = \operatorname{arc\,tan} \frac{\eta}{|\theta|}$. We must now distinguish two cases. If $|\theta| \leqq \eta$, we have

$$|Y| > A\eta, \; \delta > A,$$

and

$$\sqrt{q} \, |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right) < A \, n^{\frac{1}{4}} \eta^{-\Theta} = A \, n^{\Theta + \frac{1}{4}}. \tag{3. 122}$$

If on the other hand $\eta < |\theta| \leqq \bar{\theta}_{p,q}$, we have

$$\delta > A \frac{\eta}{|\theta|} > \frac{A}{n}, \; |Y| > A \, |\theta|,$$

$$\sqrt{q} \, |Y|^{-\Theta} \delta^{-\Theta-\frac{1}{2}} \log\left(\frac{1}{\delta} + 2\right) < A \sqrt{q} \, . \, |\theta|^{-\Theta} . \, \eta^{-\Theta-\frac{1}{2}} |\theta|^{\Theta+\frac{1}{2}} . \log n \tag{3. 123}$$

$$= A n^{\Theta+\frac{1}{2}} \log n \, (q \, |\theta|)^{\frac{1}{2}} < A n^{\Theta+\frac{1}{2}} \log n \, . \, n^{-\frac{1}{4}} = A n^{\Theta+\frac{1}{4}} \log n,$$

since $q \, |\theta| \leqq q \, \bar{\theta}_{p,q} < A n^{-\frac{1}{2}}$. Thus (3. 123) holds in either case. Also $\Theta \geqq \frac{1}{2}$ and so, by (3. 121),

$$|\Phi| < A n^{\Theta + \frac{1}{4}} (\log n)^A \tag{3. 124}$$

3. 13. Now, remembering that $r \geqq 3$, we have

$$\int_{-\theta'_{p,q}}^{\theta_{p,q}} |\varphi|^{r-1} d\theta < B h^{-(r-1)} \int_{-\theta'_{p,q}}^{\theta_{p,q}} |Y|^{-(r-1)} d\theta$$

$$< B h^{-(r-1)} \int_0^{\infty} (\eta^2 + \theta^2)^{-\frac{1}{2}(r-1)} \, d\theta$$

$$< B h^{-(r-1)} n^{r-2};$$

and so

$$(3.\ 131)\qquad \sum_{p,q}\int_{-\theta'_{p,q}}^{\theta_{p,q}} |\Phi\varphi^{r-1}|\,d\theta < Bn^{r-2}\,(\operatorname{Max}|\Phi|)\sum_q h^{-(r-2)}$$

$$< Bn^{r-2+\theta+\frac{1}{4}}(\log n)^B = Bn^{r-1+\left(\theta-\frac{3}{4}\right)}(\log n)^B,$$

by (3. 124) and (2. 91).

3. 14. Again, if $\arg x = \psi$, we have

$$\sum\int_{-\theta'_{p,q}}^{\theta_{p,q}} |f|^2\,d\theta = \int_0^{2\pi}|f|^2\,d\psi$$

$$= 2\pi\sum_{\varpi}(\log\varpi)^2|x|^{2\varpi} < A\sum_{m=2}^{\infty}\log m\,\Lambda(m)\,|x|^{2m}$$

$$< A(1-|x|^2)\sum_{m=2}^{\infty}\left(\sum_{k=2}^{m}\log k\,\Lambda(k)\right)|x|^{2m}$$

$$< A(1-|x|)\sum_{m=2}^{\infty} m\log m\,|x|^{2m}$$

$$< \frac{A}{1-|x|}\log\left(\frac{1}{1-|x|}\right) < An\log n.$$

Similarly

$$|f| \leq \sum_{\varpi}\log\varpi\,|x|^{\varpi} < \sum_m \Lambda(m)\,|x|^m < \frac{A}{1-|x|} < An.$$

Hence

$$(3.\ 141)\qquad \sum_{p,q}\int_{-\theta'_{p,q}}^{\theta_{p,q}} |f|^{r-1}|\Phi|\,d\theta \leq \operatorname{Max}|\Phi f^{r-3}|\int_0^{2\pi}|f|^2\,d\psi$$

$$< Bn^{\theta+\frac{1}{4}}\log n\,.\,n^{r-3}\,.\,n\log n$$

$$< Bn^{r-1+\left(\theta-\frac{3}{4}\right)}(\log n)^B.$$

From (3. 116), (3. 117), (3. 119), (3. 131) and (3. 141) we deduce

$$(3.\ 142)\qquad \nu_r(n) = \sum e_q(-np)\, l_{p,q} + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)}(\log n)^B\right),$$

where $l_{p,q}$ is defined by (3. 118).

3. 15. In $l_{p,q}$ we write $X = e^{-Y}$, $dX = -e^{-Y}dY$, so that Y varies on the straight line from $\eta + i\theta_{p,q}$ to $\eta - i\theta'_{p,q}$. Then, by (2. 822) and (3. 118),

$$(3.\ 151)\qquad l_{p,q} = -\frac{1}{2\pi i}\left(\frac{\mu(q)}{h}\right)^r \int_{\eta+i\theta_{p,q}}^{\eta-i\theta'_{p,q}} Y^{-r} e^{nY}\, dY.$$

Now

$$(3.\ 152)\qquad -\int_{\eta+i\theta_{p,q}}^{\eta-i\theta'_{p,q}} = \int_{\eta-i\infty}^{\eta+i\infty} Y^{-r} e^{nY}\, dY + O\left(\int_{\theta_q}^{\infty} |\eta + i\theta|^{-r}\, d\theta\right)$$

$$= 2\pi i \frac{n^{r-1}}{(r-1)!} + O\left(\int_{\theta_q}^{\infty} |\eta + i\theta|^{-r}\, d\theta\right),$$

where

$$\theta_q = \operatorname*{Min}_{p<q}\,(\theta_{p,q}, \theta'_{p,q}) \geqq \frac{1}{2qN}.$$

Also

$$(3.\ 153)\qquad \int_{\theta_q}^{\infty} (\eta + i\theta)^{-r}\, d\theta < \int_{\theta_q}^{\infty} \theta^{-r}\, d\theta < B\theta_q^{1-r} < B(q\sqrt{n})^{r-1}.$$

From (3. 151), (3. 152) and (3. 153), we deduce

$$(3.\ 154)\qquad \sum e_q(-np)\, l_{p,q} = \frac{n^{r-1}}{(r-1)!}\sum_{p,q}\left(\frac{\mu(q)}{\varphi(q)}\right)^r e_q(-np) + Q,$$

where

$$(3.\ 155)\qquad |Q| < B\sum_{p,q} h^{-r} q^{r-1} n^{\frac{1}{2}(r-1)} < Bn^{\frac{1}{2}(r-1)} \sum_q \left(\frac{q}{h}\right)^{r-1}$$

$$< Bn^{\frac{1}{2}(r-1)} \sum_{q=1}^{N} (\log q)^B < Bn^{\frac{1}{2}r}(\log n)^B.$$

Since $r \geqq 3$ and $\Theta \geqq \frac{1}{2}$, $\frac{1}{2} r < r - 1 - \frac{1}{4} \leqq r - 1 + \left(\Theta - \frac{3}{4}\right)$; and from (3. 142), (3. 154), and (3. 155) we obtain

$$(3.\ 156) \qquad \nu_r(n) = \frac{n^{r-1}}{(r-1)!} \sum_{p,q} \left(\frac{\mu(q)}{\varphi(q)}\right)^r e_q(-np) + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)} (\log n)^B\right)$$

$$= \frac{n^{r-1}}{(r-1)!} \sum_{q \leqq N} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n) + O\left(n^{r-1+\left(\Theta-\frac{3}{4}\right)} (\log n)^B\right).$$

3. 16. In order to complete the proof of Theorem A, we have merely to show that the finite series in (3. 156) may be replaced by the infinite series S_r. Now

$$\left| n^{r-1} \sum_{q>N} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n) \right| < B n^{r-1} \sum_{q>N} q^{1-r} (\log q)^B < B n^{\frac{1}{2}r} (\log n)^B,$$

and $\frac{1}{2} r < r - 1 + \left(\Theta - \frac{3}{4}\right)$. Hence this error may be absorbed in the second term of (3. 156), and the proof of the theorem is completed.

Summation of the singular series.

3. 21. *Lemma 11. If*

$$(3.\ 211) \qquad c_q(n) = \sum e_q(np),$$

where n is a positive integer and the summation extends over all positive values of p less than and prime to q, $p = 0$ being included when $q = 1$, but not otherwise, then

$$(3.\ 212) \qquad c_q(-n) = c_q(n);$$

$$(3.\ 213) \qquad c_{qq'}(n) = c_q(n)\, c_{q'}(n)$$

if $(q, q') = 1$; and

$$(3.\ 214) \qquad c_q(n) = \sum \delta \mu \left(\frac{q}{\delta}\right),$$

where δ is a common divisor of q and n.

The terms in p and $q - p$ are conjugate. Hence $c_q(n)$ is real. As $c_q(n)$ and $c_q(-n)$ are conjugate we obtain (3. 212).[1]

[1] The argument fails if $q = 1$ or $q = 2$; but $c_1(n) = c_1(-n) = 1$, $c_2(n) = c_2(-n) = -1$.

Again

$$c_q(n)\,c_{q'}(n) = \sum_{p,p'} \exp\left(2n\pi i\left(\frac{p}{q}+\frac{p'}{q'}\right)\right) = \sum_{p,p'} \exp\left(\frac{2nP\pi i}{qq'}\right),$$

where

$$P = pq' + p'q.$$

When p assumes a set of $\varphi(q)$ values, positive, prime to q, and incongruent to modulus q, and p' a similar set of values for modulus q', then P assumes a set of $\varphi(q)\varphi(q') = \varphi(qq')$ values, plainly all positive, prime to qq' and incongruent to modulus qq'. Hence we obtain (3. 213).

Finally, it is plain that

$$\sum_{d\mid q} c_d(n) = \sum_{h=0}^{q-1} e_q(nh),$$

which is zero unless $q\mid n$ and then equal to q. Hence, if we write

$$\eta(q) = q \quad (q\mid n), \qquad \eta(q) = 0 \quad (q \nmid n),$$

we have

$$\sum_{d\mid q} c_d(n) = \eta(q),$$

and therefore

$$c_q(n) = \sum_{d\mid q} \eta(d)\,\mu\left(\frac{q}{d}\right)$$

by the well-known inversion formula of Möbius.[1] This is (3. 214).[2]

3. 22. *Lemma 12. Suppose that* $r \geqq 2$ *and*

(3. 221) $$S_r = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n).$$

Then

(3. 222) $$S_r = 0$$

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[1] Landau, p. 577.

[2] The formula (3. 214) is proved by Ramanujan ('On certain trigonometrical sums and their applications in the theory of numbers', *Trans. Camb. Phil. Soc.*, vol. 22 (1918), pp. 259—276 (p. 260)). It had already been given for $n = 1$ by Landau (*Handbuch* (1909), p. 572: Landau refers to it as a known result), and in the general case by Jensen ('Et nyt Udtryk for den talteoretiske Funktion $\sum \mu(n) = M(n)$', *Den 3. Skandinaviske Matematiker-Kongres, Kristiania 1913,* Kristiania (1915), p. 145). Ramanujan makes a large number of very beautiful applications of the sums in question, and they may well be associated with his name.

if n and r are of opposite parity. But if n and r are of like parity then

(3. 223)
$$S_r = 2C_r \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p}-1)^r + (-1)^r(\mathfrak{p}-1)}{(\mathfrak{p}-1)^r - (-1)^r}\right),$$

where $\mathfrak{p}$ *is an odd prime divisor of* n *and*

(3. 224)
$$C_r = \prod_{\varpi=3}^{\infty} \left(1 - \frac{(-1)^r}{(\varpi-1)^r}\right).$$

Let

(3. 225)
$$\left(\frac{\mu(q)}{\varphi(q)}\right)^r c_q(-n) = A_q.$$

Then

$$\mu(qq') = \mu(q)\mu(q'),\ \varphi(qq') = \varphi(q)\varphi(q'),\ c_{qq'}(-n) = c_q(-n)\,c_{q'}(-n)$$

if $(q, q') = 1$; and therefore (on the same hypothesis)

(3. 226)
$$A_{qq'} = A_q A_{q'}.$$

Hence[1]

$$S_r = A_1 + A_2 + A_3 + \cdots = 1 + A_2 + \cdots = \prod_{\varpi} \chi_\varpi$$

where

(3. 227)
$$\chi_\varpi = 1 + A_\varpi + A_{\varpi^2} + A_{\varpi^3} + \cdots = 1 + A_\varpi,$$

since $A_{\varpi^2}, A_{\varpi^3}, \cdots$ vanish in virtue of the factor $\mu(q)$.

3. 23. If $\varpi \nmid n$, we have

$$\mu(\varpi) = -1,\ \varphi(\varpi) = \varpi - 1,\ c_\varpi(n) = \mu(\varpi) = -1,$$

(3. 231)
$$A_\varpi = -\frac{(-1)^r}{(\varpi-1)^r}.$$

If on the other hand $\varpi \mid n$, we have

$$c_\varpi(n) = \mu(\varpi) + \varpi\mu(1) = \varpi - 1,$$

(3. 232)
$$A_\varpi = \frac{(-1)^r}{(\varpi-1)^{r-1}}.$$

[1] Since $|c_q(n)| \leq \sum \delta$, where $\delta \mid n$, we have $c_q(n) = O(1)$ when n is fixed and $q \to \infty$. Also by Lemma 10, $\varphi(q) > Aq(\log q)^{-A}$. Hence the series and products concerned are absolutely convergent.

Hence

$$S_r = \prod_{\varpi \mid n} \left(1 + \frac{(-1)^r}{(\varpi-1)^{r-1}}\right) \prod_{\varpi \nmid n} \left(1 - \frac{(-1)^r}{(\varpi-1)^r}\right). \tag{3.233}$$

If n is even and r is odd, the first factor vanishes in virtue of the factor for which $\varpi = 2$; if n is odd and r even, the second factor vanishes similarly. Thus $S_r = 0$ whenever n and r are of opposite parity.

If n and r are of like parity, the factor corresponding to $\varpi = 2$ is in any case 2; and

$$S_r = 2 \prod_{\varpi=3}^{\infty} \left(1 - \frac{(-1)^r}{(\varpi-1)^r}\right) \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p}-1)^r + (-1)^r(\mathfrak{p}-1)}{(\mathfrak{p}-1)^r - (-1)^r}\right),$$

as stated in the lemma.

Proof of the final formulae.

3. 3. **Theorem B.** *Suppose that* $r \geqq 3$. *Then, if n and r are of unlike parity,*

$$\nu_r(n) = o(n^{r-1}). \tag{3.31}$$

But if n and r are of like parity then

$$\nu_r(n) \sim \frac{2C_r}{(r-1)!} n^{r-1} \prod_{\mathfrak{p}} \left(\frac{(\mathfrak{p}-1)^r + (-1)^r(\mathfrak{p}-1)}{(\mathfrak{p}-1)^r - (-1)^r}\right), \tag{3.32}$$

where $\mathfrak{p}$ is an odd prime divisor of n and

$$C_r = \prod_{\varpi=3}^{\infty} \left(1 - \frac{(-1)^r}{(\varpi-1)^r}\right). \tag{3.33}$$

This follows immediately from Theorem A and Lemma 12.[1]

3. 4. *Lemma 13. If $r \geqq 3$ and n and r are of like parity, then*

$$\nu_r(n) > Bn^{r-1},$$

for $n \geqq n_0(r)$.

[1] Results equivalent to these are stated in equations (5. 11)—(5. 22) of our note 2, but incorrectly, a factor

$$(\log n)^{-r}$$

being omitted in each, owing to a momentary confusion between $\nu_r(n)$ and $N_r(n)$. The $\nu_r(n)$ of 2 is the $N_r(n)$ of this memoir.

This lemma is required for the proof of Theorem C. If r is *even*

$$\prod\left(\frac{(\mathfrak{p}-1)^r+\mathfrak{p}-1}{(\mathfrak{p}-1)^r-1}\right)>1.$$

If r is *odd*

$$\prod\left(\frac{(\mathfrak{p}-1)^r-\mathfrak{p}+1}{(\mathfrak{p}-1)^r+1}\right)>\prod\left(\frac{(\mathfrak{p}-1)^r-\mathfrak{p}}{(\mathfrak{p}-1)^r}\right)>\prod_{\varpi=3}^{\infty}\left(1-\frac{\varpi}{(\varpi-1)^3}\right)=A.$$

In either case the conclusion follows from (3. 32).

3. 5. **Theorem C.** *If $r \geqq 3$ and n and r are of like parity, then*

(3. 51)
$$N_r(n) \sim \frac{\nu_r(n)}{(\log n)^r}.$$

We observe first that

$$N_r(n) = \sum_{\varpi_1+\varpi_2+\cdots+\varpi_r=n} 1 \leqq \sum_{m_1+m_2+\cdots+m_r=n} 1 < Bn^{r-1}$$

and

(3. 511)
$$\nu_r(n) = \sum_{\varpi_1+\varpi_2+\cdots+\varpi_r=n} \log\varpi_1\cdots\log\varpi_r \leqq (\log n)^r N_r(n) < Bn^{r-1}(\log n)^r.$$

Write now

(3. 512)
$$\nu_r = \nu'_r + \nu''_r, \quad N_r = N'_r + N''_r,$$

where ν'_r and N'_r include all terms of the summations for which

$$\varpi_s \geqq n^{1-\delta} \quad (0<\delta<1,\ s=1,2,\ldots,r).$$

Then plainly

(3. 513)
$$\nu'_r(n) \geqq (1-\delta)^r (\log n)^r N'_r(n).$$

Again

$$N''_r(n) \leqq r\sum_{\varpi_r<n^{1-\delta}}\left(\sum_{\varpi_1+\varpi_2+\cdots+\varpi_{r-1}=n-\varpi_r} 1\right)$$

$$< B\sum_{\varpi_r<n^{1-\delta}} N_{r-1}(n-\varpi_r) < Bn^{1-\delta}\,.\,n^{r-2} < Bn^{r-1-\delta},$$

$$\nu''_r(n) \leqq (\log n)^r N''_r(n) < Bn^{r-1-\delta}(\log n)^r.$$

But $\nu_r(n) > Bn^{r-1}$ for $n \geqq n_0(r)$, by Lemma 13; and so

(3. 514)
$$(\log n)^r N''_r(n) = o(\nu_r(n)), \qquad \nu''_r(n) = o(\nu_r(n)),$$

for every positive δ.

From (3. 511), (3. 512), (3. 513), and (3. 514) we deduce

$$(1-\delta)^r (\log n)^r (N_r - N''_r) \leq \nu_r - \nu''_r \leq (\log n)^r N_r,$$

$$(1-\delta)^r (\log n)^r N_r \leq \nu_r + o(\nu_r) \leq (\log n)^r N_r,$$

$$(1-\delta)^r \leq \varliminf \frac{\nu_r}{(\log n)^r N_r}, \quad \varlimsup \frac{\nu_r}{(\log n)^r N_r} \leq 1.$$

As δ is arbitrary, this proves (3. 51).

3. 6. **Theorem D.** *Every large odd number n is the sum of three odd primes. The asymptotic formula for the number of representations $\bar{N}_3(n)$ is*

$$\bar{N}_3(n) \sim C_3 \frac{n^2}{(\log n)^3} \prod \left(\frac{(\mathfrak{p}-1)(\mathfrak{p}-2)}{\mathfrak{p}^2 - 3\mathfrak{p} + 3}\right), \tag{3. 61}$$

where $\mathfrak{p}$ is a prime divisor of n and

$$C_3 = \prod_{\varpi=3}^{\infty} \left(1 + \frac{1}{(\varpi - 1)^3}\right). \tag{3. 62}$$

This is an almost immediate corollary of Theorems B and C. These theorems give the corresponding formula for $N_3(n)$. If not all the primes are odd, two must be 2 and $n-4$ a prime. The number of such representations is one at most.

Theorem E. *Every large even number n is the sum of four odd primes (of which one may be assigned.) The asymptotic formula for the total number of representations is*

$$\bar{N}_4(n) \sim \frac{1}{3} C_4 \frac{n^3}{(\log n)^4} \prod \left(\frac{(\mathfrak{p}-1)(\mathfrak{p}^2 - 3\mathfrak{p} + 3)}{(\mathfrak{p}-2)(\mathfrak{p}^2 - 2\mathfrak{p} + 2)}\right), \tag{3. 63}$$

where $\mathfrak{p}$ is an odd prime divisor of n and

$$C_4 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi - 1)^4}\right). \tag{3. 64}$$

This is a corollary of the same two theorems. We have only to observe that the number of representations by four primes which are not all odd is plainly $O(n)$. There are evidently similar theorems for any greater value of r.

4. Remarks on 'Goldbach's Theorem'.

4. 1. Our method fails when $r = 2$. It does not fail *in principle*, for it leads to a definite result which appears to be correct; but we cannot overcome the difficulties of the proof, even if we assume that $\Theta = \frac{1}{2}$. The best upper bound that we can determine for the error is too large by (roughly) a power $n^{\frac{1}{4}}$.

The formula to which our method leads is contained in the following

Conjecture A. *Every large even number is the sum of two odd primes. The asymptotic formula for the number of representatives is*

$$N_2(n) \sim 2C_2 \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right) \tag{4. 11}$$

where $\mathfrak{p}$ *is an odd prime divisor of* n, *and*

$$C_2 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi-1)^2}\right). \tag{4. 12}$$

We add a few words as to the history of this formula, and the empirical evidence for its truth.[1]

The first definite formulation of a result of this character appears to be due to Sylvester[2], who, in a short abstract published in the *Proceedings of London Mathematical Society* in 1871, suggested that

$$N_2(n) \sim \frac{2n}{\log n} \prod \left(\frac{\varpi-2}{\varpi-1}\right), \tag{4. 13}$$

where

$$3 \leqq \varpi < \sqrt{n}, \quad \varpi \nmid n.$$

Since

$$\prod_{\varpi < \sqrt{n}} \left(\frac{\varpi-2}{\varpi-1}\right) = \prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{(\varpi-1)^2}\right) \prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{\varpi}\right) \sim C_2 \prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{\varpi}\right),$$

[1] As regards the earlier history of 'Goldbach's Theorem', see L. E. Dickson, *History of the Theory of Numbers*, vol. 1 (Washington 1919), pp. 421—425.

[2] J. J. Sylvester, 'On the partition of an even number into two primes', *Proc. London Math. Soc.*, ser. 1, vol. 4 (1871), pp. 4—6 (*Math. Papers*, vol. 2, pp. 709—711). See also 'On the Goldbach-Euler Theorem regarding prime numbers', *Nature*, vol. 55 (1896—7), pp. 196—197, 269 (*Math. Papers*, vol. 4, pp. 734—737).

We owe our knowledge of Sylvester's notes on the subject to Mr. B. M. Wilson of Trinity College, Cambridge. See, in connection with all that follows, Shah and Wilson, 1, and Hardy and Littlewood, 2.

and [1]

$$\prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{\varpi}\right) \sim \frac{2e^{-C}}{\log n}, \tag{4. 14}$$

where C is Euler's constant, (4. 13) is equivalent to

$$N_2(n) \sim 4e^{-C} C_2 \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right), \tag{4. 15}$$

and contradicts (4. 11), the two formulae differing by a factor $2e^{-C} = 1.123\ldots$ We prove in 4. 2 that (4. 11) is the only formula of the kind that can possibly be correct, so that Sylvester's formula is erroneous. But Sylvester was the first to identify the factor

$$\prod \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right), \tag{4. 16}$$

to which the *irregularities* of $N_2(n)$ are due. There is no sufficient evidence to show how he was led to his result.

A quite different formula was suggested by STÄCKEL[2] in 1896, viz.,

$$N_2(n) \sim \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p}}{\mathfrak{p}-1}\right).$$

This formula does not introduce the factor (4. 16), and does not give anything like so good an approximation to the facts; it was in any case shown to be incorrect by LANDAU[3] in 1900.

In 1915 there appeared an uncompleted essay on Goldbach's Theorem by MERLIN.[4] MERLIN does not give a complete asymptotic formula, but recognises (like Sylvester before him) the importance of the factor (4. 16).

About the same time the problem was attacked by BRUN[5]. The formula to which Brun's argument naturally leads is

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[1] Landau, p. 218.

[2] P. STÄCKEL, 'Über Goldbach's empirisches Theorem: Jede grade Zahl kann als Summe von zwei Primzahlen dargestellt werden', *Göttinger Nachrichten*, 1896, pp. 292—299.

[3] E. LANDAU, 'Über die zahlentheoretische Funktion $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz', *Göttinger Nachrichten*, 1900, pp. 177—186.

[4] J. MERLIN, 'Un travail sur les nombres premiers', *Bulletin des sciences mathématiques*, vol. 39 (1915), pp. 121—136.

[5] V. BRUN, 'Über das Goldbachsche Gesetz und die Anzahl der Primzahlpaare', *Archiv for Mathematik* (Christiania), vol. 34, part 2 (1915), no. 8, pp. 1—15. The formula (4. 18) is not actually formulated by Brun: see the discussion by Shah and Wilson, 1, and Hardy and Littlewood, 2. See also a second paper by the same author, 'Sur les nombres premiers de la forme $ap+b$', *ibid.*, part. 4 (1917), no. 14, pp. 1—9; and the postscript to this memoir.

$$N_2(n) \sim 2Hn \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right), \tag{4. 17}$$

where

$$H = \prod_{3 \leqq \varpi < \sqrt{n}} \left(1 - \frac{2}{\varpi}\right). \tag{4. 171}$$

This is easily shown to be equivalent to

$$N_2(n) \sim 8 e^{-2\gamma} C_2 \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right), \tag{4. 18}$$

and differs from (4. 11) by a factor $4e^{-2C} = 1.263\ldots$ The argument of 4. 2 will show that this formula, like Sylvester's, is incorrect.

Finally, in 1916 Stäckel[1] returned to the subject in a series of memoirs published in the *Sitzungsberichte der Heidelberger Akademie der Wissenschaften*, which we have until very recently been unable to consult. Some further remarks concerning these memoirs will be found in our final postscript.

4. 2. We proceed to justify our assertion that the formulae (4. 15) and (4. 18) cannot be correct.

Theorem F. *Suppose it to be true that*[2]

$$N_2(n) \sim A \frac{n}{(\log n)^2} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right) \tag{4. 21}$$

if

$$n = 2^{\alpha} \mathfrak{p}^{a} \mathfrak{p}'^{a'} \ldots \qquad (\alpha > 0,\ a, a', \ldots > 0),$$

and

$$N_2(n) = o\left(\frac{n}{(\log n)^2}\right) \tag{4. 22}$$

if n is odd. Then

$$A = 2C_2 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi-1)^2}\right). \tag{4. 23}$$

[1] P. Stäckel, 'Die Darstellung der geraden Zahlen als Summen von zwei Primzahlen', 8 August 1916; 'Die Lückenzahlen r-ter Stufe und die Darstellung der geraden Zahlen als Summen und Differenzen ungerader Primzahlen', I. Teil 27 Dezember 1917, II. Teil 19 Januar 1918, III. Teil 19 Juli 1918.

[2] Throughout 4.2 A is *the same* constant.

Write

$$(4.\ 24)\qquad \Omega(n)=An\prod_{\mathfrak{p}}\left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right)\ (n\ \textit{even}),\ \Omega(n)=0\ (n\ \textit{odd}).$$

Then, by (4. 21) and Theorem C, now valid in virtue of (4. 21),

$$(4.\ 25)\qquad \nu_2(n)=\sum_{\varpi+\varpi'=n}\log\varpi\log\varpi'\sim\Omega(n),$$

it being understood that, when n is odd, this formula means

$$\nu_2(n)=o(n).$$

Further let

$$f(s)=\sum\frac{\Omega(n)}{n^s}=\sum\frac{\Omega(n)}{n^{1+u}},$$

these series being absolutely convergent if $\Re(s)>2$, $\Re(u)>1$. Then

$$(4.\ 26)\qquad f(s)=A\sum_{n\equiv 0\ (\mathrm{mod.}\ 2)}n^{-u}\prod_{\mathfrak{p}}\left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right)$$

$$=A\sum_{a>0}2^{-au}\,\mathfrak{p}^{-au}\,\mathfrak{p}'^{-a'u}\cdots\frac{(\mathfrak{p}-1)(\mathfrak{p}'-1)\ldots}{(\mathfrak{p}-2)(\mathfrak{p}'-2)\ldots}$$

$$=\frac{2^{-u}A}{1-2^{-u}}\prod_{\varpi=3}^{\infty}\left(1+\frac{\varpi-1}{\varpi-2}\,\frac{\varpi^{-u}}{1-\varpi^{-u}}\right)=\frac{2^{-u}A}{1-2^{-u}}\xi(u),$$

say. Suppose now that $u\to 1$, and let

$$\eta(u)=\prod_{\varpi=3}^{\infty}\left(1+\frac{\varpi^{-u}}{1-\varpi^{-u}}\right)=\prod_{\varpi=3}^{\infty}\left(\frac{1}{1-\varpi^{-u}}\right)=(1-2^{-u})\zeta(u).$$

Then

$$\frac{\xi(u)}{\eta(u)}=\prod\left(\left(1+\frac{\varpi-1}{\varpi-2}\,\frac{\varpi^{-u}}{1-\varpi^{-u}}\right)\Big/\left(1+\frac{\varpi^{-u}}{1-\varpi^{-u}}\right)\right)$$

$$\to\prod\left(\left(1+\frac{1}{\varpi-2}\right)\Big/\left(1+\frac{1}{\varpi-1}\right)\right)=\prod\left(\frac{(\varpi-1)^2}{\varpi(\varpi-2)}\right)$$

$$=\prod\left(\frac{(\varpi-1)^2}{(\varpi-1)^2-1}\right)=\frac{1}{C_2}.$$

Hence

$$(4.\ 27)\qquad f(s)\sim A\xi(u)\sim\frac{A}{C_2}\eta(u)\sim\frac{A}{2C_2}\zeta(u)\sim\frac{A}{2C_2(u-1)}=\frac{A}{2C_2(s-2)}.$$

On the other hand, when $x \to 1$,

$$\sum \nu_2(n) x^n \sim \left(\sum \log \varpi\, x^{\varpi}\right)^2 \sim \frac{1}{(1-x)^2},$$

and so[1]

$$\nu_2(1) + \nu_2(2) + \cdots + \nu_2(n) \sim \frac{1}{2} n^2. \tag{4. 28}$$

It is an elementary deduction[2] that

$$g(s) = \sum \frac{\nu_2(n)}{n^s} \sim \sum \frac{1}{n^{s-1}} \sim \frac{1}{s-2}$$

when $s \to 2$; and hence[2] that (under the hypotheses (4. 21) and (4. 22))

$$f(s) \sim \frac{1}{s-2}. \tag{4. 29}$$

Comparing (4. 27) and (4. 29), we obtain the result of the theorem.

4. 3. The fact that both Sylvester's and Brun's formulae contain an erroneous constant factor, and that this factor is in each case a simple function of the number e^{-C}, is not so remarkable as it may seem.

In the first place we observe that any formula in the theory of primes, *deduced from considerations of probability,* is likely to be erroneous in just this way. Consider, for example, the problem '*what is the chance that a large number n should be prime?*' We know that the answer is that the chance is approximately $\frac{1}{\log n}$.

Now the chance that n should not be divisible by any prime less than a *fixed* number x is asymptotically equivalent to

$$\prod_{\varpi < x} \left(1 - \frac{1}{\varpi}\right);$$

[1] We here use Theorem 8 of our paper 'Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive', *Proc. London Math. Soc.,* ser. 2, vol. 13, pp. 174—192. This is the quickest proof, but by no means the most elementary. The formula (4. 28) is equivalent to the formula

$$\sum_1^n N_2(m) \sim \frac{n^2}{2(\log n)^2}$$

used by Landau in his note quoted on p. 33.

[2] For general theorems including those used here as very special cases, see K. Knopp, 'Divergenzcharactere gewisser Dirichlet'scher Reihen', *Acta Mathematica,* vol. 34, 1909, pp. 165—204 (*e. g.* Satz III, p. 176).

and it would be natural to infer[1] that the chance required is asymptotically equivalent to

$$\prod_{\varpi < \sqrt{n}} \left(1 - \frac{1}{\varpi}\right).$$

But[2]

$$\prod_{\varpi > \sqrt{n}} \left(1 - \frac{1}{\varpi}\right) \sim \frac{2e^{-C}}{\log n};$$

and our inference is incorrect, to the extent of a factor $2e^{-C}$.

It is true that Brun's argument is not stated in terms of probabilities[3], but it involves a heuristic passage to the limit of exactly the same character as that in the argument we have just quoted. Brun finds first (by an ingenious use of the 'sieve of Eratosthenes') an asymptotic formula for the number of representations of n as the sum of two numbers, *neither divisible by any fixed number of primes*. This formula is correct and the proof valid. So is the first stage in the argument above; it rests on an enumeration of cases, and all reference to 'probability'[4] is easily eliminated. It is in the passage to the limit that error is introduced, and the nature of the error is the same in one case as in the other.

4. 4. SHAH and WILSON have tested Conjecture A extensively by comparison with the empirical data collected by CANTOR, AUBRY, HAUSSNER, and RIPERT. We reprint their table of results; but some preliminary remarks are required. In the first place it is essential, in a numerical test, to work with a formula $N_2(n)$, such as (4. 11), and not with one for $\nu_2(n)$, such as (4. 25). In our analysis, on the other hand, it is $\nu_2(n)$ which presents itself first, and the formula for $N_2(n)$ is secondary. In order to derive the asymptotic formula for $N_2(n)$, we write

$$\nu_2(n) = \sum_{\varpi + \varpi' = n} \log \varpi \log \varpi' \sim (\log n)^2 N_2(n).$$

The factor $(\log n)^2$ is certainly in error to an order $\log n$, and it is more natural[5] to replace $\nu_2(n)$ by

$$((\log n)^2 - 2 \log n + \cdots) N_2(n).$$

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[1] One might well replace $\varpi < \sqrt{n}$ by $\varpi < n$, in which case we should obtain a probability half as large. This remark is in itself enough to show the unsatisfactory character of the argument.

[2] Landau, p. 218.

[3] Whether Sylvester's argument was or was not we have no direct means of judging.

[4] *Probability* is not a notion of pure mathematics, but of philosophy or physics.

[5] Compare Shah and Wilson, *l. c.*, p. 238. The same conclusion may be arrived at in other ways.

For the *asymptotic* formula, naturally, it is indifferent which substitution we adopt. But, for purposes of *verification within the limits of calculation*, it is by no means indifferent, for the term in $\log n$ is by no means of negligible importance; and it will be found that is makes a vital difference in the plausibility of the results. Bearing these considerations in mind, Shah and Wilson worked, not with the formula (4. 11), but with the modified formula

$$N_2(n) \backsim \varrho(n) = 2\, C_2 \frac{n}{(\log n)^2 - 2\log n} \prod_{\mathfrak{p}} \left(\frac{\mathfrak{p}-1}{\mathfrak{p}-2}\right).$$

Failure to make allowances of this kind has been responsible for a good deal of misapprehension in the past. Thus (as is pointed out by Shah and Wilson[1]) Sylvester's erroneous formula gives, for values of n within the limits of Table I, decidedly *better* results than those obtained from the *unmodified* formula (4. 11).

There is another point of less importance. The function which presents itself most naturally in our analysis is not

$$f(x) = \sum \log \varpi\, x^{\varpi}$$

but

$$g(x) = \sum \Lambda(n)\, x^n = \sum_{\varpi,\, l} \log \varpi\, x^{\varpi^l}.$$

The corresponding numerical functions are not $\nu_2(n)$ and $N_2(n)$, but

$$g_2(n) = \sum_{m+m'=n} \Lambda(m)\, \Lambda(m'),\quad Q_2(n) = \sum_{\varpi^l + \varpi'^{l'} = n} 1$$

(so that $Q_2(n)$ is the number of decompositions of n *into two primes or two powers of primes*). Here again, $N_2(n)$ and $Q_2(n)$ are asymptotically equivalent; the difference between them is indeed of lower order than errors which we are neglecting in any case; but there is something to be said for taking the latter as the basis for comparison, when (as is inevitable) the values of n are not very large.

In the table the decompositions into primes, and powers of primes, are reckoned separately; but it is the total which is compared with $\varrho(n)$. The value of the constant $2\,C_2$ is 1.3203. It will be seen that the correspondence between the calculated and actual values is excellent.

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[1] *l. c.*, p. 242.

Table I.

n	$Q_2(n)$	$\rho(n)$	$Q_2(n):\rho(n)$
$30 = 2.3.5$	$6 + 4 = 10$	22	0.45
$32 = 2^5$	$4 + 7 = 11$	8	1.38
$34 = 2.17$	$7 + 6 = 13$	9	1.44
$36 = 2^2.3^2$	$8 + 8 = 16$	17	0.94
$210 = 2.3.5.7$	$42 + 0 = 42$	49	0.85
$214 = 2.107$	$17 + 0 = 17$	16	1.07
$216 = 2^3.3^3$	$28 + 0 = 28$	32	0.88
$256 = 2^8$	$16 + 3 = 19$	17	1.10
$2{,}048 = 2^{11}$	$50 + 17 = 67$	63	1.06
$2{,}250 = 2.3^2.5^3$	$174 + 26 = 200$	179	1.11
$2{,}304 = 2^8.3^2$	$134 + 8 = 142$	136	1.04
$2{,}306 = 2.1153$	$67 + 20 = 87$	69	1.26
$2{,}310 = 2.3.5.7.11$	$228 + 16 = 244$	244	1.00
$3{,}888 = 2^4.3^5$	$186 + 24 = 210$	197	1.06
$3{,}898 = 2.1949$	$99 + 6 = 105$	99	1.06
$3{,}990 = 2.3.5.7.19$	$328 + 20 = 348$	342	1.02
$4{,}096 = 2^{12}$	$104 + 5 = 109$	102	1.06
$4{,}996 = 2^2.1249$	$124 + 16 = 140$	119	1.18
$4{,}998 = 2.3.7^2.17$	$228 + 20 = 308$	305	1.01
$5{,}000 = 2^3.5^4$	$150 + 26 = 176$	157	1 12
$8{,}190 = 2.3^2.5.7.13$	$578 + 26 = 604$	597	1.01
$8{,}192 = 2^{13}$	$150 + 32 = 182$	171	1.06
$8{,}194 = 2.17.241$	$192 + 10 = 202$	219	0.92
$10{,}008 = 2^3.3^2.139$	$388 + 30 = 418$	396	1.06
$10{,}010 = 2.5.7.11.13$	$384 + 36 = 420$	384	1.09
$10{,}014 = 2.3.1669$	$408 + 8 = 416$	396	1.05
$30{,}030 = 2.3.5.7.11.13$	$1{,}800 + 54 = 1854$	1795	1.03
$36{,}960 = 2^5.3.5.7.11$	$1{,}956 + 38 = 1994$	1937	1.03
$39{,}270 = 2.3.5.7.11.17$	$2{,}152 + 36 = 2188$	2213	0.99
$41{,}580 = 2^2.3^3.5.7.11$	$2{,}140 + 44 = 2184$	2125	1.03
$50{,}026 = 2.25013$	$702 + 8 = 710$	692	1.03
$50{,}144 = 2^5.1567$	$607 + 32 = 706$	694	1.02
$170{,}166 = 2.3.79.359$	$3{,}734 + 46 = 3780$	3762	1.00
$170{,}170 = 2.5.7.11.13.17$	$3{,}784 + 8 = 3792$	3841	0.99
$170{,}172 = 2^2.3^2.29.163$	$3{,}732 + 48 = 3780$	3866	0.98

5. Other problems.

5. 1. This last section is frankly conjectural, and is not to be judged by the same standards as §§ 1—3.

The problems to which we have applied our method may be divided roughly into three classes. The typical problem of the first class is Waring's Problem. Our solution of this problem is not yet as conclusive as we should desire, and we have not exhausted the possibilities of our method, even when allowance is made for still unpublished work; we cannot at present prove, for example, that every large number is the sum of 7 cubes or 16 biquadrates. But our proofs, so far as they go, are complete.

The typical problem of the second class is that considered in §§ 1—3. The arguments by which we prove our results are rigorous, but the results depend upon the unproved hypothesis R.

There is a third class of problems, of which Goldbach's Problem is typical. Here we are unable (with or without Hypothesis R) to offer anything approaching to a rigorous proof. What our method yields is a *formula*, and one which seems to stand the test of comparison with the facts. In this concluding section we propose to state a number of further formulae of the same kind. Our apology for doing so must be (1) that no similar formulae have been suggested before, and that the process by which they are deduced has at any rate a certain algebraical interest, and (2) that it seems to us very desirable that (in default of proof) the formulae should be checked, and that we hope that some of the many mathematicians interested in the computative side of the theory of numbers may find them worthy of their attention.

Conjugate problems: the problem of prime-pairs.

5. 2. The problems to which our method is applicable group themselves in pairs in an interesting manner which will be most easily understood by an example.

In Goldbach's Problem we have to study the integral

$$\frac{1}{2\pi i}\int (f(x))^2 \frac{dx}{x^{n+1}},$$

where

$$f(x) = \sum \log \varpi x^{\varpi}, \quad x = Re^{i\varphi} = e^{-\frac{1}{n}+i\varphi},$$

or

$$(5.\ 21) \qquad \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\psi})\, f(Re^{i\psi})\, e^{1-ni\psi}\, d\psi.$$

The formal transformations of this integral to which we are led may be stated shortly as follows. We divide up the range of integration into a large number of pieces by means of the Farey arcs $\xi_{p,q}$, ψ varying over the interval $\left(\frac{2p\pi}{q}-\theta'_{p,q},\ \frac{2p\pi}{q}+\theta_{p,q}\right)$ when x varies over $\xi_{p,q}$. We then replace $f(x)$ by the appropriate approximation

$$\frac{\mu(q)}{\varphi(q)}\,\frac{1}{\log\left(\frac{e_q(p)}{x}\right)} = \frac{\mu(q)}{\varphi(q)}\,\frac{1}{\frac{1}{n}-i\left(\psi-\frac{2p\pi}{q}\right)},$$

$\psi-\frac{2p\pi}{q}$ by u, and the integral

$$(5.\ 22) \qquad e_q(-np)\int_{-\theta'_{p,q}}^{\theta_{p,q}} \frac{e^{1-niu}}{\left(\frac{1}{n}-iu\right)^2}\,du$$

by

$$(5.\ 23) \qquad n\, e_q(-np)\int_{-\infty}^{\infty}\frac{e^{1-iw}}{(1-iw)^2}\,dw = 2\pi n\, e_q(-np).$$

We are thus led to the singular series S_2.

Now suppose that, instead of the integral (5. 21), we consider the integral

$$(5.\ 24) \qquad J(R) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\psi})\, f(Re^{-i\psi})\, e^{ki\psi}\, d\psi,$$

where now k is a *fixed* positive integer. Instead of (5. 22), we have now

$$e_q(kp)\int_{-\theta'_{p,q}}^{\theta_{p,q}}\frac{e^{kiu}}{\left(\frac{1}{n}-iu\right)\left(\frac{1}{n}+iu\right)}\,du \sim e_q(kp)\int_{-\infty}^{\infty}\frac{du}{\frac{1}{n^2}+u^2} = \pi n\, e_q(kp).$$

We are thus led to suppose that

$$J(R) \sim \frac{1}{2} n \sum \left(\frac{\mu(q)}{\varphi(q)}\right)^2 e_q(kp) \tag{5.25}$$

when $R = e^{-\frac{1}{n}}$, $n \to \infty$.

The series here (which we call for the moment S'_2) is the singular series S_2 with $-k$ in the place of n. On the other hand

$$J(R) = \frac{1}{2\pi}\int_0^{2\pi} \sum \log \varpi R^{\varpi} e^{\varpi i \psi} \,.\, \sum \log \varpi R^{\varpi} e^{-\varpi i \psi} \,.\, e^{k i \psi} d\psi = R^k \sum a_{\varpi} R^{2\varpi},$$

where

$$a_{\varpi} = \log \varpi \log (\varpi + k)$$

if both ϖ and $\varpi + k$ are prime, and $a_{\varpi} = 0$ otherwise. Hence we obtain

$$\sum a_{\varpi} R^{2\varpi} \sim \frac{1}{1 - R^2} S'_2.$$

Here $R = e^{-\frac{1}{n}}$, but the result is easily extended to the case of continuous approach to the limit 1, and we deduce[1]

$$\sum_{\varpi < n} a_{\varpi} \sim n S'_2. \tag{5.26}$$

And from this it would be an easy deduction that the number of prime pairs differing by k, and less than a large number n, is asymptotically equivalent to

$$\frac{n}{(\log n)^2} S'_2.$$

We are thus led to the following

Conjecture B. *There are infinitely many prime pairs*

$$\varpi, \varpi' = \varpi + k,$$

for every even k. If $P_k(n)$ is the number of pairs less than n, then

$$P_k(n) \sim 2 C_2 \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p} - 1}{\mathfrak{p} - 2}\right),$$

where C_2 is the constant of § 4 and $\mathfrak{p}$ is an odd prime divisor of k.

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[1] We appeal again here to the Tauberian theorem referred to at the end of 4. 2 (f. n. 1). This time, of course, there is no question of an alternative argument.

[2] Note that $S'_2 = 0$ if k is odd, as it should be.

It will be observed that the analysis connected with Conjectures A and B, which deal respectively with the equations

$$n = \varpi + \varpi', \quad \varpi' = \varpi + k,$$

is substantially the same. It is pairs of problems connected in this manner that we call *conjugate* problems.

Numerical verifications.

5. 31. For $k = 2, 4, 6$ we obtain

(5. 311) $$P_2(n) \sim \frac{2C_2 n}{(\log n)^2},$$

(5. 312) $$P_4(n) \sim \frac{2C_2 n}{(\log n)^2},$$

(5. 313) $$P_6(n) \sim \frac{4C_2 n}{(\log n)^2},$$

Thus there should be approximately equal numbers of prime-pairs differing by 2 *and by* 4, *but about twice as many differing by* 6. The actual numbers of pairs, below the limits

100, 500, 1000, 2000, 3000, 4000, 5000

are

9	24	35	61	81	103	125
9	26	41	63	86	107	121
16	47	73	125	168	201	241

The correspondence is as accurate as could be desired.

5. 32. The first formula (5. 311) has been verified much more systematically. A little caution has to be exercised in undertaking such a verification. The formula (5. 26) is equivalent, when $k = 2$, to

(5. 321) $$\sum_{m<n} \Lambda(m)\,\Lambda(m+2) \sim 2C_2 n;$$

and, when we pass from this formula to one for the number of prime-pairs, the formula which arises most naturally is not (5. 311) but[1]

[1] This formula follows from (5. 321) in exactly the same way that

$$\pi(x) \sim Li\,x$$

follows from

$$\sum_{m<x} \Lambda(m) \sim x.$$

$$P_2(n) \sim 2\,C_2 \int^n \frac{dx}{(\log x)^2}; \tag{5. 322}$$

indeed it is not unreasonable to expect this approximation to be a really good one, and much better than the formulae of 4. 4. The formula (5. 322) is naturally equivalent to (5. 311). But

$$\int^n \frac{dx}{(\log x)^2} = \frac{n}{(\log n)^2}\left(1 + \frac{2\,!}{\log n} + \frac{3\,!}{(\log n)^2} + \cdots\right),[1]$$

and the second factor on the right hand side is (for such values of n as we have to consider) far from negligible. It is for this reason that Brun, when he attempted to deduce a value of the constant in (5. 311) from the statistical results, was led to a value seriously in error.

We therefore take the formula (5. 322) as our basis for comparison, choosing the lower limit to be 2. For our statistics as to the actual number of prime-pairs we are indebted to (*a*) a count up to 100,000 made by Glaisher in 1878[2] and (*b*) a much more extensive count made for us recently by Mrs. G. A. Streatfeild. The results obtained by Mrs. Streatfeild are as follows.

n	$P_2(n)$	$2\,C_2 \int_2^n \frac{dx}{(\log x)^2}$	Ratio
100000	1224	1246.3	1.018
200000	2159	2179.5	1.009
300000	2992	3035.4	1.015
400000	3801	3846.1	1.012
500000	4562	4625.6	1.014
600000	5328	5381.5	1.010
700000	6058	6118.7	1 010
800000	6763	6840.2	1.011
900000	7469	7548.6	1.011
1000000	8164	8245.6	1.010

[1] The series is of course divergent, and must be regarded as closed after a finite number of terms, with an error term of lower order than the last term retained.

[2] J. W. L. Glaisher, 'An enumeration of prime-pairs', *Messenger of Mathematics*, vol. 8 (1878), pp. 28—33. Glaisher counts (1, 3) as a pair, so that his figure exceeds ours by 1.

5. 33. Similar reasoning leads us to the following more general results.

Conjecture C. *If a, b are fixed positive integers and $(a, b) = 1$, and $N(n)$ is the number of representations of n in the form*

$$n = a\varpi + b\varpi',$$

then

$$N(n) = o\left(\frac{n}{(\log n)^2}\right)$$

unless $(n, a) = 1$, $(n, b) = 1$, and one and only one of n, a, b is even.[1] *But if these conditions are satisfied then*

$$N(n) \sim \frac{2C_2}{ab} \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p} - 1}{\mathfrak{p} - 2}\right),$$

where C_2 is the constant of § 4, and the product extends over all odd primes $\mathfrak{p}$ which divide n, a, or b.

Conjecture D. *If $(a, b) = 1$ and $P(n)$ is the number of pairs of solutions of*

$$a\varpi' - b\varpi = k$$

such that $\varpi' < n$, then

$$P(n) = o\left(\frac{n}{(\log n)^2}\right)$$

unless $(k, a) = 1$, $(k, b) = 1$, and just one of k, a, b is even. But if these conditions are satisfied then

$$P(n) \sim \frac{2C_2}{a} \frac{n}{(\log n)^2} \prod \left(\frac{\mathfrak{p} - 1}{\mathfrak{p} - 2}\right),$$

where $\mathfrak{p}$ is an odd prime factor of k, a, or b.

It should be clear that the theorems proved in §§ 1—3 must be capable of a similar generalisation. Thus we might investigate the number of representations of n in the form

$$n = a\varpi + b\varpi' + c\varpi'';$$

and here proof would be possible, though only with the assumption of hypothesis R. We have not performed the actual calculations.

[1] This is trivial. If n and a have a common factor, it divides $b\varpi'$, and must therefore be ϖ', which is thus restricted to a finite number of values. If n, a, b are all odd, ϖ or ϖ' must be 2.

Primes of the forms m^2+1, am^2+bm+c.

5.41. Of the four problems mentioned by Landau in his Cambridge address, two were Goldbach's problem and the problem of the prime-pairs. The third was that of *the existence of an infinity of primes of the form* m^2+1.[1]

Our method is applicable to this problem also. We have now to consider the integral

$$J(R)=\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\psi})\,\vartheta(Re^{-i\psi})\,e^{-i\psi}\,d\psi,$$

where $f(x)$ is the same function as before and

$$\vartheta(x)=\sum_{m=1}^{\infty} x^{m^2}.$$

The approximation for $\vartheta(\bar{x})=\vartheta(Re^{-i\psi})$ on $\xi_{p,q}$ is

$$\vartheta(Re^{-i\psi})\backsim\frac{1}{2}\sqrt{\pi}\,\frac{\bar{S}_{p,q}}{q}\left(\frac{1}{n}+i\left(\psi-\frac{2p\pi}{q}\right)\right)^{-\frac{1}{2}},$$

where

$$S_{p,q}=\sum_{h=1}^{q} e_q(h^2p)$$

and $\bar{S}_{p,q}$ is the conjugate of $S_{p,q}$; and we find, as an approximation for $J(R)$,

$$\frac{1}{4\sqrt{\pi}}\sum_{p,q}\frac{\mu(q)}{q\,\varphi(q)}\bar{S}_{p,q}\,e_q(-p)\int_{-\theta'_{p,q}}^{\theta_{p,q}}\frac{e^{-iu}\,du}{\left(\frac{1}{n}-iu\right)\sqrt{\frac{1}{n}+iu}}.$$

We replace the ntegral here by

$$\int_{-\infty}^{\infty}\frac{du}{\left(\frac{1}{n}-iu\right)\sqrt{\frac{1}{n}+iu}}=\pi\sqrt{2n};$$

[1] The fourth was that of the existence of a prime between n^2 and $(n+1)^2$ for every $n>0$. The problem of primes am^2+bm+c must not be confused with the much simpler (though still difficult) problem of primes included in the definite quadratic form $ax^2+bxy+cy^2$ in two independent variables. This problem, of course, was solved in the classical researches of de la Vallée Poussin. Our method naturally leads to de la Vallée Poussin's results, and the formal verification of them in this manner is not without interest. Here, however, our method is plainly not the right one, and could lead at best to a proof encumbered with an unnecessary hypothesis and far more difficult than the accepted proof.

and we are led to the formula

(5. 411)
$$J(R) \sim \frac{1}{4}\sqrt{2\pi n}\,S,$$

where S is the singular series

(5. 412)
$$S = \sum_{p,q} \frac{\mu(q)}{q\,\varphi(q)} \bar{S}_{p,q}\, e_q(-p).$$

Repeating the arguments of § 5. 2, we conclude that *the number* $P(n)$ *of primes of the form* m^2+1 *and less than* n *is given asymptotically by*

(5. 413)
$$P(n) \sim \frac{\sqrt{n}}{\log n} S.$$

5. 42. The singular series (5. 412) may be summed by the method of § 3. 2. Writing

$$S = \sum A_q = 1 + A_2 + A_3 + \cdots,$$

there is no difficulty in proving that $A_{qq'} = A_q A_{q'}$ if $(q, q') = 1$. Hence we write[1]

$$S = \prod \chi_\varpi,$$

where

$$\chi_\varpi = 1 + A_\varpi + A_{\varpi^2} + \cdots = 1 + A_\varpi.$$

If $\varpi = 2$, $A_\varpi = 0$, $\chi_\varpi = 1$. If $\varpi > 2$,[2]

$$S_{p,\varpi} = \left(\frac{p}{\varpi}\right) i^{\frac{1}{4}(\varpi-1)^2} \sqrt{\varpi},\ \bar{S}_{p,\varpi} = \left(\frac{p}{\varpi}\right) i^{-\frac{1}{4}(\varpi-1)^2} \sqrt{\varpi},$$

and

$$A_\varpi = -\frac{1}{(\varpi-1)\sqrt{\varpi}} i^{-\frac{1}{4}(\varpi-1)^2} \sum_{p=1}^{\varpi-1} \left(\frac{p}{\varpi}\right) e_\varpi(-p)$$

$$= -\frac{(-1)^{\frac{1}{2}(\varpi-1)}}{\varpi-1} = -\frac{1}{\varpi-1}\left(\frac{-1}{\varpi}\right).$$

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[1] Even this is a formal process, for (5. 412) is not absolutely convergent.

[2] See Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie*, ed. 4 (1894), pp. 293 *et seq.*

Thus finally we are led to

Conjecture E. *There are infinitely many primes of the form m^2+1. The number $P(n)$ of such primes less than n is given asymptotically by*

$$P(n) \sim C\frac{\sqrt{n}}{\log n},$$

where

$$C = \prod_{\varpi=3}^{\infty}\left(1 - \frac{1}{\varpi-1}\left(\frac{-1}{\varpi}\right)\right).$$

5. 43. Generalising the analysis of §§ 5. 41, 5. 42, we arrive at

Conjecture F. *Suppose that a, b, c are integers and a is positive; that $(a, b, c) = 1$; that $a+b$ and c are not both even; and that $D = b^2 - 4ac$ is not a square. Then there are infinitely many primes of the form am^2+bm+c. The number $P(n)$ of such primes less than n is given asymptotically by*

$$P(n) \sim \frac{\varepsilon C}{\sqrt{a}}\frac{\sqrt{n}}{\log n}\prod\left(\frac{\mathfrak{p}}{\mathfrak{p}-1}\right),$$

where $\mathfrak{p}$ is a common odd prime divisor of a and b, ε is 1 if $a+b$ is odd and 2 if $a+b$ is even, and

$$C = \prod_{\varpi \geqq 3,\, \varpi \nmid a}\left(1 - \frac{1}{\varpi-1}\left(\frac{D}{\varpi}\right)\right). \tag{5. 4321}$$

It is instructive here to observe the genesis of the exceptional cases. If $(a, b, c) = d > 1$, there can obviously be at most one prime of the form required. In this case χ_ϖ vanishes for every ϖ for which $\varpi \mid d$. If $a+b$ and c are both even, am^2+bm+c is always even: in this case χ_2 vanishes. If $D = k^2$, then

$$4a(am^2+bm+c) = (2am+b)^2 - k^2,$$

and

$$4a\varpi = (2am+b)^2 - k^2$$

involves $2am+b \pm k \mid 4a$, which can be satisfied by at most a finite number of values of m. In this case no factor χ_ϖ vanishes, but the product (5. 4321) diverges to zero.

5. 44. The conjugate problem is that of the expression of a number n in the form

$$n = am^2 + bm + \varpi. \tag{5. 441}$$

Here we are led to

Conjecture G. *Suppose that a and b are integers, and $a > 0$, and let $N(n)$ be the number of representations of n in the form $am^2 + bm + \varpi$. Then if n, a, b have a common factor, or if n and $a+b$ are both even, or if $b^2 + 4an$ is a square, then*

$$N(n) = o\left(\frac{\sqrt{n}}{\log n}\right). \tag{5.442}$$

In all other cases

$$N(n) \sim \frac{\varepsilon}{\sqrt{a}} \frac{\sqrt{n}}{\log n} \prod \left(\frac{\mathfrak{p}}{\mathfrak{p}-1}\right) \prod_{\varpi \geqq 3,\, \varpi \nmid a} \left(1 - \frac{1}{\varpi - 1}\left(\frac{b^2 + 4an}{\varpi}\right)\right), \tag{5.443}$$

where $\mathfrak{p}$ is a common odd prime divisor of a and b, and ε is 1 if $a+b$ is odd and 2 if $a+b$ is even.

The following are particularly interesting special cases of this proposition.

Conjecture H. *Every large number n is either a square or the sum of a prime and a square. The number $N(n)$ of representations is given asymptotically by*

$$N(n) \sim \frac{\sqrt{n}}{\log n} \prod_{\varpi = 3}^{\infty} \left(1 - \frac{1}{\varpi - 1}\left(\frac{n}{\varpi}\right)\right). \tag{5.444}$$

There does not seem to be anything in mathematical literature corresponding to this conjecture: probably because the idea that *every* number is a square, or the sum of a prime and a square, is refuted (even if 1 is counted as a prime) by such immediate examples as 34 and 58. But the problem of the representation of an *odd* number in the form $\varpi + 2m^2$ has received some attention; and it has been verified that the only odd numbers less than 9000, and not of the form desired, are 5777 and 5993.[1]

Conjecture I. *Every large odd number n is the sum of a prime and the double of a square. The number $N(n)$ of representations is given asymptotically by*

$$N(n) \sim \frac{\sqrt{2n}}{\log n} \prod_{\varpi = 3}^{\infty} \left(1 - \frac{1}{\varpi - 1}\left(\frac{2n}{\varpi}\right)\right). \tag{5.445}$$

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[1] By Stern and his pupils in 1856. See Dickson's *History* (referred to on p. 32) p. 424. The tables constructed by Stern were preserved in the library of Hurwitz, and have been very kindly placed at our disposal by Mr. G. Pólya. These manuscripts also contain a table of decompositions of primes $q = 4m + 3$ into sums $q = p + 2p'$, where p and p' are primes of the form $4m + 1$, extending as far as $q = 20983$. The conjecture that such a decomposition is always possible (1 being counted as a prime) was made by Lagrange in 1775 (see Dickson, *l. c.*, p. 424).

5.45 We may equally work out the number of representations of n as the sum of a prime and any number of squares. Thus, for example, we find

Conjecture J. *The numbers of representations of n in the forms*

$$n = \varpi + m_1^2 + m_2^2, \quad n = \varpi + m_1^2 + m_2^2 + m_3^2 + m_4^2,$$

are given asymptotically by the formulae

$$N(n) \sim C\pi n \prod_{\mathfrak{p} \equiv 1 \pmod 4} \left(\frac{(\mathfrak{p}-1)^2}{\mathfrak{p}^2 - \mathfrak{p} + 1}\right) \prod_{\mathfrak{p} \equiv 3 \pmod 4} \left(\frac{\mathfrak{p}^2 - 1}{\mathfrak{p}^2 - \mathfrak{p} - 1}\right), \tag{5.451}$$

where

$$C = \sum_{\varpi = 3}^{\infty} \left(1 + \frac{1}{\varpi(\varpi - 1)}\left(\frac{-1}{\varpi}\right)\right); \tag{5.4511}$$

and

$$N(n) \sim \frac{1}{2} C\pi^2 n^2 \prod \left(\frac{(\mathfrak{p}-1)^2(\mathfrak{p}+1)}{\mathfrak{p}^3 - \mathfrak{p}^2 + 1}\right), \tag{5.452}$$

where

$$C = \prod_{\varpi = 3}^{\infty} \left(1 + \frac{1}{\varpi^2(\varpi - 1)}\right), \tag{5.4521}$$

Here $\mathfrak{p}$ is an odd prime divisor of n, and representations which differ only in the sign or order of the numbers $m_1, m_2, \ldots$ are counted as distinct.

The last pair of formulae should be capable of rigorous proof.

Problems with cubes.

5.5. The corresponding problems with cubes have, so far as we are aware, never been formulated. The problem which suggests itself first is that of the existence of an infinity of primes of the form $m^3 + 2$ or, more generally, $m^3 + k$, where k is any number other than a (positive or negative) cube.

Here again our method may be used, but the algebraical transformations, depending, as obviously they must, on the theory of cubic residuacity, are naturally a little more complex. As there is in any case no question of proof, we content ourselves with stating a few of the results which are suggested.

Conjecture K. *If k is any fixed number other than a (positive or negative) cube, then there are infinitely many primes of the form $m^3 + k$. The number $P(n)$ of such primes less than n is given asymptotically by*

$$P(n) \sim \frac{n^{\frac{1}{3}}}{\log n} \prod_{\varpi} \left(1 - \frac{2}{\varpi - 1}(-k)_{\varpi}\right), \tag{5.51}$$

where

$$\varpi \equiv 1 \pmod{3}, \quad \varpi \nmid k,$$

and $(-k)_\varpi$ *is equal to* 1 *or to* $-\frac{1}{2}$ *according as* $-k$ *is or is not a cubic residue of* ϖ.

Conjecture L. *Every large number* n *is either a cube or the sum of a prime and a (positive) cube. The number* $N(n)$ *of representations is given asymptotically by*

$$N(n) \sim \frac{n^{\frac{1}{3}}}{\log n} \prod_{\varpi} \left(1 - \frac{2}{\varpi - 1} (n)_\varpi\right),$$

the range of values of ϖ *being defined as in* K.

Conjecture M. *If* k *is any fixed number other than zero, there are infinitely many primes of the form* $l^3 + m^3 + k$, *where* l *and* m *are both positive. The number* $P(n)$ *of such primes less than* n, *every prime being counted multiply according to its number of representations, is given asymptotically by*

$$P(n) \sim \frac{\left(\Gamma\left(\frac{4}{3}\right)\right)^2}{\Gamma\left(\frac{5}{3}\right)} \frac{n^{\frac{2}{3}}}{\log n} \prod_{\mathfrak{p}} \left(1 - \frac{2}{\mathfrak{p}}\right) \prod_{\varpi} (1 + A_\varpi),$$

where $\mathfrak{p}$ *and* ϖ *are odd primes of the form* $3r + 1$, $\mathfrak{p} \mid k$, $\varpi \nmid k$, *and*

$$A_\varpi = -\frac{A - 2}{\varpi(\varpi - 1)}$$

if $-k$ *is a cubic residue of* ϖ,

$$A_\varpi = \frac{\frac{1}{2}A \pm \frac{9}{2}B - 2}{\varpi(\varpi - 1)}$$

in the contrary case. The positive sign is to be chosen if

$$\left(\frac{-k}{\omega}\right)_3 = e^{\frac{2}{3}\pi i} = \varrho,$$

$\omega = a + b\varrho$ *being that complex prime factor of* ϖ *for which* $a \equiv -1$, $b \equiv 0 \pmod{3}$; *the negative in the contrary event. And* A *and* B *are defined by*

$$A = 2a - b, \quad 3B = b, \quad 4\varpi = A^2 + 27B^2.$$

In particular, when $k = 1$, the number of primes $l^3 + m^3 + 1$ is given asymptotically by

$$P(n) \sim \frac{\left(\Gamma\left(\frac{4}{3}\right)\right)^2}{\Gamma\left(\frac{5}{3}\right)} \frac{n^{\frac{2}{3}}}{\log n} \prod_{\varpi} \left(1 - \frac{A-2}{\varpi(\varpi - 1)}\right),$$

primes susceptible of multiple representation being counted multiply.

Conjecture N. *There are infinitely many primes of the form $k^3 + l^3 + m^3$, where k, l, m are all positive. The number $P(n)$ of such primes less than n, primes susceptible of multiple representation being counted multiply, is given asymptotically by*

$$P(n) \sim \left(\Gamma\left(\frac{4}{3}\right)\right)^3 \frac{n}{\log n} \prod_{\varpi} \left(1 - \frac{A}{\varpi^2}\right),$$

where ϖ is a prime of the form $3m + 1$, and A has the meaning explained under **M**.

Triplets and other sequences of primes.

5. 61. It is plain that the numbers ϖ, $\varpi + 2$, $\varpi + 4$ cannot all be prime, for at least one of the three is divisible by 3. But it is possible that ϖ, $\varpi + 2$, $\varpi + 6$ or ϖ, $\varpi + 4$, $\varpi + 6$ should all be prime. It is natural to enquire whether our method is applicable in principle to the investigation of the distribution of triplets and longer sequences.

The general case raises very interesting questions as to the density of the distribution of primes, and it will be convenient to begin by discussing them.

We write

$$\varrho(x) = \overline{\lim_{n \to \infty}} (\pi(n + x) - \pi(n)), \tag{5. 611}$$

so that $\varrho(x) = \varrho([x])$ is the greatest number of primes that occurs indefinitely often in a sequence $n + 1, n + 2, \ldots, n + [x]$ of $[x]$ consecutive integers. The existence of an infinity of primes shows that $\varrho(x) \geqq 1$ for $x \geqq 1$, and nothing more than this is known; but of course Conjecture B involves $\varrho(x) \geqq 2$ for $x \geqq 3$. It is plain that the determination of a lower bound for $\varrho(x)$ is a problem of exceptional depth.

The problem of an upper bound has greater possibilities. We proceed to prove, by a simple extension of an argument due to Legendre[1],

[1] See Landau, p. 67.

Theorem G. *If* $\varepsilon > 0$ *then*

$$\varrho(x) < (1+\varepsilon)\, e^{-C} \frac{x}{\log\log x} \quad (x > x_0 = x_0(\varepsilon)),$$

where C *is Euler's constant. More generally, if* $N(x, n)$ *is the number of the integers* $n+1, n+2, \ldots, n+[x]$ *that are not divisible by any prime less than or equal to* $\log x$, *then*

$$\sigma(x) = \overline{\lim_{n\to\infty}}\, N(x, n) < (1+\varepsilon)\, e^{-C} \frac{x}{\log\log x} \quad (x > x_0(\varepsilon)).$$

It is well-known that the number of the integers $1, 2, \ldots, [y]$, not divisible by any one of the primes $p_1, p_2, \ldots, p_\nu$, is

$$[y] - \sum \left[\frac{y}{p_r}\right] + \sum \left[\frac{y}{p_r p_s}\right] - \cdots$$

where the i-th summation is taken over all combinations of the ν primes i at a time. Since the number of terms in the total summation is 2^ν, this is

$$y - \sum \frac{y}{p_r} + \sum \frac{y}{p_r p_s} - \cdots + O(2^\nu) = y\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_\nu}\right) + O(2^\nu).$$

We now take $p_1, p_2, \ldots, p_\nu$ to be the first ν primes, write $n+x$ and n successively for y, subtract, and take the upper limit of the difference as $n \to \infty$. We obtain

$$\sigma(x) \leq x \prod_{r=1}^{\nu} \left(1 - \frac{1}{p_r}\right) + O(2^\nu).$$

But

$$\prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-C}}{\log y}$$

as $y \to \infty$.[1] If we take $y = \log x$, and p_ν to be the greatest prime not less than y, we have

$$\nu < p_\nu \leq \log x, \quad 2^\nu = o\left(\frac{x}{\log\log x}\right),$$

$$\sigma(x) < (1+\varepsilon)\, e^{-C} \frac{x}{\log\log x} \quad (x > x_0(\varepsilon)),$$

the desired result.

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[1] Landau, p. 140.

An examination of the primes less than 200 suggests forcibly that

$$\varrho(x) \leq \pi(x) \qquad (x \geq 2).$$

But although the methods we are about to explain lead to striking conjectural lower bounds, they throw no light on the problem of an upper bound. We have not succeeded in proving, even with our additional hypothesis, more than the »elementary» Theorem G. We pass on therefore to our main topic.

5. 62. We consider now the problem of the occurrence of groups of primes of the form

$$n,\ n+a_1,\ n+a_2,\ \ldots,\ n+a_m,$$

where $a_1, a_2, \ldots, a_m$ are distinct positive integers. We write for brevity

$$f_m(x) = \sum_{\varpi=2}^{\infty} \Lambda(\varpi)\Lambda(\varpi+a_1)\ldots\Lambda(\varpi+a_m)x^{\varpi}.$$

Then, if $(h, k) = 1$, we have

$$(5.\ 621) \qquad r^{a_m} f_m(r^2 e_k(h)) = \sum \Lambda(\varpi)\Lambda(\varpi+a_1)\ldots\Lambda(\varpi+a_m) r^{2\varpi+a_m} e_k(\varpi h)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum \Lambda(\varpi)\ldots\Lambda(\varpi+a_{m-1}) r^{\varpi} e^{\varpi i\varphi} e_k(\varpi h) \,.\, \sum \Lambda(\varpi) r^{\varpi} e^{-\varpi i\varphi} \,.\, e^{a_m i\varphi}\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f_{m-1}\left(r e^{i\left(\varphi+\frac{2h\pi}{k}\right)}\right) f(re^{-i\varphi}) e^{a_m i\varphi}\, d\varphi.$$

If $\varphi = \frac{2p\pi}{q} + \theta$, $r \to 1$, $\theta \to 0$, and θ is sufficiently small in comparison with $1 - r$, then

$$f(re^{-i\varphi}) \sim \frac{\chi(q)}{1 - re^{-i\theta}},$$

where

$$\chi(q) = \frac{\mu(q)}{\varphi(q)}.$$

Let us assume for the moment that

$$f_{m-1}(re^{i\psi}) \sim g_{m-1}\left(\frac{p'}{q'}\right)\frac{1}{1 - re^{i\theta}}$$

if $\psi = \frac{p'}{q'} + \theta$, $r \to 1$, and θ is sufficiently small. Then our method leads us to write

$$\frac{1}{2\pi}\int_0^{2\pi} f_{m-1}\left(r e^{i\left(\varphi+\frac{2h\pi}{k}\right)}\right) f(re^{-i\varphi})\, e^{a_m i\varphi}\, d\varphi$$

$$\sim \sum_{p\, q} \chi(q)\, g_{m-1}\left(\frac{p}{q}+\frac{h}{k}\right) e\left(\frac{a_m p}{q}\right) \frac{1}{2\pi}\int_{\xi_{p,q}} \frac{d\theta}{(1-re^{i\theta})(1-re^{-i\theta})}$$

$$\sim \frac{1}{1-r^2}\sum_{p,q} \chi(q)\, g_{m-1}\left(\frac{p}{q}+\frac{h}{k}\right) e\left(\frac{a_m p}{q}\right),$$

on replacing the integral by one extended from $-\pi$ to π. Thus (5. 621) suggests that

(5. 622) $$f_m(r) \sim \frac{g_m(0)}{1-r},$$

where g_m is determined by the recurrence formula

(5. 623) $$g_m\left(\frac{h}{k}\right) = \sum_{p,q} \chi(q)\, g_{m-1}\left(\frac{p}{q}+\frac{h}{k}\right) e\left(\frac{a_m p}{q}\right)$$

and

(5. 624) $$g_0\left(\frac{h}{k}\right) = \chi(k).$$

From this recurrence formula we obtain without difficulty

(5. 625) $$g_m(0) = S_m = \sum_{p_1, q_1, \ldots, p_m, q_m} \prod_{r=1}^{m} \chi(q_r)\, \chi(Q)\, e\left(\sum_{r=1}^{m} \frac{a_r p_r}{q_r}\right),$$

where q_r runs through all positive integral values, p_r through all positive values less than and prime to q_r, and Q is the number such that

$$\frac{P}{Q} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \cdots + \frac{p_m}{q_m}, \qquad (P, Q) = 1.$$

If we sum with respect to the p's, we obtain a result which we shall write in the form

(5. 6251) $$S_m = \sum_{q_1, q_2, \ldots, q_m} A(q_1, q_2, \ldots, q_m).$$

We shall see presently that the multiple series (5. 6251) is absolutely convergent.

For greater precision of statement we now introduce a definite hypothesis.

Hypothesis X. *If $m \geqq 0$, and $r \to 1$, then*

$$f_m(r) \sim \frac{S_m}{1-r}, \tag{5.626}$$

where S_m is given by (5. 625) *and* (5. 6251).

Our deductions from this hypothesis will be made rigorously, and we shall describe the results as Theorems X 1, X 2, ...

5. 63. From (5. 626) it follows, by the argument of 4. 2, that

$$P(x; 0, a_1, a_2, \ldots, a_m) \sim S_m \frac{x}{(\log x)^m}, \tag{5.631}$$

as $x \to \infty$, where the left-hand side denotes the number of groups of $m+1$ primes $n, n+a_1, \ldots, n+a_m$ of which all the members are less than x.

We proceed to evaluate S_m. In the first place we observe that $A(q_1, q_2, \ldots, q_m)$ is zero if any q has a square factor. Next we have

$$A(q_1 q'_1, q_2 q'_2, \ldots, q_m q'_m) = A(q_1, q_2, \ldots q_m)\, A(q'_1, q'_2, \ldots, q'_m), \tag{5.632}$$

provided $(q_r, q'_s) = 1$ for all values of r and s. For, if we write

$$\frac{p_r}{q_r} + \frac{p'_r}{q'_r} = \frac{p_r q'_r + p'_r q_r}{q_r q'_r} = \frac{\mathfrak{p}_r}{\mathfrak{q}_r},$$

so that $\mathfrak{q}_r = q_r q'_r$, and suppose that p_r and p'_r run through complete sets of residues prime to q_r (or q'_r) and incongruent to modulus q_r (or q'_r), then $\mathfrak{p}_r$ runs through a similar set of residues for modulus $\mathfrak{q}_r$. Also $(Q, Q') = 1$ and so $(PQ' + P'Q, QQ') = 1$. Hence, since

$$\sum \frac{\mathfrak{p}_r}{\mathfrak{q}_r} = \frac{P}{Q} + \frac{P'}{Q'} = \frac{PQ' + P'Q}{QQ'},$$

the Q associated with $\sum \frac{\mathfrak{p}_r}{\mathfrak{q}_r}$ is QQ'. Since $\chi(qq') = \chi(q)\chi(q')$ if $(q, q') = 1$, (5. 632) follows at once.

Assuming then the absolute convergence, more conveniently established later, of the series and the product, we have

$$S_m = \sum A(q_1, q_2, \ldots, q_m) = \Pi X(\varpi) = \Pi X_m(\varpi) = \Pi X_m(\varpi; a_1, \ldots, a_m), \tag{5.633}$$

where

$$(5.634)\qquad X(\varpi) = 1 + \sum\nolimits_1 A(\varpi, 1, 1, \ldots, 1) + \sum\nolimits_2 A(\varpi, \varpi, 1, \ldots, 1) + \cdots$$
$$+ \sum\nolimits_r A(\varpi, \varpi, \varpi, \ldots 1) + \cdots + A(\varpi, \varpi, \varpi, \ldots, \varpi),$$

and where $\sum_r$ is extended over all A's in which r of the m places are filled by ϖ's and the remaining $m-r$ by 1's.

Our next step is to evaluate the A's corresponding to a prime ϖ. Writing $x = \chi(\varpi) = -\dfrac{1}{\varpi - 1}$, we have first, when only one place, say the first, is filled by a ϖ,

$$q_1 = \varpi,\ q_r = 1\,(r > 1),\ p_r = 0\,(r > 1),\ Q = \varpi,$$

and so

$$(5.635)\qquad A(\varpi, 1, 1, \ldots, 1) = (\chi(\varpi))^2 \sum_{(p,\varpi)=1} e_\varpi(a_1 p) = x^2 c_\varpi(a_1).$$

When $r > 1$ places, say the first r, are filled by ϖ's, we have similarly

$$A(\varpi, \varpi, \varpi, \ldots, 1) = x^r \sum_{p_1, p_2, \ldots, p_r} e_\varpi(a_1 p_1 + \cdots + a_r p_r)\, \chi(Q),$$

where the p's run through the numbers $1, 2, \ldots, \varpi - 1$, and Q is determined by

$$(P, Q) = 1,\quad \frac{P}{Q} = \frac{p_1 + p_2 + \cdots + p_r}{\varpi}.$$

Clearly

$$Q = 1\left(\sum p \equiv 0 \pmod{\varpi}\right),\ Q = \varpi\left(\sum p \not\equiv 0 \pmod{\varpi}\right).$$

Hence

$$(5.636)\qquad A(\varpi, \varpi, \varpi, \ldots, 1) = x^{r+1}\Bigg[\sum_{p_1,\ldots,p_r} e_\varpi\left(\sum_{s=1}^{r} a_s p_s\right) +$$
$$+ \frac{\chi(1) - \chi(\varpi)}{\chi(\varpi)} \sum_{p_1+p_2+\cdots\equiv 0} e_\varpi\left(\sum_{s=1}^{r} a_s p_s\right)\Bigg]$$
$$= x^{r+1}\left[\prod_{s=1}^{r} c_\varpi(a_s) - \varpi \sum_{p_1+p_2+\cdots\equiv 0} e_\varpi\left(\sum a_s p_s\right)\right].$$

Now

$$B = \sum_{p_1+p_2+\cdots=0} e\varpi\left(\sum a_s p_s\right)$$

is evidently a function of $\varpi, a_1, \ldots, a_r$ which is unaltered by a permutation of $a_1, \ldots, a_r$. We denote it (dropping the reference to ϖ) by $B_m(a_1, a_2, \ldots, a_r)$, the suffix m being used to recall that $a_1, a_2, \ldots, a_r$, or rather the a's that replace them in the general case, are selected from $a_1, a_2, \ldots, a_m$.

Then

$$(5.637)\quad B_m(a_1, a_2, \ldots, a_r) = \left(\sum_{p_2, p_3, \ldots, p_r} - \sum_{p_2+p_3+\cdots=0}\right) e\varpi(a_2 p_2 + \cdots + a_r p_r - a_1(p_2 + \cdots + p_r))$$

$$= \sum_{p_2, \ldots, p_r} e\varpi((a_2 - a_1) p_2 + \cdots + (a_r - a_1) p_r) - \sum_{p_3, \ldots, p_r} e\varpi((a_3 - a_2) p_3 + \cdots + (a_r - a_2) p_r) + \cdots$$

$$= \prod_{s=2}^{r} c\varpi(a_s - a_1) - \prod_{s=3}^{r} c\varpi(a_s - a_2) + \cdots$$

Here we are supposing $r \geqq 2$. We shall adopt the convention $B_m(a_s) = 0$.

5. 64. We now digress for a moment to establish the absolute convergence of our product and multiple series. We have

$$(5.\ 641)\qquad c\varpi(k) = \varpi - 1 \quad (\varpi \mid k),\ c\varpi(k) = -1 \quad (\varpi \nmid k).$$

Hence, when ϖ is large, every $c\varpi$ occurring in (5. 635), (5. 636), or (5. 637) is equal to -1.[1] It follows that

$$|A(\varpi, \varpi, \varpi, \ldots, 1)| < Kx^2 < \frac{K}{\varpi^2} \quad (r \geqq 1);$$

and so, since $A(q_1, q_2, \ldots)$ is the product of A's each involving only a single prime ϖ, that the multiple series and the product in (5. 633) are absolutely convergent.

5. 65. Returning now to $X(\varpi)$, we have, for $r \geqq 1$,

$$A(\varpi, \varpi, \varpi, \ldots 1) = x^{r+1}\left(\prod_{s=1}^{r} c\varpi(a_s) - \varpi B_m(a_1, a_2, \ldots, a_r)\right),$$

the result being true for $r = 1$ in virtue of (5. 635) and our convention as to $B_m(a_s)$. Hence

[1] It is here that we use the condition $a_r \neq a_s$.

$$(5.651)\qquad X_m(\varpi) = 1 + \sum_{r=1}^{m} x^{r+1} \prod_{s=1}^{r} c_\varpi(a_s) - \varpi \sum_{r=2}^{m} x^{r+1} \sum_r B_m(a_1, a_2, \ldots, a_r)$$

$$= 1 + x\left(\prod_{r=1}^{m}(1 + x c_\varpi(a_r)) - 1\right) - \varpi x \sum_{r=2}^{m} x^r C_{m,r}$$

$$= Y_m - \varpi x Z_m,$$

say, where

$$(5.652)\qquad C_{m,r} = \sum_r B_m(a_1, a_2, \ldots, a_r),$$

the summation being taken over all combinations (without reference to order) of $a_1, \ldots, a_m$ taken r at a time.

Now

$$(5.653)\qquad Y_{m+1} - (1-x) Y_m = 1 - x - (1-x)^2 + x\prod_{r=1}^{m}(1 + x c_\varpi(a_r))(1 + x c_\varpi(a_{m+1}) - 1 + x)$$

$$= x(1-x) + x^2(1 + c_\varpi(a_{m+1})) \prod_{r=1}^{m} (1 + x c_\varpi(a_r)).$$

Also

$$C_{m+1,r} = C_{m,r} + \sum{}' B(a_{m+1}, a_1, a_2, \ldots, a_{r-1}) \quad (r \geq 2).$$

Here $\sum'$ denotes a sum taken over the combinations of $a_1, a_2, \ldots, a_m$, $r-1$ at a time; and the equation holds even for $r = m+1$ if we interpret $C_{m,m+1}$ as zero. Hence, by (5. 637),

$$C_{m+1,r} = C_{m,r} + \sum{}'\left(\prod_{s=1}^{r} c_\varpi(a_s - a_{m+1}) - B_m(a_1, a_2, \ldots, a_{r-1})\right)$$

$$= C_{m,r} + \sum{}' \prod_{s=1}^{r} c_\varpi(a_s - a_{m+1}) - C_{m,r-1};$$

and therefore

$$(5.654)\qquad Z_{m+1} = \sum_{r=2}^{m+1} x^r C_{m+1,r} = Z_m + \sum_{r=2}^{m+1} x^r \sum{}' \prod_{s=1}^{r-1} c_\varpi(a_s - a_{m+1}) - \sum_{r=2}^{m+1} x^r C_{m,r-1}$$

$$= (1-x) Z_m + x\left(\prod_{r=1}^{m}(1 + x c_\varpi(a_r - a_{m+1})) - 1\right).$$

Using (5. 651), (5. 653), and (6. 654), and observing that $x(1-x) = -\varpi x^2$, we obtain

$$(5.\ 655)\quad X_{m+1}(\varpi) - (1-x)\,X_m(\varpi) = x^2(1 + c_\varpi(a_{m+1}))\prod_{r=1}^{m}(1 + x c_\varpi(a_r)) - $$

$$- \varpi x^2 \prod_{r=1}^{m}(1 + x c_\varpi(a_r - a_{m+1})).$$

To this recurrence formula we add the value of $X_m(\varpi)$ for $m = 1$, viz.

$$(5.\ 656)\qquad X_1(\varpi) = 1 + A(\varpi) = 1 + x^2 c_\varpi(a_1).$$

5. 66. We can now deduce an exceedingly simple formula for $X_m(\varpi)$, viz.

$$(5.\ 661)\qquad X_m(\varpi) = \left(\frac{\varpi}{\varpi - 1}\right)^m \frac{\varpi - \nu}{\varpi - 1},$$

where

$$(5.\ 662)\qquad \nu = \nu_m = \nu(\varpi;\ 0, a_1, a_2, \ldots, a_m)$$

is the number of distinct residues of $0, a_1, a_2, \ldots, a_m$ (mod. ϖ).

This is readily proved by induction. Let us denote the right hand side of (5. 661) by X'_m; and let us consider first the case $m = 1$.

If $a_1 \equiv 0$ (mod. ϖ) we have $\nu = 1$, $c_\varpi(a_1) = \varpi - 1$; if $a_1 \not\equiv 0$ we have $\nu = 2$, $c_\varpi(a_1) = -1$. In either case $X_1 = X'_1$.

Now suppose the result true for m, and consider X_{m+1}. There are three cases:

(i) $a_{m+1} \equiv 0$ (mod. ϖ). Here

$$\nu_{m+1} = \nu_m,\quad X'_{m+1} = \frac{\varpi}{\varpi - 1} X'_m = (1-x)\,X'_m.$$

On the other hand $1 + c_\varpi(a_{m+1}) = \varpi$, $c_\varpi(a_r - a_{m+1}) = c_\varpi(a_r)$; the right hand side of (5. 655) vanishes; and so

$$X_{m+1} = (1-x)\,X_m = (1-x)\,X'_m = X'_{m+1}.$$

(ii) $a_{m+1} \equiv a_{r_1} \not\equiv 0$ for some $r_1 \leq m$. Here again $\nu_{m+1} = \nu_m$. On the one hand we have, as before, $X'_{m+1} = (1-x)X'_m$. On the other

$$1 + c_\varpi(a_{m+1}) = 0,\ 1 + x c_\varpi(a_{r_1} - a_{m+1}) = 1 - \frac{1}{\varpi - 1} c_\varpi(0) = 0;$$

the right hand side of (5. 665) vanishes, and $X_{m+1} = X'_{m+1}$ as before.

(iii) $a_{m+1} \neq 0$, $a_{m+1} \neq a_r (r \leq m)$. Here $\nu_{m+1} = \nu_m + 1 = \nu + 1$. Also all the c's concerned are equal to -1. Hence

$$X_{m+1} - (1-x) X_m = -\varpi x^2 (1-x)^m = x(1-x)^{m+1},$$

or, since $X_m = X'_m$,

$$\begin{aligned} X_{m+1} &= (1-x) \, . \, (1-x)^m (1 + (\nu - 1)x) + x(1-x)^{m+1} \\ &= (1-x)^{m+1}(1 + \nu x) = X'_{m+1}. \end{aligned}$$

This completes the proof.

We now restate our conclusions in a more symmetrical form.

Theorem X 1.[1] *Let $b_1, b_2, \ldots, b_m$ be m distinct integers, and $P(x; b_1, b_2, \ldots, b_m)$ the number of groups $n + b_1, n + b_2, \ldots\ldots, n + b_m$ between 1 and x and consisting wholly of primes. Then*

(5. 663) $$P(x) \backsim G(b_1, b_2, \ldots, b_m) \, Li_m(x)$$

when $x \to \infty$, where

(5. 664) $$G(b_1, b_2, \ldots, b_m) = \prod_{\varpi \geq 2} \left(\left(\frac{\varpi}{\varpi - 1} \right)^{m-1} \frac{\varpi - \nu}{\varpi - 1} \right),$$

$\nu = \nu(\varpi; b_1, b_2, \ldots, b_m)$ is the number of distinct residues of $b_1, b_2, \ldots, b_m$ to modulus ϖ, and

$$Li_m(x) = \int_2^x \frac{du}{(\log u)^m}.$$

Further

(5. 665) $$G(b_1, b_2, \ldots, b_m) = C_m H(b_1, b_2, \ldots, b_m)$$

where

(5. 666) $$C_m = \prod_{\varpi > m} \left(\left(\frac{\varpi}{\varpi - 1} \right)^{m-1} \frac{\varpi - m}{\varpi - 1} \right),$$

(5. 667) $$H(b_1, b_2, \ldots, b_m) = \prod_{\varpi \leq m} \left(\left(\frac{\varpi}{\varpi - 1} \right)^{m-1} \frac{\varpi - \nu}{\varpi - 1} \right) \prod_{\substack{\varpi | \Delta \\ \varpi > m}} \left(\frac{\varpi - \nu}{\varpi - m} \right),$$

and Δ is the product of the differences of the b's.

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[1] To avoid any possible misunderstanding, we repeat that these theorems are *consequences of Hypothesis X.*

The change from $0, a_1, \ldots, a_m$ to $b_1, b_2, \ldots, b_m$ is obtained by writing $n - b_1$ for n and m for $m+1$. The expression of G as the product of the constant C_m (depending only on m) and the finite expression H follows immediately from the fact that $\nu = m$ if $\varpi \nmid \Delta$, $\varpi > m$.

5.67. There are plainly many directions in which it would be possible to extend these investigations. We may ask, for example, whether there are indefinitely recurring pairs, triplets, or longer sequences of primes subject to further restrictions, such as that of belonging to specified quadratic forms. We have considered one problem of this character only, which is interesting in that it combines those contemplated in Conjectures B and E. Is there an infinity of pairs of primes of the forms m^2+1, m^2+3? The result suggested is as follows.

Conjecture P. *There are infinitely many prime pairs of the form m^2+1, m^2+3. The number of such pairs less than n is given asymptotically by*

$$Q(n) \sim \frac{3C\sqrt{n}}{(\log n)^2} = \frac{3\sqrt{n}}{(\log n)^2} \prod_{\varpi \geq 5} \frac{\varpi(\varpi-\nu)}{(\varpi-1)^2},$$

where ν is 0, 2, or 4 according as neither, one, or both of -1 and -3 are quadratic residues of ϖ.

Numerical verifications.

5.68. A number of our conjectures have been tested numerically by Mrs. Streatfeild, Dr. A. E. Western, and Mr. O. Western. We state here a few of their results, reserving a fuller discussion of them for publication elsewhere.

The first of these numerical tests apply to conjectures E and P. In applying such tests we work (for reasons similar to those explained in 4.4 and 5.32) not with the actual formulae stated in the ennnciations of those conjectures, but with the asymptotically equivalent formulae

$$P(n) \sim \frac{1}{2} C \int^{n} \frac{dx}{\sqrt{x}\log x} \sim \frac{1}{2} C \, Li\sqrt{n} \tag{5.681}$$

and

$$Q(n) \sim \frac{3}{2} C \int^{n} \frac{dx}{\sqrt{x}(\log x)^2} \sim \frac{3}{4} C \, Li_2\sqrt{n}. \tag{5.682}$$

The number of primes less than 9000000 and of the prime form m^2+1 is 301. The number given by (5.681) is 302.6. The ratio is 1.005, and the agreement is all that could be desired.

The number of prime-pairs m^2+1 and m^2+3, both of whose members are less than 9000000, is 57. The value given by (5.682) is 48.9. The ratio is

.858. The numbers concerned are naturally rather small, but the result is perhaps a little disappointing.

A more systematic test has been applied to the formulae for triplets and quadruplets of primes, the particular groups considered being

$$\varpi,\ \varpi+2,\ \varpi+6;\ \varpi,\ \varpi+4,\ \varpi+6;$$
$$\varpi,\ \varpi+2,\ \varpi+6,\ \varpi+8;\ \varpi,\ \varpi+4,\ \varpi+6,\ \varpi+10.$$

The two kinds of triplets should occur with the same frequency. On the other hand there should be twice as many quadruplets of the second type as of the first. For 0, 2, 6, 8 have 4 distinct residues to modulus 5 and 0, 4, 6, 10 but 3, while for all other moduli the number of residues is the same; and the ratio 5—3: 5—4 is 2. The actual results are shown in the following table.

Triplets.

x	$P_3(x; 0, 2, 6)$	$\frac{9}{2} C_3 Li_3(x)$	Ratio	$P_3(x; 0, 4, 6)$	Ratio
10^5	260	270.78	1.041	249	1.087
2.10^5	417	440.71	1.057	425	1.037
3.10^5	566	589.89	1.042	588	1.003
4.10^5	718	727.43	1.013	748	0.972
5.10^5	833	857.10	1.029	881	0.973
6.10^5	950	980.92	1.033	1008	0.973
7.10^5	1073	1100.16	1.025	1133	0.971
8.10^5	1195	1215.64	1.017	1231	0.988
9.10^5	1295	1327.97	1.025	1331	0.998
10^6	1398	1437.59	1.028	1443	0.996

Quadruplets.

x	$P_4(x; 0, 2, 6, 8)$	$\frac{27}{2} C_4 Li_4(x)$	Ratio	$P_4(x; 0, 4, 6, 10)$	$27\, C_4 Li_4(x)$	Ratio
10^5	38	40.41	1.063	80	80.82	1.010
2.10^5	52	61.18	1.177	124	122.35	0.987
3.10^5	70	78.62	1.123	160	157.24	0.983
4.10^5	87	94.28	1.084	194	188.55	0.972
5.10^5	103	108.75	1.056	219	217.50	0.993
6.10^5	117	122.36	1.045	239	244.71	1.024
7.10^5	133	135.29	1.017	263	270.59	1.029
8.10^5	141	147.69	1.047	285	295.39	1.036
9.10^5	156	159.64	1.023	299	319.29	1.068
10^6	166	171.21	1.031	316	342.42	1.084

Here C_2 and C_4 are the constants of Theorem X 1. The results are on the whole very satisfactory, though there is a curious deficiency of quadruplets of the second type between 800000 and 1000000.

5. 691. We return to the problems connected with the function $\varrho(x) = \overline{\lim}_{n \to \infty} (\pi(n+x) - \pi(n))$. We shall adhere to the notation of Theorem X 1, and shall suppose in addition that x is integral and that $0 \leqq b_1 < b_2 < \cdots < b^m$. It follows from Theorem X 1 that, if $H(b_1, b_2, \ldots, b_m) \neq 0$, groups $n+b_1, n+b_2, \ldots, n+b_m$ consisting wholly of primes continually recur, and we shall say, when this happens, that $b_1, b_2, \ldots, b_m$ is a *possible* m-group of b's. We say also that the $n+b_1, \ldots, n+b_m$ is an m-group of primes. If, in a possible group, $m = \varrho(x)$, where $x = b_m - b_1 + 1$, we shall call the group, either of primes or of b's, a *maximum* group. A set of x consecutive positive integers we call an x-sequence; and we say that the group $n+b_1, \ldots, n+b_m$ is *contained in* the $(b_m - b_1 + 1)$-sequence $b_1 \leqq c \leqq b_m$, and that its *length* is $b_m - b_1 + 1$.

Theorem X 2. *If* $b_1, b_2, \ldots, b_m$ *have a missing residue* (mod. ϖ) *for each* $\varpi \leqq m$, *then these* b's *form a possible group.*

This is an immediate consequence of Theorem X 1, since $\nu \leqq \varpi - 1$ for $\varpi > m$.

Theorem X 3. *Let* $M(x, n)$ *be the number of distinct integers* $c_1, c_2, \ldots, c_M$, *in the interval* $n < c \leqq n + x$, *which are not divisible by any prime less than or equal to*

$$\bar{\varrho}(x) = \varrho(x) + \left[\frac{x}{\varrho(x)}\right] + 1,$$

and let

$$\varrho_1(x) = \operatorname*{Max}_{(n)} M(x, n).$$

Then

$$\varrho_1(x) = \varrho(x).$$

Let $\varrho(x, \mu)$ be the number obtained in place of $\varrho_1(x)$ when the $\bar{\varrho}(x)$ that occurs in the definition of $\varrho_1(x)$ is replaced by μ. Clearly we have

$$\varrho(x, \mu - 1) \geqq \varrho(x, \mu) \geqq \varrho(x) \tag{5. 6911}$$

and

$$\varrho(x, \mu) \geqq \varrho(x, \mu - 1) - \left[\frac{x}{\mu}\right] - 1. \tag{5. 6912}$$

Further,

(5. 6913) $$\tau = \varrho(x, \mu) \leqq \mu$$

implies

$$\varrho(x, \mu) = \varrho(x).$$

For let $d_1, d_2, \ldots, d_\tau$ be an increasing set of positive integers with the properties (characteristic of a set of $\tau = \varrho(x, \mu)$ such integers) that (*a*) there is an n such that $n + d_1, \ldots, n + d_\tau$ are not divisible by any prime less than or equal to μ, and (*b*) $d_\tau - d_1 + 1 \leqq x$. Then $n + d_1, \ldots, n + d_\tau$ form a 'possible' group of b's, since they lack the residue 0 for every prime less than or equal to τ. Hence $\varrho(x) \geqq \tau = \varrho(x, \mu)$, and so, by (5. 6911), $\varrho(x) = \varrho(x, \mu)$.

Next we observe that $\varrho(x, \mu) = \varrho(x)$ for $\mu = x$, since the inequality $\tau \leqq \mu$ is clearly satisfied in this case. Let now μ_0 be the least μ, greater than or equal to $\varrho(x)$, for which $\varrho(x, \mu_0) = \varrho(x)$. Then $\varrho(x) \leqq \mu_0 \leqq x$. We have then

(5. 6914) $$\varrho(x, \mu_0) = \varrho(x), \quad \varrho(x, \mu_0 - 1) > \varrho(x),$$

and so

$$\varrho(x, \mu_0 - 1) > \mu_0 - 1,$$

by (5. 6913). Thus

$$\begin{aligned} \mu_0 \leqq \varrho(x, \mu_0 - 1) \leqq \varrho(x, \mu_0) + \left[\frac{x}{\mu_0}\right] + 1 &= \varrho(x) + \left[\frac{x}{\mu_0}\right] + 1 \\ \leqq \varrho(x) + \left[\frac{x}{\varrho(x)}\right] + 1 &= \bar{\varrho}(x). \end{aligned}$$

Hence

$$\varrho(x) = \varrho(x, \mu_0) \geqq \varrho(x, \bar{\varrho}(x)) = \varrho_1(x).$$

But it is evident that $\varrho_1(x) \geqq \varrho(x)$, and therefore $\varrho_1(x) = \varrho(x)$.

It follows from the theorem that, *in a maximum group of primes of length x, the remaining numbers of the x-sequence are all divisible by primes less than or equal to $\bar{\varrho}(x)$*. We shall see presently that (on hypothesis X) $\bar{\varrho}(x) \leqq \varrho(x) + \log x$ for large values of x.

5. 692. We consider now the problem of a lower bound for $\varrho(x)$. Let p_s denote the s-th prime.

Theorem X 4. *Let $r = r(n)$ be defined, for every value of n, by*

$$p_r \leqq n < p_{r+1}.$$

Then $p_{r+1}, p_{r+2}, \ldots, p_{r+n}$ is a possible n-group of b's.

For the primes less than or equal to n are $p_1, p_2, \ldots, p_r$ and the b's lack the residue 0 for each of them.

From Theorem X 4 we deduce at once

Theorem X 5. *If* $x = p_{r+n} - p_{r+1} + 1$, $p_r \leqq n < p_{r+1}$, *then*

$$\varrho(x) \geqq n.$$

As a numerical example, let $n = 76501$. We have $p_{7525} = 76493$, $p_{7526} = 76507$. Hence

$$r = 7525,\ n + r = 84026,\ p_{n+r} = 1076503$$

$$x = 1076503 - 76507 + 1 = 999997.$$

Thus

$$\varrho(1000000) \geqq 76501.$$

We may compare this with the numbers of primes in the first, second, and third millions, viz.

$$78498,\ 70433,\ 67885.$$

Theorem X 5 provides a lower limit for $\varrho(x)$ when x has a certain form: we proceed to consider the case when x is unrestricted.

Theorem X 6. *We have*

$$\varrho(x) > \frac{x}{\log x}$$

for sufficiently large values of x.

When m is large

$$p_m = m(\log m + \log\log m) - m + O\left(\frac{m \log\log m}{\log m}\right).$$

Let

$$r = \left[\frac{y}{(\log y)^2}\left(1 + \frac{\log\log y}{\log y}\right)\right].$$

Then we have, by straightforward calculations,

$$p_r = \frac{y}{\log y}\left(1 - \frac{1}{\log y} + O\left(\frac{\log\log y}{\log y}\right)^2\right).$$

Take $n = p_r$. Then

$$n + r = \frac{y}{\log y}\left(1 + O\left(\frac{\log\log y}{\log y}\right)^2\right).$$

$$p_{n+r}=y\left(1-\frac{1}{\log y}+O\left(\frac{\log\log y}{\log y}\right)^2\right)$$

$$x=p_{n+r}-p_{r+1}+1<p_{n+r}-p_r$$

$$=y\left(1-\frac{2}{\log y}+O\left(\frac{\log\log y}{\log y}\right)^2\right)<y-\frac{3}{2}\frac{y}{\log y}=z,$$

when y is large. Thus

$$\varrho(z)\geqq\varrho(x)\geqq n=p_r=\frac{y}{\log y}-\frac{y}{(\log y)^2}+O\left(\frac{y(\log\log y)^2}{(\log y)^3}\right)$$

$$>\frac{z}{\log z}.$$

Since y is arbitrary, so is z, and the theorem is proved.

5. 693. We conclude our discussion of $\varrho(x)$ with an account of one or two particular cases. For a given x it is, of course, theoretically possible to determine the maximum number of integers in an x-sequence that are not divisible by any prime less than x. On hypothesis X, this number is $\varrho(x)$. Thus L. Aubry[1] has shown that 30 consecutive *odd* integers cannot contain more than 15 primes (or more than 15 numbers not divisible by 2, 3, 5, or 7). Thus $\varrho(59)\leqq 15$. On the other hand if we take, in Theorem X 5, $n=15$, $r=6$, we see that the 15 primes from 17 to 73 give a possible group of b's. Hence, on hypothesis X,

$$\varrho(59)\geqq\varrho(57)=\varrho(73-17+1)\geqq 15;$$

and so $\varrho(59)=15$. The value of $\pi(59)$ is 17.

Similarly a 35-sequence cannot contain more than 10 numbers not divisible by 2, 3, or 5, but the 10 primes from 13 to 47, and therefore the numbers 0, 4, 6, 10, 16, 18, 24, 28, 30, 34, form a possible 10-group of b's, so that $\varrho(35)=10=$ $=\pi(35)-1$. A striking example of a maximum prime group $n+b_1,\ldots,n+b_{10}$, corresponding to this group of b's, is provided by $n=113143$.

The best example of a close approach by $\varrho(x)$ to $\pi(x)$ occurs when $x=97$. Consider the 24 primes from 17 to 113. They are a possible group of b's if they have a missing residue for each prime less than 24. We have only to test 17, 19, 23, and we find that 17 lacks the residue 8, 19 lacks 1 and 11, and 23 lacks 3, 12, 16, and 22. Hence on hypothesis X, $\varrho(97)\geqq 24$. On the other hand it

[1] L. E. Dickson, *l. c.*, vol. 1, p. 355.

may be shown that a 97-sequence cannot contain 25 numbers not divisible by 2, 3, 5, 7, 11, or 13. Let us denote the range $n \leqq x \leqq n+96$ by R_n. There is one and only one value of n, not greater than $2.3.5.7 = 210$, for which R_n contains 25 numbers not divisible by 2, 3, 5, or 7, viz. $n = 101$. If then $n \leqq 2.3.5.7.11$, and R_n contains 25 numbers not divisible by 2, 3, 5, 7, or 11, n must be one of the numbers $101 + 210\,m$ $(m = 0, 1, \ldots, 10)$; and on examination it proves that we may exclude all cases but $m = 10$. Repeating the argument we see that, if $n \leqq 2.3.5.7.11.13$, and R_n contains 25 numbers not divisible by 2, 3, 5, 7, 11, or 13, then n must be one of the numbers $n = 2201 + 2310\,m$ $(m = 0, 1, \ldots, 12)$. All these turn out to be impossible and, since any R_n may be reduced (mod. $2.3\ldots13$), it follows that no R_n can contain more than 24 numbers not divisible by a prime less than or equal to 13. *A fortiori* it follows that $\varrho(97) \leqq 24$, and so (on hypothesis X) $\varrho(97) = 24$. Since $\pi(97) = 25$, the difference $\varrho - \pi$ is here unity. Beyond $x = 97$ it would seem that $\varrho(x)$ falls further below $\pi(x)$, at least within any range in which calculation is practicable.

Conclusion.

5. 7. We trust that it will not be supposed that we attach any exaggerated importance to the speculations which we have set out in this last section. We have not forgotten that in pure mathematics, and in the Theory of Numbers in particular, 'it is only proof that counts'. It is quite possible, in the light of the history of the subject, that the whole of our speculations may be ill founded. Such evidence as there is points, for what it is worth, in the opposite direction. In any case it may be useful that, finding ourselves in possession of an apparently fruitful method, we should develop some of its consequences to the full, even where accurate investigation is beyond our powers.

Postscript.

(1). Prof. Landau has called our attention to the following passage in the *Habilitationsschrift* of Piltz ('Über die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze', Jena, 1884), pp. 46-47: —

'Ferner wiederholen sich gewisse Gruppierungen der Primzahlen mit gewisser Regelmässigkeit, so ist z. B. die durchschnittliche Häufigkeit der Gruppen von

je 2 Primzahlen, die in gegebenem Abstand aufeinanderfolgen, für die ungefähre Grösse x der Primzahlen, proportional $\frac{1}{(l\,x)^2}$, wobei allerdings dieser Ausdruck je nach dem gegebenen Abstand mit verschiedenen constanten Faktoren behaftet ist, die Häufigkeit einer Gruppe von 3 Primzahlen proportional $\frac{1}{(l\,x)^3}$ und so fort..... Die nähere Ausführung dieser und andrer Gesetze ... werde ich ein andres Mal folgen lassen.'

All of this is of course in perfect agreement with the results suggested in our concluding section.

(2). We must add a few words concerning the memoirs of Stäckel referred to on p. 34. These have only become accessible to us during the printing of the present memoir, and it is not possible for us even now to give any satisfactory summary of their contents; but Stäckel considers the problem of 'prime-groups' in much detail, and it is clear that he has anticipated some at any rate of the speculations of 5.6. The method of Stäckel, like that of Brun, rests on the use of the sieve of Eratosthenes, followed by a heuristic passage to the limit; but Stäckel's problem is much more general, and he has gone much further than Brun in the determination of the constants in the asymptotic formulae. It seems to be the principal advantage of our transcendental method, considered merely as a machine for the production of heuristic formulae, that these constants are determined naturally in the course of the analysis.

(3). We should also refer to a later memoir of Brun ('Le crible d'Eratosthène et le théorème de Goldbach', *Videnskapsselskapets Skrifter, Mat.-naturv. Klasse*, Kristiania, 1920, No. 3). Brun proves, by elementary methods, (1) that every large even number is the sum of two numbers, each composed of at most 9 prime factors, (2) that the number of prime-pairs ϖ, $\varpi + 2$, less than x, cannot exceed a constant multiple of $x(\log x)^{-2}$.

Brun's work enables us to make a substantial improvement in the elementary theorem G. Using the inequalities proved on pp. 32—34 of his memoir, we can show that

$$\varrho(x) < \frac{A\,x}{\log x}.$$

(4). Prof. Landau has pointed out to us an error on p. 9. It is not necessarily true that $C_k = 0$ when χ_k is imprimitive: our argument is only valid when Q is divisible by every prime factor of q.

The inequality (2. 16) is however correct. Suppose first that $q = \varpi^{\lambda}$ $(\lambda > 0)$. Our argument then holds unless $Q = 1$; in this case χ_k is the principal character and

$$\left|\sum_{m=1}^{q} e_q(m)\,\bar{\chi}_k(m)\right| = 1 \leqq \sqrt{q}.$$

This inequality is then easly generalised to all values of q. If $q = q_1 q_2$, where $(q_1, q_2) = 1$, then every χ (mod. q) is the product of a χ_1 (mod. q_1) and a χ_2 (mod. q_2). and it is easily proved that

$$\left|\sum_{m} e_q(m)\,\chi(m)\right| = \left|\chi_1(q_2)\,\chi_2(q_1)\sum_{m_1} e_{q_1}(m_1)\,\chi_1(m_1)\sum_{m_2} e_{q_2}(m_2)\,\chi_2(m_2)\right|$$

$$\leqq \sqrt{q_1}\,\sqrt{q_2} = \sqrt{q}.$$

The conclusion now follows by induction.

CORRECTIONS

p. 5, footnote 2. Read: p. 492.

p. 64. On the third line of § 5.691, for b^m read b_m.

p. 67. The statement on the last line but one, that the primes from 17 to 113 lack the residue 8 (mod 17), is incorrect since $59 \equiv 8$ (mod 17).

COMMENTS

Lemma 11 and footnote on p. 27. There is a simple closed expression for $c_q(n)$, namely

$$c_q(n) = \frac{\mu(N)\phi(q)}{\phi(N)}, \quad \text{where } N = \frac{q}{(q, n)},$$

which escaped the notice of both Ramanujan and Hardy, and was discovered by Hölder in 1936. See Hardy and Wright (4th ed.), p. 238.

Recent researches of Turán (in course of publication) have enabled him to prove some of Hardy and Littlewood's results under considerably weaker hypotheses.

Summation of a certain Multiple Series

Prof. G. H. Hardy *and* Mr. J. E. Littlewood.

The series in question is

$$S_m = \sum_{p_1, q_1;\, p_2, q_2;\, \dots;\, p_m, q_m} \chi(q_1)\chi(q_2)\dots\chi(q_m)\chi(Q)\, e\left(\frac{a_1 p_1}{q_1}+\frac{a_2 p_2}{q_2}+\dots+\frac{a_m p_m}{q_m}\right).$$

Here q_r runs through all positive integral values, and p_r through all such values less than and prime to q_r, and Q is the denominator of

$$\frac{p_1}{q_1}+\frac{p_2}{q_2}+\dots+\frac{p_m}{q_m} = \frac{P}{Q},$$

expressed in its lowest terms. The arithmetical function $\chi(q)$ is defined by

$$\chi(q) = \frac{\mu(q)}{\phi(q)},$$

where $\mu(q)$ and $\phi(q)$ are the well known functions of Möbius and Euler. Finally, the a's are unequal positive integers, and

$$e(x) = e^{2\pi i x}.$$

The sum of the series is

$$S_m = \prod_{\varpi} \left\{ \left(\frac{\varpi}{\varpi-1}\right)^m \left(\frac{\varpi-\nu}{\varpi-1}\right) \right\},$$

where ϖ assumes all prime values, and ν is the number of distinct residues of the group of numbers $0, a_1, a_2, \dots, a_m$ to modulus ϖ. It is plain that $\nu = m+1$ from a certain point onwards.

The series is of very great interest, for it is the series on which the asymptotic distribution of groups of primes

$$p,\ p+a_1,\ p+a_2,\ \dots,\ p+a_m$$

appears to depend. The details of the summation, and some indication of the concordance of the results suggested with the evidence of computation, are included in a memoir to appear in the *Acta Mathematica*.

1922, 8 (with J. E. Littlewood) *Proceedings of the London Mathematical Society*, (2) 20, xxx.

SOME PROBLEMS OF "PARTITIO NUMERORUM" (V) : A FURTHER CONTRIBUTION TO THE STUDY OF GOLDBACH'S PROBLEM

By G. H. Hardy *and* J. E. Littlewood

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1. *Introduction.*

1.1. This paper is a sequel to the third of the series.* We proved there that if a certain hypothesis, which we called Hypothesis R*, and which is a natural generalisation of the hypothesis of Riemann concerning the zeros of $\zeta(s)$, is true, then *every large odd number is the sum of three primes*; and we determined an asymptotic formula for the number of such representations. But our analysis broke down when the number of primes considered is less than three, and we were unable to make any rigorous contribution to the study of Goldbach's Problem itself.

In the present paper we prove (still, naturally, on the assumption of Hypothesis R*) that *almost all even numbers are sums of two primes*, that is to say that the number of numbers less than n, for which "Goldbach's Theorem" is false, is $o(n)$ when n is large. We prove, in fact, considerably more, but this is the essential result.

1.2. We recall the terminology of P. N. 3, in so far as it is relevant here.† Our fundamental hypothesis‡ is

Hypothesis R*.—*Every zero of every Dirichlet's function*

$$L(s) = \Sigma \frac{\chi(m)}{m^s},$$

* G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum' (III): On the expression of a number as a sum of primes", *Acta Mathematica*, Vol. 44 (1922), pp. 1–70. We refer to this memoir as P. N. 3. The analysis which it contains has been considerably simplified by Landau ["Zur Additive Primzahltheorie", *Rend. di Palermo* Vol. 46 (1922), pp. 349–356].

† We have modified the notation of P. N. 3, since we prefer now to denote a typical prime, as is usual, by p. What were there p, q are now h, k; what was there h is now $\phi(k)$; and what was there ϕ is now ψ.

‡ In P. N. 3 we did not actually use Hypothesis R*, but a slightly less drastic hypothesis (Hypothesis R) which proved sufficient for our purpose.

1924, 6 (with J. E. Littlewood) *Proceedings of the London Mathematical Society*, (2) 22, 46–56.

where $\chi(m)$ is a character to modulus k, has a real part less than or equal to $\frac{1}{2}$.

We write p for a prime,

$$f(x) = \Sigma \log p \, x^p \quad (|x| < 1),$$

$$\big(f(x)\big)^2 = \Sigma \nu_2(m) x^m,$$

$$\nu(m) = \nu_2(m) = \sum_{p+p'=m} \log p \log p',$$

$$F(x) = \Sigma \Omega(m) x^m = \Sigma m S(m) x^m,$$

where
$$S(m) = \sum_k A_k(m) = \sum_k \left(\frac{\mu(k)}{\phi(k)}\right)^2 c_k(-m),$$

$$c_k(-m) = \sum_h e^{-2hm\pi i/k} = \sum_h e_k(-hm).$$

Here $m = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots$, and h is positive, less than k, and prime to k, except when $k = 1$, when $h = 0$, $c_1(-m) = 1$ and $A_1 = 1$. $S(m)$ is the "singular series", and its sum is given by*

$$S(m) = 0 \quad (m \text{ odd}),$$

$$S(m) = 2C \Pi \left(\frac{P-1}{P-2}\right) \quad (m \text{ even}),$$

where P is an odd prime divisor of m, and

$$C = \prod_{p \geqslant 3} \left(1 - \frac{1}{(p-1)^2}\right).$$

We use the machinery of the "Farey dissection", explained in our memoirs on Waring's Problem. The circle Γ to which it is applied is defined by
$$|x| = e^{-H} = e^{-1/n},$$

and the "Farey arcs" are denoted by $\xi_{h,k}$ or ξ. The dissection is of order

$$N = [\sqrt{n}],$$

and there is, in this problem, no distinction of "major" and "minor" arcs.

When we are studying the arc $\xi_{h,k}$, we write

$$x = e_k(h) e^{-Y}, \quad Y = \frac{1}{n} + i\theta, \quad -\theta_0' \leqslant \theta \leqslant \theta_0,$$

* For the formal summation of the singular series see P. N. 3, pp. 26-29.

where θ_0 and θ_0' lie between $2\pi/kN$ and π/kN. The function which affords an approximation to $f(x)$ on $\xi_{h,k}$ is

$$\psi = \psi_{h,k} = \frac{\mu(k)}{\phi(k)}\frac{1}{Y}.$$

We write further

$$F_{h,k} = \left(\frac{\mu(k)}{\phi(k)}\right)^2 \Sigma m\left(xe_k(-h)\right)^m = \left(\frac{\mu(k)}{\phi(k)}\right)^2 g\left(xe_k(-h)\right),$$

where
$$g(z) = \Sigma m z^m = \frac{z}{(1-z)^2},$$

so that

$$F = F(x) = \sum_m mx^m \sum_k A_k(m) = \sum_m mx^m \sum_k \left(\frac{\mu(k)}{\phi(k)}\right)^2 \sum_h e_k(-hm)$$

$$= \sum_{k,h}\left(\frac{\mu(k)}{\phi(k)}\right)^2 \sum_m m\left(xe_k(-h)\right)^m = \sum_{k,h} F_{h,k}.$$

Finally, we write

$$S(m) = S_1(m)+S_2(m),$$

$$F = \Sigma mS(m)\,x^m = \Sigma mS_1(m)\,x^m+\Sigma mS_2(m)\,x^m,$$

where the suffix 1 limits k to the range $k \leqslant \nu$, and the suffix 2 to the range $k > \nu$.* Then

$$F_1 = \sum_{k\leqslant\nu,\,h} F_{h,k}, \qquad F_2 = \sum_{k>\nu,\,h} F_{h,k}.$$

2. *Preliminary lemmas.*

2.1. In all that follows A denotes an absolute constant, and O's and o's are uniform in all parameters that occur, except ϵ.

LEMMA 1.—*On* $\xi_{h,k}$,

$$\Psi = f-\psi = f(x)-\psi_{h,k} = O\left(n^{\frac{3}{4}}(\log n)^A\right).$$

See P. N. 3, p. 23, equation (3.124), and take $\Theta = \frac{1}{2}$.

LEMMA 2.—*If* $x = |x|e^{i\omega} = e^{-H+i\omega}$, *so that x lies on* Γ, *then*

$$\int_0^{2\pi} |f(x)|^2\,d\omega = O(n\log n).$$

See P. N. 3, p. 24.

---

* ν is an integer as yet undetermined. We ultimately take $\nu = [n^{\frac{1}{4}}]$.

LEMMA 3.—*We have*

$$\Sigma \int_{\xi} |\psi|^2 \, d\theta = O\left(n (\log n)^A\right).$$

From P. N. 3, p. 23 (bottom), with $r = 3$, we have

$$\int_{\xi} |\psi|^2 \, d\theta < A \frac{n}{\left(\phi(k)\right)^2}.$$

Hence

$$\Sigma \int_{\xi} |\psi|^2 \, d\theta < An \sum_{k,h} \frac{1}{\left(\phi(k)\right)^2} < An \sum_{k} \frac{1}{\phi(k)} < An \sum_{k} \frac{(\log k)^A}{k} < An (\log n)^A.$$

LEMMA 4.—*We have*

$$\Sigma \int_{\xi} |f^2 - \psi^2|^2 \, d\theta = O(n^{\frac{5}{2}+\epsilon})$$

for every positive ϵ.

For

$$\Sigma \int_{\xi} |f^2 - \psi^2|^2 \, d\theta < A \Sigma \int_{\xi} |f - \psi|^2 |f|^2 \, d\theta + A \Sigma \int_{\xi} |f - \psi|^2 |\psi|^2 \, d\theta$$
$$= A \Sigma_1 + A \Sigma_2,$$

say. But

$$\Sigma_1 < An^{\frac{3}{2}} (\log n)^A \Sigma \int_{\xi} |f|^2 \, d\theta = O\left(n^{\frac{5}{2}} (\log n)^A\right) = O(n^{\frac{5}{2}+\epsilon}),$$

by Lemmas 1 and 2; and

$$\Sigma_2 < An^{\frac{3}{2}} (\log n)^A \Sigma \int_{\xi} |\psi|^2 \, d\theta = O\left(n^{\frac{5}{2}} (\log n)^A\right) = O(n^{\frac{5}{2}+\epsilon}),$$

by Lemmas 1 and 3; whence the conclusion.

2.2. LEMMA 5.—*If* $k > 1$, $|\mu(k)| > 0$,* *and* $k = k_1 k_2$, *where* $(k_2, m) = 1$, *then*

$$|A_k(m)| < \frac{A (\log k)^A}{k_1 k_2^2}.$$

For $$|A_k(m)| = \frac{|c_k(-m)|}{\left(\phi(k)\right)^2} < \frac{A (\log k)^A}{k^2} |c_k(-m)|.$$

---

* So that k has no repeated factor (is *quadratfrei*). Otherwise $A_k(m) = 0$.

But, if $k = \Pi p_s$, where the p's are primes, then*

$$|c_k(-m)| = \prod_{p_s|m} (p_s-1) < \prod_{p_s|m} p_s = k_1,$$

whence the result of the lemma.

LEMMA 6.—*If* $m > 1$, $\nu > 1$, *then*

$$|S_2(m)| < A(\log m)^A\, d(m)\, \frac{(\log \nu)^A}{\nu},$$

where $d(m)$ *is the number of divisors of* m.

By Lemma 5,

$$|S_2(m)| \leqslant \sum_{k>\nu} |A_k(m)| < A \sum \frac{(\log k)^A}{k_1 k_2^2},$$

the summation extending over those values of k for which $k > \nu$, $|\mu(k)| > 0$.

Suppose that $\pi_1, \pi_2, \ldots, \pi_t$ are the prime divisors of m, and that $\delta = \Pi \pi_i$ is a typical *quadratfrei* divisor of m; and consider those terms of Σ for which

$$k > \nu, \quad |\mu(k)| > 0, \quad k = \delta k_2, \quad (k_2, m) = 1,$$

δ is fixed and k_2 varies. The contribution of these terms is less than

$$\begin{aligned}\sum_{\delta k_2 > \nu} \frac{(\log \delta k_2)^A}{\delta k_2^2} &< \frac{A(\log m)^A}{\delta} \sum_{k_2 > \nu/\delta} \frac{(\log k_2)^A}{k_2^2} \\ &< \frac{A(\log m)^A}{\delta}\, \frac{A\delta}{\nu} \left(\log\left(\frac{\nu}{\delta} + A\right)\right)^A \\ &< A(\log m)^A \frac{(\log \nu)^A}{\nu}.\end{aligned}$$

Hence $$\sum \frac{(\log k)^A}{k_1 k_2^2} < A(\log m)^A \frac{(\log \nu)^A}{\nu} \sum_{\delta|m} 1,$$

which proves the lemma.

2.3. LEMMA 7.—*If* $x = |x|\, e^{i\omega}$, *as in Lemma* 2, *then*

$$\int_0^{2\pi} |F_2(x)|^2\, d\omega = O\left(n^3 (\log n)^A \frac{(\log \nu)^A}{\nu^2}\right).$$

---

* See P. N. 3, p. 28 (bottom).

For $F_2(x) = \Sigma m S_2(m) x^m$, and so

$$(2.31)\qquad \int_0^{2\pi} |F_2(x)|^2\, d\omega = A\Sigma\left(mS_2(m)\right)^2 |x|^{2m}$$

$$= O\left(\frac{(\log\nu)^A}{\nu^2}\Sigma\, m^2(\log m)^A\left(d(m)\right)^2 |x|^{2m}\right),$$

by Lemma 6. But

$$\sum_{m<n}\left(d(m)\right)^2 = O\left(n(\log n)^A\right),^*$$

$$\sum_{m<n} m^2(\log m)^A\left(d(m)\right)^2 = O\left(n^3(\log n)^A\right),$$

and so $\left(\text{since } |x| < 1-\dfrac{A}{n}\right)$

$$(2.32)\qquad \Sigma m^2(\log m)^A\left(d(m)\right)^2|x|^{2m} = O\left(n^3(\log n)^A\right).$$

The lemma follows from (2.31) and (2.32).

LEMMA 8.—*We have*

$$\Sigma\int_\xi |F_{h,k}-\psi^2_{h,k}|^2\, d\theta = O(1).$$

This is trivial, since

$$F_{h,k}-\psi^2_{h,k} = \left(\frac{\mu(k)}{\phi(k)}\right)^2\left(g\left(xe_k(-h)\right)-\frac{1}{Y^2}\right)$$

$$= \left(\frac{\mu(k)}{\phi(k)}\right)^2\left\{\frac{xe_k(-h)}{\left(1-xe_k(-h)\right)^2}-\frac{1}{\left(\log\dfrac{1}{xe_k(-h)}\right)^2}\right\} = O(1).$$

LEMMA 9.—*If ξ_2 denotes a typical Farey arc for which $k>\nu$, then*

$$\Sigma\int_{\xi_2}|\psi|^4\, d\theta = O\left(n^3\frac{(\log\nu)^A}{\nu^2}\right).$$

We have

$$\int_\xi |\psi|^4\, d\theta \leqslant \frac{1}{\left(\phi(k)\right)^4}\int_\xi\frac{d\theta}{|Y|^4} = \frac{1}{\left(\phi(k)\right)^4}\int_\xi\frac{d\theta}{(n^{-2}+\theta^2)^2}$$

---

* In fact the sum is $O\left(n(\log n)^3\right)$. See S. Ramanujan, "Some formulæ in the analytic theory of numbers", *Messenger of Mathematics*, Vol. 45 (1916), pp. 81–84.

$$< \frac{A(\log k)^A}{k^4} \int_{-\infty}^{\infty} \frac{d\theta}{(n^{-2}+\theta^2)^2} < An^3 \frac{(\log k)^A}{k^4}.$$

Hence $$\Sigma \int_{\xi_2} |\psi|^4 \, d\theta < An^3 \sum_{k>\nu} \frac{(\log k)^A}{k^3} < An^3 \frac{(\log \nu)^A}{\nu^2}.$$

2.4. LEMMA 10.—*Suppose that x lies on $\xi_{h,k}$ and that $\xi_{H,K}$ is a Farey arc different from $\xi_{h,k}$. Then*

$$|F_{H,K}| < \frac{An(\log K)^A}{(hK-kH)^2}.$$

We have $$x = |x| \, e_k(h) e^{-i\theta} = |x| \, e_K(H) e^{-i\omega},$$

where $$\omega = 2\pi \left(\frac{H}{K} - \frac{h}{k} \right) + \theta,$$

and

$$|F_{H,K}| = \left(\frac{\mu(K)}{\phi(K)} \right)^2 |g(xe_K(-H))| = \left(\frac{\mu(K)}{\phi(K)} \right)^2 \left| g(|x| e^{-i\omega}) \right| < \frac{A(\log K)^A}{K^2 |\omega|^2}.$$

Suppose first that the arcs $\xi_{h,k}$, $\xi_{H,K}$ are not adjacent in the dissection. Then

$$\left| 2\pi \left(\frac{h}{k} - \frac{H}{K} \right) \right| \geqslant \frac{4\pi}{kK} \geqslant 2 \frac{2\pi}{kN} \geqslant 2|\theta|,$$

and so $$|\omega| > A \left| \frac{h}{k} - \frac{H}{K} \right| = A \frac{|hK-kH|}{kK},$$

$$|F_{H,K}| < \frac{A(\log K)^A}{K^2} \frac{k^2K^2}{(hK-kH)^2} < \frac{An(\log K)^A}{(hK-kH)^2},$$

since $k^2 \leqslant N^2 \leqslant n$. The argument fails when the arcs are adjacent. In this case $|hK-kH| = 1$. As x is outside the arc $\xi_{H,K}$, $|\omega| > A/KN$; and so

$$|F_{H,K}| < A \frac{(\log K)^A}{K^2} K^2N^2 < An(\log K)^A = \frac{An(\log K)^A}{(hK-kH)^2}.$$

The result of the lemma is therefore true in any case.

2.5. LEMMA 11.—*Let*

$$G = \bar{\Sigma} \, |F_{H,K}|,$$

where the sign $\bar{\Sigma}$ implies a summation over those pairs of values (H, K),

* So that G is a function of h and k, as well as of x and ν.

distinct from (h, k), *for which* $K \leqslant \nu$.* *Then*

$$\Sigma \int_{\xi_{h,k}} G^2 d\theta = O\left(n^{\frac{3}{2}} (\log n)^A \nu^A\right).$$

Since G is a sum of less than ν^2 terms, we have

$$G^2 < A\nu^2 \bar{\Sigma} \, | F_{H,K} |^2 < An^2\nu^2 \bar{\Sigma} \frac{(\log K)^A}{(hK-kH)^4},$$

by Lemma 10. Hence

$$\Sigma \int_{\xi_{h,k}} G^2 d\theta < An^2\nu^2 \sum_{h,k} \frac{1}{kN} \bar{\sum_{H,K}} \frac{(\log K)^A}{(hK-kH)^4}$$

$$< An^{\frac{3}{2}} (\log n)^A \nu^2 \sum_{h,k} \frac{1}{k} \bar{\sum_{H,K}} \frac{1}{(hK-kH)^4}$$

$$< An^{\frac{3}{2}} (\log n)^A \nu^2 \sum_{H,K} \sum_{h,k} \frac{1}{k(hK-kH)^4},$$

where now the inner summation is defined by

$$k \leqslant N, \quad (h, k) \neq (H, K),$$

and the outer summation by $K \leqslant \nu$. But

$$\sum_{h,k} \frac{1}{k(hK-kH)^4} = \frac{1}{K^4} \sum_k \frac{1}{k} \sum_h \frac{1}{\left(h - \dfrac{kH}{K}\right)^4}$$

$$< \frac{1}{K^4} \sum_k \frac{A}{k} \left(\frac{1}{\left(\dfrac{1}{K}\right)^4} + \frac{1}{\left(1+\dfrac{1}{K}\right)^4} + \frac{1}{\left(2+\dfrac{1}{K}\right)^4} + \dots \right)$$

$$< \frac{A}{K^4} \sum_k \frac{K^4}{k} < A \sum_k \frac{1}{k} < A \log n;$$

and so
$$\sum_{H,K} \sum_{h,k} \frac{1}{k(hK-kH)^4} < A\nu^2 \log n,$$

$$\Sigma \int_{\xi_{h,k}} G^2 d\theta < An^{\frac{3}{2}} (\log n)^A \nu^2 . A\nu^2 \log n < An \; (\log n)^A \nu^4,$$

the result of the lemma.

3. *The main theorems.*

3.1. THEOREM A.—*If Hypothesis* R* *is true, then*

$$\sum_1^n \left(\nu(m) - \Omega(m)\right)^2 = O(n^{\frac{1}{2}+\epsilon})$$

for every positive ϵ.

It is sufficient* to prove that

$$\Sigma\left(\nu(m)-\Omega(m)\right)^2 |x|^{2m} = O(n^{\frac{5}{2}+\epsilon}),$$

when $|x| = e^{-H}$, or that

$$\int_0^{2\pi} |f^2-F|^2 \, d\omega = O(n^{\frac{5}{2}+\epsilon}),$$

when $$x = |x| e^{i\omega} = e^{-H+i\omega}.$$

Now $$\int_0^{2\pi} |f^2-F|^2 \, d\omega = \Sigma \int_\xi |f^2-F|^2 \, d\theta$$

$$< A\Sigma \int_\xi |f^2-\psi^2|^2 \, d\theta + A\Sigma \int_\xi |\psi^2-F|^2 \, d\theta;$$

and $$\Sigma \int_\xi |f^2-\psi^2|^2 \, d\theta = O(n^{\frac{5}{2}+\epsilon}),$$

by Lemma 4. It is therefore sufficient to prove that

$$\Sigma \int_\xi |\psi^2-F|^2 \, d\theta = O(n^{\frac{5}{2}+\epsilon}). \tag{3.11}$$

3.2. Since $F = F_1+F_2$, we have

$$\Sigma \int_\xi |\psi^2-F|^2 \, d\theta < A\Sigma \int_\xi |\psi^2-F_1|^2 \, d\theta + A\Sigma \int_\xi |F_2|^2 \, d\theta; \tag{3.21}$$

and

$$\Sigma \int_\xi |F_2|^2 \, d\theta < An^3(\log n)^A \frac{(\log \nu)^A}{\nu^2}, \tag{3.22}$$

by Lemma 7. Next

$$F_1 = \sum_{H,\, K \leqslant \nu} F_{H,\,K} = F_{h,\,k} + \bar{\Sigma} F_{H,\,K}$$

or $$F_1 = \bar{\Sigma} F_{H,\,K},$$

according as $k \leqslant \nu$ or $k > \nu$, the range of summation indicated by $\bar{\Sigma}$ being defined as in Lemma 11. If $k \leqslant \nu$, then, we have

$$|\psi^2-F_1|^2 < A\,|\psi^2-F_{h,\,k}|^2 + A\,|\bar{\Sigma} F_{H,\,K}|^2 < A\,|\psi^2-F_{h,\,k}|^2 + AG^2;$$

---

* Since $$\sum_1^n a_m^2 \leqslant e^{2nH} \sum_1^n a_m^2 e^{-2mH} \leqslant A \sum_1^\infty a_m^2 |x|^{2m}.$$

while if $k > \nu$ we have

$$|\psi^2-F_1|^2 < A|\psi|^4+A|F_1|^2 < A|\psi|^4+A|\bar{\Sigma} F_{H,K}|^2 < A|\psi|^4+AG'$$

Hence

$$(3.23)\qquad \Sigma\int_\xi |\psi^2-F_1|^2\,d\theta < A\sum_{k\leqslant\nu}\int_\xi |\psi^2-F_{h,k}|^2\,d\theta$$

$$+A\sum_{k>\nu}\int_\xi |\psi|^4\,d\theta+A\Sigma\int_\xi G^2 d\theta$$

$$< A+A\frac{n^3(\log\nu)^A}{\nu^2}+An^{\frac{5}{2}}(\log n)^A\nu^4,$$

by Lemmas 8, 9, and 11. From (3.21), (3.22), and (3.23), we conclude that

$$\Sigma\int_\xi |\psi^2-F|^2\,d\theta < An^3(\log n)^A\frac{(\log\nu)^A}{\nu^2}+An^{\frac{5}{2}}(\log n)^A\nu^4.$$

Taking $\nu = [n^{\frac{1}{12}}]$, we obtain (3.11), and complete the proof of the theorem.

3.2. THEOREM B.—*If Hypothesis* R* *is true, then the number of even numbers less than* n, *for which Goldbach's Theorem is false, is* $O(n^{\frac{1}{2}+\epsilon})$ *for every positive* ϵ.

We have $\Omega(m) = 0$, if m is odd, and

$$\Omega(m) = Am\,\Pi\left(\frac{P-1}{P-2}\right) > Am,$$

if m is even; and so

$$\big(\nu(m)-\Omega(m)\big)^2 = \big(\Omega(m)\big)^2 > Am^2,$$

for every even number m for which Goldbach's Theorem is false. The number of such numbers between $\frac{1}{2}n$ and n (inclusive of the limits) is therefore less than

$$B(\epsilon)n^{\frac{1}{2}+\epsilon},$$

where $B(\epsilon)$ is a function of ϵ only; and the number of such numbers less than n is less than

$$B(\epsilon)\,n^{\frac{1}{2}+\epsilon}\big(1+(\tfrac{1}{2})^{\frac{1}{2}+\epsilon}+(\tfrac{1}{4})^{\frac{1}{2}+\epsilon}+\dots\big) < B(\epsilon)n^{\frac{1}{2}+\epsilon}.$$

4. *Conclusion.*

4.1. It may be observed that the main conclusion of P. N. 3, viz. that *every large odd number is* (on Hypothesis R*) *the sum of three primes*, is

an immediate corollary of Theorem B. For, if n is a large odd number which is not the sum of three primes, then none of the numbers

$$n-p \quad (2 < p < n)$$

can be the sum of two, which obviously contradicts Theorem B.

The method which we have followed here has other important applications. It enables us, for example, to prove that almost all numbers are sums of

five cubes,

sixteen biquadrates,

two squares and a cube (or any odd power),

two squares and a prime,

—the last theorem being, naturally, subject to Hypothesis R*. We shall prove the first two of these assertions in the next memoir of the series, in which we return to Waring's Problem.

(*d*) Inaugural Lecture

(Oxford 1920)

SOME FAMOUS PROBLEMS

of the

THEORY OF NUMBERS

and in particular

Waring's Problem

An Inaugural Lecture delivered before the

University of Oxford

BY

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SOME FAMOUS PROBLEMS OF THE THEORY OF NUMBERS.

It is expected that a professor who delivers an inaugural lecture should choose a subject of wider interest than those which he expounds to his ordinary classes. This custom is entirely reasonable; but it leaves a pure mathematician faced by a very awkward dilemma. There are subjects in which only what is trivial is easily and generally comprehensible. Pure mathematics, I am afraid, is one of them; indeed it is more: it is perhaps the one subject in the world of which it is true, not only that it is genuinely difficult to understand, not only that no one is ashamed of inability to understand it, but even that most men are more ready to exaggerate than to dissemble their lack of understanding.

There is one method of meeting such a situation which is sometimes adopted with considerable success. The lecturer may set out to justify his existence by enlarging upon the overwhelming importance, both to his University and to the community in general, of the particular studies on which he is engaged. He may point out how ridiculously inadequate is the recognition at present afforded to them; how urgent it is in the national interest that they should be largely and immediately re-endowed; and how immensely all of us would benefit were we to entrust him and his colleagues with a predominant voice in all questions of educational administration. I have observed friends of my own, promoted to chairs of various subjects in various

Universities, addressing themselves to this task with an eloquence and courage which it would be impertinent in me to praise. For my own part, I trust that I am not lacking in respect either for my subject or myself. But, if I am asked to explain how, and why, the solution of the problems which occupy the best energies of my life is of importance in the general life of the community, I must decline the unequal contest: I have not the effrontery to develop a thesis so palpably untrue. I must leave it to the engineers and the chemists to expound, with justly prophetic fervour, the benefits conferred on civilization by gas-engines, oil, and explosives. If I could attain every scientific ambition of my life, the frontiers of the Empire would not be advanced, not even a black man would be blown to pieces, no one's fortune would be made, and least of all my own. A pure mathematician must leave to happier colleagues the great task of alleviating the sufferings of humanity.

I suppose that every mathematician is sometimes depressed, as certainly I often am myself, by this feeling of helplessness and futility. I do not profess to have any very satisfactory consolation to offer. It is possible that the life of a mathematician is one which no perfectly reasonable man would elect to live. There are, however, one or two reflections from which I have sometimes found it possible to extract a certain amount of comfort. In the first place, the study of mathematics is, if an unprofitable, a perfectly harmless and innocent occupation, and we have learnt that it is something to be able to say that at any rate we do no harm. Secondly, the scale of the universe is large, and, if we are wasting our time, the waste of the lives of a few university dons is no such overwhelming catastrophe. Thirdly, what we do may be small, but it has a certain character of permanence; and to have pro-

duced anything of the slightest permanent interest, whether it be a copy of verses or a geometrical theorem, is to have done something utterly beyond the powers of the vast majority of men. And, finally, the history of our subject does seem to show conclusively that it is no such mean study after all. The mathematicians of the past have not been neglected or despised; they have been rewarded in a manner, undiscriminating perhaps, but certainly not ungenerous. At all events we can claim that, if we are foolish in the object of our devotion, we are only in our small way aping the folly of a long line of famous men, and that, in these days of conflict between ancient and modern studies, there must surely be something to be said for a study which did not begin with Pythagoras, and will not end with Einstein, but is the oldest and the youngest of all.

It seemed to me for a moment, when I was considering what subject I should choose, that there was perhaps one which might, in a philosophic University like this, be of wider interest than ordinary technical mathematics. If modern pure mathematics has any important applications, they are the applications to philosophy made by the mathematical logicians of the last thirty years. In the sphere of philosophy we mathematicians put forward a strictly limited but absolutely definite claim. We do not claim that we hold in our hands the key to all the riddles of existence, or that our mathematics gives us a vision of reality to which the less fortunate philosopher cannot attain; but we do claim that there are a number of puzzles, of an abstract and elusive kind, with which the philosophers of the past have struggled ineffectually, and of which we now can give a quite definite and explicit solution. There is a certain region of philosophical territory which it is our intention to annex. It is a strictly

demarcated region, but it has suffered under the misrule of philosophers for generations, and it is ours by right; we propose to accept the mandate for it, and to offer it the opportunity of self-determination under the mathematical flag. Such at any rate is the thesis which I hope it may before long be my privilege to defend.

It seemed to me, however, when I considered the matter further, that there are two fatal objections to mathematical philosophy as a subject for an inaugural address. In the first place the subject is one which requires a certain amount of application and preliminary study. It is not that it is a subject, now that the foundations have been laid, of any extraordinary difficulty or obscurity; nor that it demands any wide knowledge of ordinary mathematics. But there are certain things that it does demand; a little thought and patience, a little respect for mathematics, and a little of the mathematical habit of mind which comes fully only after long years spent in the company of mathematical ideas. Something, in short, may be learnt in a term, but hardly in a casual hour.

In the second place, I think that a professor should choose, for his inaugural lecture, a subject, if such a subject exists, to which he has made himself some contribution of substance and about which he has something new to say. And about mathematical philosophy I have nothing new to say; I can only repeat what has been said by the men, Cantor and Frege in Germany, Peano in Italy, Russell and Whitehead in England, who have originated the subject and moulded it now into something like a definite form. It would be an insult to my new University to offer it a watered synopsis of some one else's work. I have therefore finally decided, after much hesitation, to take a subject which is quite frankly mathematical, and to give a summary account of the results of some researches which,

whether or no they contain anything of any interest or importance, have at any rate the merit that they represent the best that I can do.

My own favourite subject has certain redeeming advantages. It is a subject, in the first place, in which a large proportion of the most remarkable results are by no means beyond popular comprehension. There is nothing in the least popular about its *methods*; as to its votaries it is the most beautiful, so by common consent it is the most difficult of all branches of a difficult science; but many of the actual results are such as can be stated in a simple and striking form. The subject has also a considerable historical connexion with this particular chair. I do not wish to exaggerate this connexion. It must be admitted that the contributions of English mathematicians to the Theory of Numbers have been, in the aggregate, comparatively slight. Fermat was not an Englishman, nor Euler, nor Gauss, nor Dirichlet, nor Riemann; and it is not Oxford or Cambridge, but Göttingen, that is the centre of arithmetical research to-day. Still, there has been an English connexion, and it has been for the most part a connexion with Oxford and with the Savilian chair.

The connexion of Oxford with the theory of numbers is in the main a nineteenth-century connexion, and centres naturally in the names of Sylvester and Henry Smith. There is a more ancient, if indirect, connexion which I ought not altogether to forget. The theory of numbers, more than any other branch of pure mathematics, has begun by being an empirical science. Its most famous theorems have all been conjectured, sometimes a hundred years or more before they have been proved; and they have been suggested by the evidence of a mass of computation. Even now there is a considerable part to be played by the computer; and a man who has to undertake

laborious arithmetical computations is hardly likely to forget what he owes to Briggs. However, this is ancient history, and it is with Sylvester and Smith that I am concerned to-day, and more particularly with Smith.

Henry Smith was very many things, but above all things a most brilliant arithmetician. Three-quarters of the first volume of his memoirs is occupied with the theory of numbers, and Dr. Glaisher, his mathematical biographer, has observed very justly that, even when he is primarily concerned with other matters, the most striking feature of his work is the strongly arithmetical spirit which pervades the whole. His most remarkable contributions to the theory are contained in his memoirs on the arithmetical theory of forms, and in particular in the famous memoir on the representation of numbers by sums of five squares, crowned by the Paris Academy and published only after his death. This memoir is peculiarly interesting to me, for the problem is precisely one of those of which I propose to speak to-day; and I may perhaps add one comment on the surprising history set out in Dr. Glaisher's introduction. The name of Minkowski is familiar to-day to many, even in Oxford, who have certainly never read a line of Smith. It is curious to contemplate at a distance the storm of indignation which convulsed the mathematical circles of England when Smith, bracketed after his death with the then unknown German mathematician, received a greater honour than any that had been paid to him in life.

The particular problems with which I am concerned belong to what is called the 'additive' side of higher arithmetic. The general problem may be stated as follows.

Suppose that n is any positive integer, and

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

positive integers of some special kind, squares, for example,

or cubes, or perfect kth powers, or primes. We consider all possible expressions of n in the form

$$n = \alpha_1 + \alpha_2 + \ldots + \alpha_s,$$

where s may be fixed or unrestricted, the α's may or may not be necessarily distinct, and order may or may not be relevant, according to the particular problem on which we are engaged. We denote by

$$r(n)$$

the number of representations which satisfy the conditions of the problem. Then *what can we say about* $r(n)$? Can we find an exact formula for $r(n)$, or an approximate formula valid for large values of n? In particular, is $r(n)$ *always positive*? Is it always possible, that is to say, to find at least *one* representation of n of the type required? Or, if this is not so, is it at any rate always possible when n is sufficiently large?

I can illustrate the nature of the general problem most simply by considering for a moment an entirely trivial case. Let us suppose that there are three different α's only, viz. the numbers 1, 2, 3; that repetitions of the same α are permissible; that the order of the α's is irrelevant; and that s, the number of the α's, is unrestricted. Then it is easy to see that $r(n)$, the number of representations, is the number of solutions of the equation

$$n = x + 2y + 3z$$

in positive integers, including zero.

There are various ways of solving this extremely simple problem. The most interesting for our present purpose is that which rests on an analytical foundation, and uses the idea of the *generating function*

$$f(x) = 1 + \sum_1^\infty r(n) x^n,$$

in which the coefficients are the values of the arithmetical function $r(n)$. It follows immediately from the definition of $r(n)$ that

$$f(x) = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)$$
$$= \frac{1}{(1-x)(1-x^2)(1-x^3)};$$

and, in order to determine the coefficients in the expansion, nothing more than a little elementary algebra is required. We find, by the ordinary theory of partial fractions, that

$$f(x) = \frac{1}{6(1-x)^3} + \frac{1}{4(1-x)^2} + \frac{17}{72(1-x)} + \frac{1}{8(1+x)} + \frac{1}{9(1-\omega x)} + \frac{1}{9(1-\omega^2 x)},$$

where ω and ω^2 denote as usual the two complex cube roots of unity. Expanding the fractions, and picking out the coefficient of x^n, we obtain

$$r(n) = \frac{(n+3)^2}{12} - \frac{7}{72} + \frac{(-1)^n}{8} + \frac{2}{9}\cos\frac{2n\pi}{3}.$$

It is easily verified that the sum of the last three terms can never be as great as $\frac{1}{2}$, so that $r(n)$ is the integer nearest to

$$\frac{(n+3)^2}{12}.$$

The problem is, as I said, quite trivial, but it is interesting none the less. A great deal of work has been done on problems similar in kind, though naturally far more complex and difficult in detail, by Cayley and Sylvester, for example, in the last century, and by Glaisher, and above all by Macmahon, in this. And even this problem,

simple as it is, has sufficient content to bring out clearly certain principles of cardinal importance.

In particular, the solution of the problem shows quite clearly that, if we are to attack these 'additive' problems by analytic methods, it is in the theory of integral power series

$$\sum a_n x^n$$

that the necessary machinery must be found. It is this characteristic which distinguishes this theory sharply from the other great side of the analytic theory of numbers, the 'multiplicative' theory, in which the fundamental idea is that of the resolution of a number into primes. In the latter theory the right weapon is generally not a power series, but what is called a Dirichlet's series, a series of the type

$$\sum a_n n^{-s}.$$

It is easy to see this by considering a simple example. One of the most interesting functions of the multiplicative theory is $d(n)$, the number of divisors of n. The associated power series

$$\sum d(n) x^n$$

is easily transformed into the series

$$\sum \frac{x^n}{1-x^n},$$

called Lambert's series. The function is an interesting one, but somewhat unmanageable, and certainly not one of the fundamental functions of analysis. The corresponding Dirichlet's series is far more fundamental; it is in fact

$$\sum \frac{d(n)}{n^s} = \left(\sum \frac{1}{n^s}\right)^2 = \left(\zeta(s)\right)^2,$$

the square of the famous Zeta function of Riemann.

The underlying reason for this distinction is fairly obvious. It is natural to *multiply* primes and unnatural to *add* them. Now

$$m^{-s} \times n^{-s} = (mn)^{-s},$$

so that, in the theory of Dirichlet's series, the terms combine naturally with one another in a 'multiplicative' manner. But

$$x^m \times x^n = x^{m+n},$$

so that the multiplication of two terms of a power series involves an additive operation on their ranks. It is thus that the Dirichlet's series rather than the power series proves to be the proper weapon in the theory of primes.

It would be difficult for anybody to be more profoundly interested in anything than I am in the theory of primes; but it is not of this theory that I propose to speak to-day, and we must return to our general additive problem. As soon as we try to specialize the problem in some more interesting manner, two problems stand out as calling for research. Each of them, naturally, is only the representative of a class.

The first of these problems is the *problem of partitions*. Let us suppose now that the α's are *any* positive integers, and that (as in the trivial problem) repetitions are allowed, order is irrelevant, and s is unrestricted. The problem is then that of expressing n in any manner as a sum of integral parts, or of solving the equation

$$n = x + 2y + 3z + 4u + 5v + \ldots,$$

and $r(n)$ or, as it is now more naturally written, $p(n)$, is the number of *unrestricted partitions* of n. Thus

$$5 = 1+1+1+1+1 = 1+1+1+2$$
$$= 1+2+2 = 1+1+3 = 2+3 = 1+4 = 5,$$

so that $p(5)=7$. The generating function in this case was found by Euler, and is

$$f(x)=1+\sum_{1}^{\infty}p(n)x^n=\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

I do not wish to discuss this problem in any detail now, but the form of the generating function calls for one or two general remarks. In the trivial problem the generating function was *rational*, with a finite number of poles all situated upon the unit circle. Here also we are led to a power series, or infinite product, convergent inside the unit circle; but there the resemblance ends. This function will be recognized by any one familiar with the theory of elliptic functions; it is an elliptic modular function; and, like all such functions, it has the unit circle as a continuous line of singularities and does not exist at all outside. It is easy to imagine the immensely increased difficulties of any analytic solution of the problem.

It was conjectured by a very brilliant Hungarian mathematician, Mr. G. Pólya, five or six years ago, that *any* function represented by a power series whose coefficients are *integers*, and which is convergent inside the unit circle, must behave, in this respect, like one or other of the two generating functions which we have considered. Either such a function is a rational function, that is to say, completely elementary; or else the unit circle is a line of essential singularities. I believe that a proof of this theorem has now been found by Mr. F. Carlson of Upsala, and is to be published shortly in the *Mathematische Zeitschrift*. It is difficult for me to give reasoned praise to a memoir which I have not seen, but I can recommend the theorem to your attention with confidence as one of the most beautiful of recent years.

The problem of partitions is one to which, in collaboration with the Indian mathematician, Mr. S. Ramanujan, I have

myself devoted a great deal of work. The principal result of our work has been the discovery of an approximate formula for $p(n)$ in which the leading term is

$$\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\frac{e^{\frac{2\pi}{\sqrt{6}}\sqrt{n-\frac{1}{24}}}}{\sqrt{n-\frac{1}{24}}},$$

and which enables us to approximate to $p(n)$ with an accuracy which is almost uncanny. We are able, for example, by using 8 terms of our formula, to calculate $p(200)$, a number of 13 figures, with an error of ·004. I have set out the details of the calculation in Table I.

TABLE I

$p(200)$

	3,972,998,993,185·896
	36,282·978
−	87·555
+	5·147
+	1·424
+	0·071
+	0·000
+	0·043
	3,972,999,029,388·004

The value of $p(200)$ was subsequently verified by Major MacMahon, by a direct computation which occupied over a month.

The formulae connected with this problem are very elaborate, and except on the purely numerical side, where the results of the theory are compared with those of computation, it is not very well suited for a hasty exposition; and I therefore pass on at once to the principal object of my lecture, the very famous problem known, after a Cambridge professor of the eighteenth century, as *Waring's Problem.*

We suppose now that every α is a perfect k-th power m^k, k being fixed in each case of the problem which we consider; m may be of either sign if k is even, but must be positive if k is odd. In either case we allow m to be zero. Repetitions are permitted, as in our previous problems; but it is more convenient now to take account of the order of the α's; and s, which was formerly unrestricted, is now fixed in each case of the problem, like k. The problem is therefore that of determining the number of representations of a number n as the sum of s positive k-th powers. Thus Henry Smith's problem, the problem of five squares, is the particular case of Waring's problem in which k is 2 and s is 5. The problem has a long history, which centres round this simplest case of squares; a history which began, I suppose, with the right-angled triangles of Pythagoras, and has been continued by a long succession of mathematicians, including Fermat, Euler, Lagrange, and Jacobi, down to the present day. I will begin by a summary of what is known in the simplest case, where the solution is practically complete.

A number n is the sum of two squares if and only if it is of the form

$$n = M^2 P,$$

where P is a product of primes, all different and all of the form $4k+1$. In particular, a prime number of the form $4k+1$ can be expressed as the sum of two squares, and substantially in only one way. Thus $5 = 1^2 + 2^2$, and there is no other solution except the solutions $(\pm 1)^2 + (\pm 2)^2$, $(\pm 2)^2 + (\pm 1)^2$, which are not essentially different, although it is convenient to count them as distinct. The number of numbers less than x, and expressible as the sum of two squares, is approximately

$$\frac{Cx}{\sqrt{\log x}},$$

where C is a certain constant. The last result was proved by Landau in 1908; all the rest belong to the classical theory.

A number is the sum of three squares unless it is of the form

$$4^a(8k+7),$$

when it is not so expressible. *Every* number may be expressed by four squares, and *a fortiori* by five or more. It is this last theorem of Lagrange that I would ask you particularly to bear in mind.

If s, the number of squares, is even and less than 10, the number of representations may be expressed in a very simple form by means of the divisors of n. Thus the number of representations by 4 squares, when n is odd, is 8 times the sum of the divisors of n; when n is even, it is 24 times the sum of the odd divisors; and there are similar results for 2 squares, or 6, or 8.

When s is 3, 5, or 7, the number of representations can also be found in a simple form, though one of a very different character. Suppose, for example, that s is 3. The problem is in this case essentially the same as that of determining the number of classes of binary arithmetical forms of determinant $-n$; and the solution depends on certain finite sums of the form

$$\sum \beta, \quad \sum \gamma,$$

extended over quadratic residues β or non-residues γ of n.

When s, whether even or odd, is greater than 8, the solution is decidedly more difficult, and it is only very recently that a uniform method of solution, for which I must refer you to some recent memoirs of Mr. L. J. Mordell and myself, has been discovered. For the moment I wish to concentrate your attention on two points: the first, that *an expression by 4 squares is always possible, while one*

by 3 *is not*; and the second, that the existence of numbers not expressible by 3 squares is revealed at once by the quite trivial observation that no number so expressible can be congruent to 7 to modulus 8.

It is plain, when we proceed to the general case, that any number n can be expressed as a sum of k-th powers; we have only to take, for example, the sum of n ones. And, when n is given, there is a *minimum* number of k-th powers in terms of which n can be expressed; thus

$$23 = 2 \cdot 2^3 + 7 \cdot 1^3$$

is the sum of 9 cubes and of no smaller number. But it is not at all plain (and this is the point) that this minimum number cannot tend to infinity with n. It does not when $k = 2$; for then it cannot exceed 4. And Waring's Problem (in the restricted sense in which the name has commonly been used) is the problem of proving that the minimum number is similarly bounded in the general case. It is not an easy problem; its difficulty may be judged from the fact that it took 127 years to solve.

We may state the problem more formally as follows. Let k be given. Then there may or may not exist a number m, the same for all values of n, and such that n can always be expressed as the sum of m k-th powers or less. If any number m possesses this property, all larger numbers plainly possess it too; and among these numbers we may select the *least*. This least number, which will plainly depend on k, we call $g(k)$; thus $g(k)$ is, by definition, the least number, if such a number exists, for which it is true that

'*every number is the sum of* $g(k)$ *k-th powers or less*'.

We have seen already that $g(2)$ exists and has the value 4.

In the third edition of his *Meditationes Algebraicae*, published in Cambridge in 1782, Waring asserted that

every number is the sum of not more than 4 squares, not more than 9 cubes, not more than 19 fourth powers, *et sic deinceps.* A little more precision would perhaps have been desirable; but it has generally been held, and I do not question that it is true, that what Waring is asserting is precisely the existence of $g(k)$. He implies, moreover, that $g(2) = 4$ and $g(3) = 9$; and both of these assertions are correct, though in the first he had been anticipated by Lagrange. Whether $g(4)$ is or is not equal to 19 is not known to-day.

Waring advanced no argument of any kind in support of his assertion, and it is in the highest degree unlikely that he was in possession of any sort of proof. I have no desire to detract from the reputation of a man who was a very good mathematician if not a great one, and who held a very honourable position in a University which not even Oxford has persuaded me entirely to forget. But there is a tendency to exaggerate the profundity implied by the enunciation of a theorem of this particular kind. We have seen this even in the case of Fermat, a mathematician of a class to which Waring had not the slightest pretensions to belong, whose notorious assertion concerning 'Mersenne's numbers' has been exploded, after the lapse of over 250 years, by the calculations of the American computer Mr. Powers. No very laborious computations would be necessary to lead Waring to a highly plausible speculation, which is all I take his contribution to the theory to be; and in the Theory of Numbers it is singularly easy to speculate, though often terribly difficult to prove; and it is only proof that counts.

The next advance towards the solution of the problem was made by Liouville, who established the existence of $g(4)$. Liouville's proof, which was first published in 1859, is quite simple and, as the simplest example of an important

type of argument, is worth reproducing here. It may be verified immediately that

$$\begin{aligned}6X^2 = 6(x^2+y^2+z^2+t^2)^2 \\ &= (x+y)^4+(x-y)^4+(z+t)^4+(z-t)^4 \\ &+ (x+z)^4+(x-z)^4+(t+y)^4+(t-y)^4 \\ &+ (x+t)^4+(x-t)^4+(y+z)^4+(y-z)^4;\end{aligned}$$

and since, by Lagrange's theorem, any number X is the sum of 4 squares, it follows that any number of the form $6X^2$ is the sum of 12 biquadrates. Hence any number of the form $6(X^2+Y^2+Z^2+T^2)$ or, what is the same thing, any number of the form $6m$, is the sum of 48 biquadrates. But *any* number n is of the form $6m+r$, where r is 0, 1, 2, 3, 4, or 5. And therefore n is, at worst, the sum of 53 biquadrates. That is to say, $g(4)$ exists, and does not exceed 53. Subsequent investigators, refining upon this argument, have been able to reduce this number to 37; the final proof that $g(4) \leqq 37$, the most that is known at present, was given by Wieferich in 1909. The number

$$79 = 4 \cdot 2^4 + 15 \cdot 1^4$$

needs 19 biquadrates, and no number is known which needs more. There is therefore still a wide margin of uncertainty as to the actual value of $g(4)$.

The case of cubes is a little more difficult, and the existence of $g(3)$ was not established until 1895, when Maillet proved that $g(3) \leqq 17$. The proof then given by Maillet rests upon the identity

$$\begin{aligned}6x(x^2+y^2+z^2+t^2) \\ &= (x+y)^3+(x-y)^3+(x+z)^3+(x-z)^3+(x+t)^3+(x-t)^3,\end{aligned}$$

and the known results concerning the expression of a number by 3 squares. It has not the striking simplicity of Liouville's proof; but it has enabled successive investigators to reduce the number of cubes, until finally Wieferich,

in 1909, proved that $g(3) \leqq 9$. As 23 and 239 require 9 cubes, the value of $g(3)$ is in fact exactly 9. It is only for $k = 2$ and $k = 3$ that the actual value of $g(k)$ has been determined. But similar existence proofs were found, and upper bounds for $g(k)$ determined, by various writers, in the cases $k = 5, 6, 7, 8$, and 10.

Before leaving the problem of the cubes I must call your attention to another very beautiful theorem of a slightly different character. The numbers 23 and 239 require 9 cubes, and it appears, from the results of a survey of all numbers up to 40,000, that no other number requires so many. It is true that this has not actually been proved; but it *has* been proved (and this is of course the essential point) that the number of numbers which require as many cubes as 9 is *finite*.

This singularly beautiful theorem, which is due to my friend Professor Landau of Göttingen, and is to me as fascinating as anything in the theory, also dates from 1909, a year which stands out for many reasons in the history of the problem. It is of exceptional interest not only in itself but also on account of the method by which it was proved, which utilizes some of the deepest results in the modern theory of the asymptotic distribution of primes, and made it, until very recently, the only theorem of its kind erected upon a genuinely transcendental foundation. To me it has a personal interest also, as being the only theorem of the kind which, up to the present, defeats the new analytic method by which Mr. Littlewood and I have recently studied the problem.

Landau's theorem suggests the introduction of another function of k, which I will call $G(k)$, of the same general character as $g(k)$, but I think probably more fundamental. This number $G(k)$ is defined as being the least number for which it is true that

'*every member* FROM A CERTAIN POINT ONWARDS *is the sum of $G(k)$ k-th powers or less.*'

It is obvious that the existence of $g(k)$ involves that of $G(k)$, and that $G(k) \leqq g(k)$. When $k = 2$, both numbers are 4; but $G(3) \leqq 8$, by Landau's theorem, while $g(3) = 9$; and doubtless $G(k) < g(k)$ in general. It is important also to observe that, conversely, the existence of $G(k)$ involves that of $g(k)$. For, if $G(k)$ exists, all numbers beyond a certain limit γ are sums of $G(k)$ k-th powers or less. But all numbers less than γ are sums of γ ones or less, and therefore $g(k)$ certainly cannot exceed the greater of $G(k)$ and γ.

I said that $G(k)$ seemed to me the more fundamental of these numbers, and it is easy to see why. Let us assume (as is no doubt true) that the only numbers which require 9 cubes for their expression are 23 and 239. This is a very curious fact which should be interesting to any genuine arithmetician; for it ought to be true of an arithmetician that, as has been said of Mr. Ramanujan, and in his case at any rate with absolute truth, that 'every positive integer is one of his personal friends'. But it would be absurd to pretend that it is one of the profounder truths of higher arithmetic: it is nothing more than an entertaining arithmetical fluke. It is Landau's 8 and not Wieferich's 9 that is the profoundly interesting number.

The real value of $G(3)$ is still unknown. It cannot be less than 4; for every number is congruent to 0, or 1, or -1 to modulus 3, and it is an elementary deduction that every cube is congruent to 0, or 1, or -1 to modulus 9. From this it follows that the sum of three cubes cannot be of the form $9m+4$ or $9m+5$: for such numbers at least 4 cubes are necessary, so that $G(3) \geqq 4$. But whether $G(3)$ is 4, 5, 6, 7, or 8 is one of the deepest mysteries of arithmetic.

It is worth while to glance at the evidence of computation.

Dase, at the instance of Jacobi, tabulated the minimum number of cubes for values of n from 1 to 12,000, and Daublensky von Sterneck has extended the table to 40,000. Some of the results are shown in Table II.

TABLE II

	1	2	3	4	5	6	7	8	9
1- 1000	10	41	122	242	293	202	73	15	2
1000- 2000	2	27	113	283	358	194	23	—	—
9000-10000	1	17	121	377	401	83	—	—	—
19000-20000	1	12	100	400	426	61	—	—	—
29000-30000	1	11	105	448	388	47	—	—	—
39000-40000	1	13	117	457	384	28	—	—	—

In each row I have shown a typical thousand numbers, classified according to the minimum number of cubes by which they can be expressed. There are 15 numbers only for which 8 are needed, the largest being 454. There are 121 for which 7 are needed, the two largest being 5818 and 8042; the distribution of these 121 numbers in the first 9 thousands is

73, 23, 7, 6, 7, 4, 0, 0, 1.

If empirical evidence means anything, it seems clear that $G(3) \leqq 6$. I am sure that Professor Townsend and Professor Lindemann have made countless generalizations on evidence far less substantial.

It is also clear that, throughout von Sterneck's tables, there is a fairly steady, though latterly very slow, decrease in the proportion of numbers for which even 6 cubes are required; but that the table is not sufficiently extensive to give any very decisive indication as to whether these numbers disappear or not. It seemed to me this was a case in which further evidence would be worth having. To calculate a *systematic* table on the scale required would be a work of years. It is possible, however, to obtain some indication

of the probable truth, without any superhuman patience, by exploring a selected stratum of much larger numbers. Dr. Ruckle of Göttingen recently undertook this task at my request, and I am glad to be able to tell you his results. He found, for the 2,000 numbers immediately below 1,000,000, the following distribution.

	1	2	3	4	5	6	7
998000–999000	0	1	98	640	262	1	0
999000–1000000	1	1	94	614	289	1	0

You will observe that the 6-cube numbers have all but disappeared, and that there is a quite marked turnover from 5 to 4. Conjecture in such a matter is extremely rash, but I am on the whole disposed to predict with some confidence that $G(3) \leqq 5$. If I were asked to choose between 5 and 4, all I could say would be this. That $G(3)$ should be 4 would harmonize admirably, so far as we can see at present, with the general trend of Mr. Littlewood's and my researches. But it is plain that, if the 5-cube numbers too do ultimately disappear, it can only be among numbers the writing of which would tax the resources of the decimal notation; and at present we cannot *prove* even that $G(3) \leqq 7$, though here success seems not impossible.

With the fourth powers or biquadrates we have been very much more successful. I have explained that $g(4)$ lies between 19 and 37. As regards $G(4)$, we have here no numerical evidence on the same scale as for cubes. Any fourth power is congruent to 0 or 1 to modulus 16, and from this it follows that no number congruent to 15 to modulus 16 can be the sum of less than 15 fourth powers. Thus $G(4) \geqq 15$; and Kempner, by a slight elaboration of this simple argument, has proved that $G(4) \geqq 16$. No better upper bound was known before than the 37 of Wieferich, but here Mr. Littlewood and I have been able

to make a very substantial improvement, first to 33 and finally to 21. Thus $G(4)$ lies between 16 and 21, and the margin is comparatively small.

I turn now to the general case. In the years up to 1909, the existence proof was effected, and upper bounds for $g(k)$ determined, for the values of k from 2 to 8 inclusive and for $k = 10$. These upper bounds are shown in the first row of Table III; that for 10, which is not included, is somewhere in the neighbourhood of 140,000.

TABLE III.

	2	3	4	5	6	7	8
$g(k) \leqq$	4	9	37	58	478	3806	31353
$g(k) \geqq [(\frac{3}{2})^k] + 2^k - 2 =$	4	9	19	37	73	143	279
$G(k) \leqq$	4	[8]	37	58	478	3806	31353
$G(k) \leqq (k-2)2^{k-1} + 5 =$	(5)	(9)	**21**	**53**	**133**	**325**	**773**
$G(k) \geqq k+1, 4k$	4	4	16	6	7	8	32

In the second row I have shown the best known lower bounds, which are given by the simple general formula which stands to the left, in which $[(\frac{3}{2})^k]$ denotes the integral part of $(\frac{3}{2})^k$. It is easily verified, in fact, that the number

$$([(\tfrac{3}{2})^k] - 1)\, 2^k + 2^k - 1,$$

which is less than 3^k, requires the number of k-th powers stated.[1] It will be observed that the first three numbers are those which occur in Waring's enunciation.

Waring's problem, as I have defined it—the problem, that is to say, of finding a general existence proof for $g(k)$, and *a fortiori* for $G(k)$—was ultimately solved by Hilbert, once more in 1909. I wish that I had time to give a proper account of his justly famous memoir, which raised the whole discussion at once on to an

[1] This observation was made by Bretschneider in 1853.

altogether higher level. As it is, I must confine myself to one or two extremely inadequate remarks. The proof falls into two parts. The first part is what I may call semi-transcendental. It is not fully transcendental in the sense in which, for example, the classical proofs in the theory of the distribution of primes are transcendental, for it does not make use of the machinery of the theory of analytic functions of a complex variable; but it uses the methods of the integral calculus, and is therefore not fully elementary. Hilbert set out with what would appear at first sight to be the singularly ill-adapted weapon of a volume integral in space of 25 dimensions, a number which he was afterwards able to reduce to 5. The formula which he ultimately used is

$$(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2)^k$$
$$= C\iiiiint (x_1t_1+x_2t_2+x_3t_3+x_4t_4+x_5t_5)^{2k}\, dt_1 \ldots dt_5,$$

where C is a certain constant, viz.

$$\frac{(2k+1)(2k+3)(2k+5)}{8\pi^2},$$

and the integration is effected over the interior of the hypersphere

$$t_1^2+t_2^2+t_3^2+t_4^2+t_5^2 = 1.$$

Starting from this formula he was able, by an exceedingly ingenious process based upon the definition of a definite integral as the limit of a finite sum, to prove the existence in the general case of algebraical identities analogous to that used by Liouville and his followers when k is 4. It should be observed that Hilbert's proof is essentially an *existence proof*; his method is not effective for the actual determination of these identities

even in the simplest cases. The identities which are known for special values of k have been obtained by common algebra, and are, after the first few values of k, excessively complicated. The simplest known identity for $k = 10$, for instance, is

$$22680\,(x_1^2+x_2^2+x_3^2+x_4^2)^5$$

$$= 9\sum^{(8)} (x_1 \pm x_2 \pm x_3 \pm x_4)^{10} + \sum^{(48)} (2x_1 \pm x_2 \pm x_3)^{10}$$

$$+\,180\sum^{(12)} (x_1 \pm x_2)^{10} + 9\sum^{(4)} (2x_1)^{10},$$

where the figures in brackets show the number of terms under the signs of summation. However, the identities exist; and it should be clear to you, after our discussion of the case $k = 4$, that they enable us at once to obtain a proof in succession for $k = 2, 4, 8, 16, \ldots$ and generally whenever k is a power of 2. This concludes the first and most characteristic part of Hilbert's argument. The second part, in which the conclusion is extended to every value of k, is purely algebraical.

Hilbert's work has been reconsidered and simplified by a number of writers, most notably by Dr. Stridsberg of Stockholm, and the ultimate result of their work has been to eliminate the transcendental elements from the proof entirely. The proof, as they have left it, is fully elementary; it does not involve any reference to integrals, or to any kind of limiting process, but depends simply on an ingenious system of equations derived by the processes of finite algebra. It remains a pure existence proof, and throws no light on the value of $g(k)$.

It would hardly be possible for me to exaggerate the admiration which I feel for the solution of this historic problem of which I have been compelled to give so bald

and summary a description. Within the limits which it has set for itself, it is absolutely and triumphantly successful, and it stands with the work of Hadamard and de la Vallée-Poussin, in the theory of primes, as one of the landmarks in the modern history of the theory of numbers. But there is an enormous amount which remains to be done, and it would seem that, if we are to interpret Waring's problem in the widest possible sense, if we are to get into real contact with the actual values of our numbers $g(k)$ and $G(k)$, still more if we are to attack all the obvious problems connected with the number of representations, then essentially different and inherently more powerful methods are required. There is one armoury only in which such more powerful weapons can be found, that of the modern theory of functions. In short we must learn how to apply Cauchy's Theorem to the problem, and that is what Mr. Littlewood and I have set out to do.

The first step is fairly obvious. The formulae are slightly simpler when k is *even*. The number of representations of n as the sum of s kth powers, which we may denote in general by

$$r_{k,s}(n),$$

is then the coefficient of x^n in the generating function

$$1 + \sum_{1}^{\infty} r_{k,s}(n)x^n = (f(x))^s,$$

where

$$f(x) = 1 + 2x^{1^k} + 2x^{2^k} + 2x^{3^k} + \dots$$

This formula involves certain conventions as to the order and sign of the numbers which occur in the representations which are to be reckoned as distinct; but the complications so introduced are trivial and I need not dwell on them. The series is convergent when $|x| < 1$, and, by Cauchy's

Theorem, we have

$$r_{k,s}(n) = \frac{1}{2\pi i}\int \frac{(f(x))^s}{x^{n+1}}\,dx,$$

the path of integration being a circle whose centre is at the origin and whose radius is less than unity.

All this is simple enough; but the further study of the integral is very intricate and difficult, and I cannot attempt to do more than to give a rough idea of the obstacles that have to be surmounted. Let us contrast the integral for a moment with that which would stand in its place in the 'trivial' problem to which I referred early in my lecture. There the subject of integration would be a *rational* function, with a finite number of poles all situated on the unit circle. We could deform the contour into one which lies wholly at a considerable distance from the origin and in which, owing to the factor x^{n+1} in the denominator, every element is very small when n is large. We should have, of course, to introduce corrections corresponding to the residues at the poles; and it is just these corrections which would give the dominant terms of an approximate formula by means of which our coefficients could be studied. In the present case we have no such simple recourse; for every point of the unit circle is a singularity of an exceedingly complicated kind, and the circle as a whole is a barrier across which it is impossible to deform the contour. It is of course for this reason that no successful application of the method has been made before.

Our fundamental idea for overcoming the difficulty is as follows. Among the continuous mass of singularities which covers up the circle, it is possible to pick out a class which to a certain extent dominates the rest. These special singularities are those associated with the *rational* points of the circle, that is to say, the points

$$x = e^{2p\pi i/q},$$

where p/q is a rational fraction in its lowest terms. This class of points is indeed an *infinite* class; but the infinity is, in Cantor's phrase, only an *enumerable* infinity; and the points can therefore be arranged in a simply infinite series, on the model of the series

$$\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{7}, \ldots.$$

In the neighbourhood of these points the behaviour of the function is, sufficiently complex indeed, but simpler than elsewhere. The function has, to put the matter in a rough and popular way, a general tendency to become large in the neighbourhood of the unit circle, but this tendency is most pronounced near these particular points. They are not only the *simplest* but also the *heaviest* singularities; their weight is greatest when the denominator q is smallest, decreases as q increases, and (as a physicist would say) becomes infinitely small when q is infinitely large. There is, therefore, at any rate, the hope that we may be able to isolate the contributions of each of these selected points, and obtain, by adding them together, a series which may give a genuine approximation to our coefficient.

I owe to Professor Harald Bohr of Copenhagen a picturesque illustration which may help to elucidate the general nature of our argument. Imagine the unit circle as a thin circular rail, to which are attached an infinite number of small lights of varying intensity, each illuminating a certain angle immediately in front of it. The brightest light is at $x = 1$, corresponding to $p = 0, q = 1$; the next brightest at $x = -1$, corresponding to $p = 1$, $q = 2$; the next at $x = e^{2\pi i/3}$ and $e^{4\pi i/3}$, and so on. We have to arrange the inner circle, the circle of integration, in the position of maximum illumination. If it is too far away the light will not reach it; if too near, the arcs which fall within the angles of illumination will be too

narrow, and the light will not cover it completely. Is it possible to place it where it will receive a satisfactorily uniform illumination?

The answer is that this is *only* possible when k is 2. Our functions are then elliptic functions; the lights are the formulae of the theory of linear transformation; and we can find a position of the inner circle in which it falls entirely under their rays. We are thus led to a solution of the problem of the squares which is in all essential respects complete. But when k exceeds 2 the result is less satisfactory. The angle of the lights is then too narrow; the beams which they emit, instead of spreading out with reasonable regularity, are shaped like torpedoes or cigars; however we move our circle a part remains in darkness. It would seem that this difficulty, which held up our researches for something like two years, is the really characteristic difficulty of the general problem. It cannot be solved until we have found some other source of light.

It was only after the most prolonged and painful efforts that we were able to discover such another source. It is possible not only to hang lights upon the rail, but also, to a certain extent, to cause the rail itself to glow. The illumination which can be induced in this manner is irritatingly faint, and it is for this reason that our results are not yet all that we desire; but it is enough to make the dark places dimly visible and to enable us to prove a great deal more than has been proved before.

The actual results which we obtain are these. We find that there is a certain series, which we call the *singular series*, which is plainly the key to the solution. This series is

$$\mathrm{S} = \sum \left(\frac{S_{p,q}}{q}\right)^s e^{2n p \pi i/q},$$

where

$$S_{p,q} = \sum_{h=0}^{q-1} e^{2h^k p\pi i/q}$$

—a sum which reduces, when $k = 2$, to one of what are known as 'Gauss's sums'—and the summation extends, first to all values of p less than and prime to q, and secondly to all positive integral values of q. The genesis of the series is this. We associate with the rational point $x = e^{2p\pi i/q}$ an auxiliary power series

$$f_{p,q}(x) = \sum_n c_{p,q,n} x^n,$$

which (a) is as simple and natural as we can make it, and (b) behaves perfectly regularly at all points of the unit circle except at the one point with which we are particularly concerned. We then add together all these auxiliary functions, and endeavour to approximate to the coefficient of our original series by summing the auxiliary coefficients over all values of p and q. The process is, at bottom, one of 'decomposition into simple elements', applied in an unusual way.

Our final formula for the number of representations is

$$r_{k,s}(n) = \frac{\left\{2\Gamma\left(1+\frac{1}{k}\right)\right\}^s}{\Gamma\left(1+\frac{s}{k}\right)} n^{\frac{s}{k}-1} \mathfrak{S} + O(n^\sigma),$$

the second term denoting an error less than a constant multiple of n^σ, and σ being a number which is less than $\frac{s}{k} - 1$ at any rate for sufficiently large values of s. The second term is then of lower order than the first. Further, the first term is real, and it may be shown, if s surpasses a certain limit, to be *positive*. If both these conditions are satisfied, and n is sufficiently large, then $r_{k,s}(n)$

cannot be zero, and representations of n by s kth powers certainly exist. The way is thus open to a proof of the existence of $G(k)$; if $G(k)$ exists, so also does $g(k)$, and Waring's problem is solved.

The structure of the dominant term in our general formula is best realized by considering some special cases. In Table IV I have written out the leading terms of $\mathfrak{S}$,

TABLE IV.

$k = 2.$

$$\mathfrak{S} = 1 + 0 + \frac{2}{3^{\frac{1}{2}s}} \cos\left(\frac{2}{3} n\pi - \frac{1}{2} s\pi\right) + \frac{2^{\frac{1}{2}s+1}}{4^{\frac{1}{2}s}} \cos\left(\frac{1}{2} n\pi - \frac{1}{4} s\pi\right)$$
$$+ \frac{2}{5^{\frac{1}{2}s}} \left\{\cos \frac{2}{5} n\pi + \cos\left(\frac{4}{5} n\pi - s\pi\right)\right\} + 0 + \ldots.$$

$k = 3, \; s = 7.$

$$\mathfrak{S} = 1 + 0{\cdot}610 \cos \tfrac{2}{9} n\pi + 0{\cdot}130 \cos \tfrac{2}{7} n\pi + 0{\cdot}078 \cos \tfrac{6}{7} n\pi + \ldots.$$

$k = 4, \; s = 33.$

$$\mathfrak{S} = 1 + 1{\cdot}054 \cos \left(\tfrac{1}{8} n\pi - \tfrac{1}{16}\pi\right) + 0{\cdot}147 \cos \left(\tfrac{1}{4} n\pi - \tfrac{1}{8}\pi\right) + \ldots.$$

$k = 4, \; s = 21.$

$$\mathfrak{S} = 1 + 1{\cdot}331 \cos \left(\tfrac{1}{8} n\pi + \tfrac{11}{16}\pi\right) + 0{\cdot}379 \cos \left(\tfrac{1}{4} n\pi - \tfrac{5}{8}\pi\right) + \ldots.$$

first when $k = 2$ and s is arbitrary, and then for 7 cubes and for 33 and 21 biquadrates. There are certain characteristics common to all these series. The terms diminish rapidly; in each case only a very few are of real importance: and they are oscillatory, with a period which increases as the amplitude of the oscillations decreases. The series for the cubes is easily shown to be positive; but we cannot deduce that $r_{3,7}(n)$ is positive, and draw consequences as to the representation of numbers by 7 cubes,

because in this case we cannot dispose satisfactorily of the error term $O(n^{\sigma})$ in the general formula. In the two cases relating to fourth powers which I have chosen, the discussion of the series itself is rather more delicate, for there is in each of them one term which can be negative and greater than 1. But the discussion can be brought to a satisfactory conclusion, and, as in this case we are able to prove that the error term is really of lower order, we obtain what we desire. *Every large number is the sum of 21 fourth powers or less*; $G(4) \leqq 21$. Further, we have obtained a genuine asymptotic formula for the number of representations, which can be used for the study of the representations of numbers of particular forms. We can show, for example, that a large number of the form $16n + 10$ can be expressed by 21 biquadrates in about 200 times more ways than one of the form $16n + 2$.

If the method of which I have tried to give some general idea is compared with those which have previously been applied to the problem, it will be found that it has three very great advantages. In the first place it is inherently very much more powerful. It brings us for the first time into relation with the series on which the solution in the last resort depends, and tells us, approximately but truly, what the number of representations really is. Secondly, it gives us numerical results which, as soon as k exceeds 3, are far in advance of any known before. These numbers are those in the fourth row of Table III.[1] It will be seen that these numbers conform to a simple law, and that is the third advantage of the method, that it is not a mere

[1] The thick type indicates a new result. The (5) and (9) in round brackets are inferior to results already known. Our method is easily adapted to deal with the case $k = 2$ completely; but it will not at present yield Landau's 8, which is therefore enclosed in square brackets.

existence proof, but gives us a definite upper bound for $G(k)$ for all values of k, viz.

$$G(k) \leqq (k-2)\, 2^{k-1} + 5.$$

In the last row of the table I have shown all that is known about $G(k)$ on the other side. In all cases $G(k) \geqq k+1$, while if k is a power of 2 we can say more, namely that $G(k) \geqq 4k$. A comparison between this row of figures and that above it is enough to show the room which remains for further research. It is beyond question that our numbers are still very much too large; and there is no sort of finality about our researches, for which the best that we claim is that they embody a method which opens the door for more.

I will conclude by one word as to the application of our method to another and a still more difficult problem. It was asserted by Goldbach in 1742 that *every even number is the sum of two odd primes.* Goldbach's assertion remains unproved; it has not even been proved that every number n is the sum of 10 primes, or of 100, or of any number independent of n. Our method is applicable in principle to this problem also. We cannot solve the problem, but we can open the first serious attack upon it, and bring it into relation with the established prime number theory. The most which we can accomplish at present is as follows. We have to assume the truth of the notorious Riemann hypothesis concerning the zeros of the Zeta-function, and indeed in a generalized and extended form. If we do this we can prove, not Goldbach's Theorem indeed, but the next best theorem of the kind, viz. that *every odd number*, at any rate from a certain point onwards, *is the sum of three odd primes.* It is an imperfect and provisional result, but it is the first serious contribution to the solution of the problem.

CORRECTION

p. 31. In the formula for $r_{k,s}(n)$ on p. 31, read $\Gamma(s/k)$ in the denominator.

COMMENT

A French translation, by A. Sallin, with notes by Hardy on subsequent developments, appeared in 1931 as *Trois problèmes célèbres de la théorie des nombres* (les Presses Universitaires de France).

ARRANGEMENT OF THE VOLUMES

VOLUME I

I. 1 Diophantine approximation

I. 2 Additive number theory
- (*a*) Combinatory analysis and sums of squares
- (*b*) Waring's Problem
- (*c*) Goldbach's Problem
- (*d*) Inaugural Lecture (Oxford, 1920)

VOLUME II

II. 1 Multiplicative number theory (including the zeta-function)

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VOLUME III

III. 1 Trigonometric series
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- (*d*) Special trigonometric series
- (*e*) Other papers on trigonometric series

III. 2 Mean values of power series

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IV. 1 Special functions
- (*a*) Zeroes and asymptotic behaviour of particular integral functions
- (*b*) Taylor series and singularities
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- (*d*) Miscellaneous

IV. 2 Theory of functions

VOLUME V

V. Integral calculus

LIST OF PAPERS BY G. H. HARDY

Abbreviations

N.I.C. Notes on some points in the integral calculus
D.A. Some problems of Diophantine approximation
P.N. Some problems of 'Partitio Numerorum'
N.S. Notes on the theory of series

1899

1. Question 13848, *Educational Times*, 70, 43.
2. Question 13917, *Educational Times*, 70, 78–79.
3. Question 14124, *Educational Times*, 71, 100–101.
4. Question 14005, *Educational Times*, 71, 111–112. } VII. 3

1900

1. On a class of definite integrals containing hyperbolic functions, *Messenger of Mathematics*, 29, 25–42. V
2. Question 14243, *Educational Times*, 72, 80–81.
3. Question 14271, *Educational Times*, 73, 36–37.
4. Question 14179, *Educational Times*, 73, 53–54.
5. Question 14317, *Educational Times*, 73, 61–63. } VII. 3

1901

1. On differentiation and integration under the integral sign, *Quarterly Journal of Mathematics*, 32, 66–140. (Corrected in 1915, 2.) V
2. General theorems in contour integration, with some applications, *Quarterly Journal of Mathematics*, 32, 369–384. V
3. N.I.C. I: On the formula for integration by parts, *Messenger of Mathematics*, 30, 185–187. V
4. N.I.C. II: Two general convergence theorems, *Messenger of Mathematics*, 30, 187–190. V
5. Question 14496, *Educational Times*, 74, 37–38.
6. Question 14447, *Educational Times*, 74, 98–100.
7. Question 14467, *Educational Times*, 74, 111–112.
8. Question 14028, *Educational Times*, 74, 122–123.
9. Question 14369, *Educational Times*, 75, 135–136. } VII. 3

1902

1. The elementary theory of Cauchy's principal values, *Proceedings of the London Mathematical Society*, (1) 34, 16–40. V
2. The theory of Cauchy's principal values (Second Paper: the use of principal values in some of the double limit problems of the integral calculus), *Proceedings of the London Mathematical Society*, (1) 34, 55–91. V

1902 (*cont.*)

3. On the Frullanian integral

$$\int_0^\infty \frac{\phi(ax^m)-\psi(bx^n)}{x}(\log x)^p\,dx,$$

Quarterly Journal of Mathematics, 33, 113–144. V

4. N.I.C. III: On the logarithmic criteria for the absolute convergence of an integral whose upper limit is ∞, *Messenger of Mathematics*, 31, 1–6. V

5. N.I.C. IV: On the integral $\int^\infty \sin x\psi(x)\,dx$, *Messenger of Mathematics*, 31, 6–8. V

6. A new proof of Kummer's series for $\log\Gamma(a)$, *Messenger of Mathematics*, 31, 31–33. IV. 1 (*d*)

7. N.I.C. V: On absolutely convergent integrals of functions which are infinitely often infinite, *Messenger of Mathematics*, 31, 73–76. V

8. N.I.C. VI: Absolute convergence of infinite multiple integrals, *Messenger of Mathematics*, 31, 125–128. V

9. N.I.C. VII: On differentiation under the integral sign, *Messenger of Mathematics*, 31, 132–134. V

10. N.I.C. VIII: Absolutely convergent integrals of irregular types, *Messenger of Mathematics*, 31, 177–183. V

11. On the zeroes of the integral function

$$x-\sin x = \sum_1^\infty (-)^{n-1}\frac{x^{2n+1}}{2n+1!},$$

Messenger of Mathematics, 31, 161–165. IV. 1 (*a*)

12. Questions 1423, 2316, 3941, 4794, *Educational Times*, (2) 1, 25.
13. Question 14851, *Educational Times*, (2) 1, 58–59.
14. Question 14055, *Educational Times*, (2) 2, 41–42.
(12–14) VII. 3

1903

1. The theory of Cauchy's principal values (Third Paper: differentiation and integration of principal values), *Proceedings of the London Mathematical Society*, (1) 35, 81–107. V

2. On the continuity and discontinuity of definite integrals which contain a continuous parameter, *Quarterly Journal of Mathematics*, 34, 28–53. V

3. Note on the limiting values of the elliptic modular functions, *Quarterly Journal of Mathematics*, 34, 76–86. IV. 1 (*d*)

4. N.I.C. IX: On the integral $\int^\infty \{A-\phi(\sin^2 x)\}\psi(x)\,dx$, *Messenger of Mathematics*, 32, 1–3. V

5. On the zeroes of certain integral functions, *Messenger of Mathematics*, 32, 36–45. IV. 1 (*a*)

6. On the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\log(ax^2+2bx+c)^2}{\alpha x^2+2\beta x+\gamma}\,dx$, *Messenger of Mathematics*, 32, 45–50. V

7. N.I.C. X: On conditionally convergent infinite multiple integrals, *Messenger of Mathematics*, 32, 92–97. V

8. N.I.C. XI: Some conditionally convergent infinite double integrals, *Messenger of Mathematics*, 32, 159–165. V

9. N.I.C. XII: On the operation which is the inverse of double integration, *Messenger of Mathematics*, 32, 187–192. V

10. Question 14988, *Educational Times*, (2) 3, 94–95.
11. Question 14989, *Educational Times*, (2) 4, 69–70.
12. Question 15019, *Educational Times*, (2) 4, 75.
13. Question 15265, *Educational Times*, (2) 4, 109–110.
(10–13) VII. 3

COMPLETE LIST OF HARDY'S MATHEMATICAL PAPERS

1904

1. On the convergence of certain multiple series, *Proceedings of the London Mathematical Society*, (2) 1, 124–128. VI
2. A general theorem concerning absolutely convergent series, *Proceedings of the London Mathematical Society*, (2) 1, 285–90. VII. 2
3. On differentiation and integration of divergent series, *Transactions of the Cambridge Philosophical Society*, 19, 297–321. VI
4. Researches in the theory of divergent series and divergent integrals, *Quarterly Journal of Mathematics*, 35, 22–66. VI
5. A theorem concerning the infinite cardinal numbers, *Quarterly Journal of Mathematics*, 35, 87–94. VII. 2
6. Note on the function $\int_x^\infty e^{\frac{1}{2}(x^2-t^2)}dt$, *Quarterly Journal of Mathematics*, 35, 193–207. VII. 1
7. The asymptotic solution of certain transcendental equations, *Quarterly Journal of Mathematics*, 35, 261–282. IV. 1 (*a*)
8. N.I.C. XIII: On differentiation under the integral sign (continued), *Messenger of Mathematics*, 33, 62–67. V
9. The cardinal number of a closed set of points, *Messenger of Mathematics*, 33, 67–69. VII. 2
10. N.I.C. XIV: Integrals whose discontinuities are everywhere dense, *Messenger of Mathematics*, 33, 80–85. V
11. Note on divergent Fourier series, *Messenger of Mathematics*, 33, 137–144. III. 1 (*b*)
12. On the zeroes of two classes of Taylor series, *British Association Report*, 441–443. IV. 1 (*a*)
13. Question 15300, *Educational Times*, (2) 5, 61.
14. Additional note on Question 15282, *Educational Times*, (2) 5, 113–114.
15. Question 15361, *Educational Times*, (2) 5, 118.
16. Question 15125, *Educational Times*, (2) 6, 31.

(13–16) VII. 3

1905

1. On the roots of the equation $\dfrac{1}{\Gamma(x+1)} = c$, *Proceedings of the London Mathematical Society*, (2) 2, 1–7. IV. 1 (*a*)
2. (With T. J. I'A. Bromwich.) Some extensions to multiple series of Abel's theorem on the continuity of power series, *Proceedings of the London Mathematical Society*, (2) 2, 161–189. VI
3. Note in addition to a former paper on conditionally convergent multiple series. *Proceedings of the London Mathematical Society*, (2) 2, 190–191. VI
4. On the zeroes of certain classes of integral Taylor series. Part I: On the integral function $\sum_{n=0}^{\infty} \dfrac{x^{\phi(n)}}{\{\phi(n)\}!}$, *Proceedings of the London Mathematical Society*, (2) 2, 332–39. IV. 1 (*a*)
5. On the zeroes of certain classes of integral Taylor series. Part II: On the integral function $\sum_{n=0}^{\infty} \dfrac{x^n}{(n+a)^s n!}$ and other similar functions, *Proceedings of the London Mathematical Society*, (2) 2, 401–431. IV. 1 (*a*)
6. A method for determining the behaviour of certain classes of power series near a singular point on the circle of convergence, *Proceedings of the London Mathematical Society*, (2) 3, 381–389. IV. 1 (*b*)
7. On a class of analytic functions, *Proceedings of the London Mathematical Society*, (2) 3, 441–460. IV. 1 (*b*)

1905 (*cont.*)

8. On certain series of discontinuous functions connected with the modular functions, *Quarterly Journal of Mathematics*, 36, 93–123. IV. 1 (*d*)
9. Note on an integral function, *Messenger of Mathematics*, 34, 1–2. IV. 1 (*a*)
10. N.I.C. XV: On upper and lower integration, *Messenger of Mathematics*, 34, 3–6. V
11. N.I.C. XVI: A class of conditionally convergent infinite multiple integrals, *Messenger of Mathematics*, 34, 6–10. V
12. A generalization of Frullani's integral, *Messenger of Mathematics*, 34, 11–18, and note, p. 102. V
13. On the zeroes of a class of integral functions, *Messenger of Mathematics*. 34, 97–101. IV. 1 (*a*)
14. On certain conditionally convergent multiple series connected with the elliptic functions, *Messenger of Mathematics*, 34, 146–153. VI
15. Question 15686, *Educational Times*, (2) 8, 74. VII. 3
16. The expression of the double zeta function and double gamma function in terms of elliptic functions, *Transactions of the Cambridge Philosophical Society*, 20, 1–35. IV. 1 (*d*)

1906

1. On Kummer's series for log $\Gamma(a)$, *Quarterly Journal of Mathematics*, 37, 49–53. IV. 1 (*d*)
2. On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters, *Quarterly Journal of Mathematics*, 37, 53–79. IV. 1 (*d*)
3. On the function $P_\rho(x)$, *Quarterly Journal of Mathematics*, 37, 146–172 (correction at end of 1906, 5). IV. 1 (*a*)
4. On certain double integrals, *Quarterly Journal of Mathematics*, 37, 360–9. V
5. On the integral function $\Phi_{a,\alpha,\beta}(x) = \sum_0^\infty \frac{x^n}{(n+a)^{\alpha n+\beta}}$, *Quarterly Journal of Mathematics*, 37, 369–378. IV. 1 (*a*)
6. N.I.C. XVII: On the integration of series, *Messenger of Mathematics*, 35, 126–130. V
7. A formula for the prime factors of any number, *Messenger of Mathematics*, 35, 145–146. II. 1
8. N.I.C. XVIII: On some discontinuous integrals, *Messenger of Mathematics*, 35, 158–166. V
9. Some notes on certain theorems in higher trigonometry, *Mathematical Gazette*, 3, 284–288. V

1907

1. The continuum and the second number class, *Proceedings of the London Mathematical Society*, (2) 4, 10–17. VII. 2
2. Some theorems connected with Abel's theorem on the continuity of power series, *Proceedings of the London Mathematical Society*, (2) 4, 247–265. VI
3. On the singularities of functions defined by Taylor's series (Remarks in addition to a former paper), *Proceedings of the London Mathematical Society*, (2) 5, 197–205. IV. 1 (*b*)
4. The singular points of certain classes of functions of several variables, *Proceedings of the London Mathematical Society*, (2) 5, 342–360. IV. 1 (*b*)
5. On certain oscillating series, *Quarterly Journal of Mathematics*, 38, 269–288. VI
6. Some theorems concerning infinite series, *Mathematische Annalen*, 64, 77–94. VI
7. N.I.C. XIX: On Abel's lemma and the second theorem of the mean, *Messenger of Mathematics*, 36, 10–13. V
8. Higher trigonometry, *Mathematical Gazette*, 4, 13–14. VII. 2
9. A curious imaginary curve, *Mathematical Gazette*, 4, 14. VII. 2
10. The line at infinity, etc., *Mathematical Gazette*, 4, 14–15. VII. 2

1908

1. Generalization of a theorem in the theory of divergent series, *Proceedings of the London Mathematical Society*, (2) 6, 255–264. VI

2. The multiplication of conditionally convergent series, *Proceedings of the London Mathematical Society*, (2) 6, 410–423. VI

3. Further researches in the theory of divergent series and integrals, *Transactions of the Cambridge Philosophical Society*, 21, 1–48. VI

4. (With T. J. I'A. Bromwich.) The definition of an infinite integral as the limit of a finite or infinite series, *Quarterly Journal of Mathematics*, 39, 222–240. V

5. Some multiple integrals, *Quarterly Journal of Mathematics*, 39, 357–375. V

6. N.I.C. XX: On double Frullanian integrals, *Messenger of Mathematics*, 37, 96–103. V

7. N.I.C. XXI: On a conditionally convergent multiple integral, *Messenger of Mathematics*, 37, 127–130. V

8. N.I.C. XXII: On double Frullanian integrals (cont.), *Messenger of Mathematics*, 37, 154–161. V

9. Question 16257, *Educational Times*, (2) 13, 79–80. VII. 3

10. Mendelian proportions in a mixed population, *Science* (American Association for the Advancement of Science), new series 28, 49–50. VII. 2

1909

1. A note on the continuity or discontinuity of a function defined by an infinite product, *Proceedings of the London Mathematical Society*, (2) 7, 40–48. VI

2. The theory of Cauchy's principal values (Fourth Paper: the integration of principal values—continued—with applications to the inversion of definite integrals), *Proceedings of the London Mathematical Society*, (2) 7, 181–208. V

3. On an integral equation, *Proceedings of the London Mathematical Society*, (2) 7, 445–472. VII. I

4. N.I.C. XXIII: On certain oscillating cases of Dirichlet's integral, *Messenger of Mathematics*, 38, 1–8. V

5. On certain definite integrals whose values can be expressed in terms of Bessel's functions, *Messenger of Mathematics*, 38, 129–132. V

6. N.I.C. XXIV: Oscillating cases of Dirichlet's integral (cont.), *Messenger of Mathematics*, 38, 176–185 (correction at end of 1911, 3). V

7. The integral $\int_0^\infty \frac{\sin x}{x}\,dx$, *Mathematical Gazette*, 5, 98–103. V

1910

1. The application to Dirichlet's series of Borel's exponential method of summation, *Proceedings of the London Mathematical Society*, (2) 8, 277–294. VI

2. The ordinal relations of the terms of a convergent sequence, *Proceedings of the London Mathematical Society*, (2) 8, 295–300. VII. 2

3. Theorems relating to the summability and convergence of slowly oscillating series, *Proceedings of the London Mathematical Society*, (2) 8, 301–320. VI

4. The maximum modulus of an integral function, *Quarterly Journal of Mathematics*, 41, 1–9. IV. 2

5. On certain definite integrals considered by Airy and by Stokes, *Quarterly Journal of Mathematics*, 41, 226–240. IV. 1 (*d*)

6. N.I.C. XXV: Absolutely convergent integrals of irregular types (cont.), *Messenger of Mathematics*, 39, 28–32. V

7. The zeroes of the integral function $\sum \frac{x^{n^2}}{n^2!}$, and of some similar functions, *Messenger of Mathematics*, 39, 88–96. IV. 1 (*a*)

1910 (*cont.*)

8. N.I.C. XXVI: On a case of term-by-term integration of an infinite series, *Messenger of Mathematics*, 39, 136–139. V
9. To find an approximation to the large positive root of the equation $e^{e^x} = 10^{10}x^{10}e^{10^{10}x^{10}}$, *Mathematical Gazette*, 5, 333–334. VII. 2

1911

1. Theorems connected with Maclaurin's test for the convergence of series, *Proceedings of the London Mathematicsl Society*, (2), 9, 126–144. VI
2. (With S. Chapman.) A general view of the theory of summable series, *Quarterly Journal of Mathematics*, 42, 181–215. VI
3. N.I.C. XXVII: Oscillating cases of Dirichlet's integral (cont.), *Messenger of Mathematics*, 40, 44–53. V
4. A class of definite integrals, *Messenger of Mathematics*, 40, 53–54. V
5. N.I.C. XXVIII: A conditionally convergent double integral, *Messenger of Mathematics*, 40, 62–69. V
6. N.I.C. XXIX: Two convergence theorems, *Messenger of Mathematics*, 40, 87–91. V
7. N.I.C. XXX: A theorem concerning summable integrals, *Messenger of Mathematics*, 40, 108–112. VI
8. N.I.C. XXXI: The uniform convergence of Borel's integral, *Messenger of Mathematics*, 40, 161–165. VI
9. Fourier's double integral and the theory of divergent integrals, *Transactions of the Cambridge Philosophical Society*, 21, 427–451. VII. 1

1912

1. Properties of logarithmico-exponential functions, *Proceedings of the London Mathematical Society*, (2) 10, 54–90. IV. 1 (*c*)
2. On the multiplication of Dirichlet's series, *Proceedings of the London Mathematical Society*, (2) 10, 396–405. VI
3. Some results concerning the behaviour at infinity of a real and continuous solution of an algebraic differential equation of the first order, *Proceedings of the London Mathematical Society*, (2) 10, 451–468. VII. 2
4. (With J. E. L.) Some problems of Diophantine approximation, *5th International Congress of Mathematicians, Cambridge*, 1, 223–229. I. 1
5. Generalizations of a limit theorem of Mr. Mercer, *Quarterly Journal of Mathematics*, 43, 143–150. VI
6. Note on Dr. Vacca's series for γ, *Quarterly Journal of Mathematics*, 43, 215–216. IV. 1 (*d*)
7. Note on a theorem of Cesàro, *Messenger of Mathematics*, 41, 17–22. VI
8. N.I.C. XXXII: On double series and double integrals, *Messenger of Mathematics*, 41, 44–48. V
9. N.I.C. XXXIII: Some cases of the inversion of the order of integration, *Messenger of Mathematics*, 41, 102–109. VII. 1

1913

1. (With J. E. L.) The relations between Borel's and Cesàro's methods of summation, *Proceedings of the London Mathematical Society*, (2) 11, 1–16. VI
2. (With J. E. L.) Contributions to the arithmetic theory of series, *Proceedings of the London Mathematical Society*, (2) 11, 411–478. VI
3. An extension of a theorem on oscillating series, *Proceedings of the London Mathematical Society*, (2) 12, 174–180. VI

1913 (*cont.*)

4. On the summability of Fourier's series, *Proceedings of the London Mathematical Society*, 12, 365–372. III. 1 (*b*)

5. Oscillating Dirichlet's integrals: an essay in the 'Infinitärcalcül' of Paul Du Bois-Reymond, *Quarterly Journal of Mathematics*, 44, 1–40. IV. 1 (*c*)

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3. (With J. E. L.) N.S. VII: On Young's convergence criterion for Fourier series, *Proceedings of the London Mathematical Society*, (2) 28, 301–311. III. 1 (*b*)
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3. (With J. E. L.) D.A.: A series of cosecants, *Bulletin of the Calcutta Mathematical Society*, 20, 251–266. I. 1
4. (With J. E. L.) N.S. XI: On Tauberian theorems, *Proceedings of the London Mathematical Society*, (2) 30, 23–37. VI
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6. (With J. E. L.) N.S. XII: On certain inequalities connected with the calculus of variations, *Journal of the London Mathematical Society*, 5, 34–39. II. 3

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6. (With E. C. Titchmarsh) A note on Parseval's theorem for Fourier transforms, *Journal of the London Mathematical Society*, 6, 44–48. VII. 1
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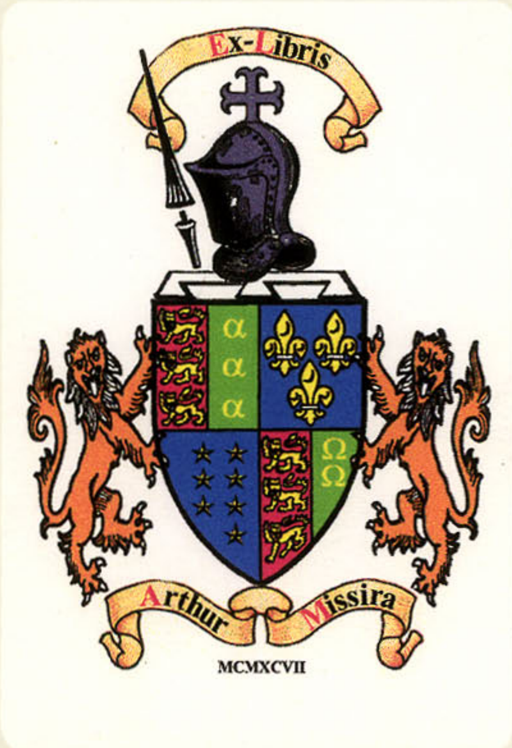
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